THE SIMULTANEOUS EQUATIONS

A. **Simultaneous Linear equations with two unknowns**

Case I: A pair of linear equations in two variables

$$\boxed{\boldsymbol{a_0x + b_0y + c_0 = 0 \ \ , \ a_1x + b_1y + c_1 = 0}}$$

NOTE: x and y are two variables and the unknowns.

Methods to be used

- Solution by substitution
- Solution by eliminating coefficients (Elimination method)
- Matrix method
- Crammer's rule method
- Determinant method
- Graphical method

The last four methods have not been discussed, but the first two are to be reviewed in the subsequent examples below.

Examples

Solve the pair of simultaneous equations

a) $5x + 2y = 14\,, 3x - 4y = 24$

b) $3x + 4y - 9 = 0\,, 2x + 3y = 8$

Solution

Solution by substitution

a) $5x + 2y = 14$.................................... (1)

$3x - 4y = 24$ .... (2)

From eqn. (1), making x the subject

$\Rightarrow x = \frac{14-2y}{5}$ .... (3)

Substituting eqn. (3) in to eqn. (2)

$\Rightarrow 3\left(\frac{14-2y}{5}\right) - 4y = 24$.............................$(\times 5)$

$\Rightarrow 3(14 - 2y) - 20y = 120$

$\Rightarrow 42 - 6y - 20y = 120 \ or - 26y = 78$

$\therefore y = -3$

Substituting $y = -3$ in two eqn. (3) (either of the original equations)

$\Rightarrow x = \frac{14-2\times-3}{5} = \frac{20}{5} = 4$

$\therefore x = 4\,, y = -3$

Solving by equating coefficients

$5x + 2y = 14$.................................... (1)

$3x - 4y = 24$ .... (2)

Multiplying both sides of eqn. (1) by 3 (the coefficient of x in eqn. (2)) and multiplying both sides of eqn. (2) by 5 (the coefficient of y in eqn. (1))

$\Rightarrow 15x + 6y = 42$

$\Rightarrow 15x - 20y = 120$

Now subtracting the equations,

$26y = -78 \therefore y = -3$

Substituting $y = -3$ in either of the original equations

$5x + 2y = 14\,, y = -3$
$\Rightarrow 5x - 6 = 14 \therefore 5x = 20 \;\; or \; x = 4$
$\therefore x = 4\,, y = -3$

b) Left as an exercise. **ANSWER**: $\boldsymbol{x = -5\,, y = 6}$
Try also to solve simultaneously
$3x + 2y = 16\,, 4x - 3y - 10 = 0.$ **ANSWER**: $\boldsymbol{x = -5\,, y = 6}$

Case II: A pair of simultaneous equations in two variables, one Quadratic and one Linear

Example

Solve the pair of simultaneous equations

a) $4x - 3y = 1\,, 12xy + 13x^2 = 25$
b) $x^2 + y^2 - 6x + 4y - 13 = 0\,, 2x = 3y - 1$

Solution

a) $4x - 3y = 1$…………………………………. (1)
$12xy + 13x^2 = 25$ ………………………… (2)
From eqn. (1)
$y = \frac{4x-1}{3}$ ……………………………………. (3)
Substitute eqn. (3) in to eqn. (2)
$\Rightarrow 12x\left(\frac{4x-1}{3}\right) + 13x^2 = 25$ ………………………$(\times 3)$
$\Rightarrow 48x^2 - 12x + 39x^2 = 75$
$\therefore 87x^2 - 12x - 75 = 0$ ……………………………$(\div 3)$
$\Rightarrow 29x^2 - 4x - 25 = 0$
$\Rightarrow (29x + 25)(x - 1) = 0$
$\therefore x = 1\,, -\frac{25}{29}$ …………………………………………. (4)
Eqn. (4) in to eqn. (3) yields
$y = \frac{4-1}{3} = 1$, for $x = 5$ and $y = \frac{-\frac{100}{29}-1}{3}$ for $x = -\frac{25}{29}$
$\therefore x = 1\,, y = 1\,; x = -\frac{25}{29}\,, y = -\frac{43}{29}$

b) $2x = 3y - 1 \Rightarrow x = \frac{3y-1}{2}$ ……………………………… (1)
$x^2 + y^2 - 6x + 4y - 13 = 0$ ……………………… (2)
Substitute eqn. (1) in to eqn. (2)
$\Rightarrow \left(\frac{3y-1}{2}\right)^2 + y^2 - 6\left(\frac{3y-1}{2}\right) + 4y - 13 = 0$
$\Rightarrow \frac{9y^2-6y+1}{4} + y^2 + 3 - 9y + 4y - 13 = 0$ ……………..$(\times 4)$
$\Rightarrow 9y^2 - 6y + 1 + 4y^2 + 12 - 36y + 16y - 52 = 0$
$\Rightarrow 13y^2 - 26y - 39 = 0 \;\; or \; y^2 - 2y - 4 = 0$
$\Rightarrow (y - 3)(y + 1) = 0$
$\therefore y = -1\,, 3$
Substitute eqn. (3) in to eqn. (1)
For $y = -1\,, x = \frac{3\times-1-1}{2} = -2$
For $y = 3\,, x = \frac{3\times3-1}{2} = 4$
$\therefore x = -2\,, y = -1; x = 4\,, y = 3$

Other techniques of solving specific set of simultaneous equations

Solve:

a) $x^2 + y^2 = 185\,, x - y = 3$
b) $x^3 + y^3 = 4914\,, x + y = 18$
c) $x^3 - y^3 = 218\,, x - y = 2$
d) $2x + 3y = 5\,, xy = 1$
e) $5x - y = 3\;,\; y^2 - 6x^2 = 25$
f) $3x - 2y = 7\;, xy = 20$

NOTE:
The above equations can also be solved using substitution as in the previous examples.

Solution

a) $x^2 + y^2 = 185$…………………………………… (1)
$x - y = 3$ …………………………………………… (2)
From eqn. (2), squaring both sides
$(x - y)^2 = 3^2$
$\Rightarrow x^2 + y^2 - 2xy = 9$
But from eqn. (1), $x^2 + y^2 = 185$
$\Rightarrow 185 - 2xy = 9 \; or \; xy = 88$ …………… (3)
Again $(x + y)^2 = x^2 + y^2 + 2xy$
From eqn. (1) and eqn. (3)
$\Rightarrow (x + y)^2 = 185 + 2 \times 88$
$\Rightarrow (x + y)^2 = 361$
$\therefore x + y = \pm 19$ ……………………………… (4)
Now solving eqn. (2) and eqn. (4) simultaneously
Adding the equations,
$\Rightarrow 2x = \pm 19 + 3$
$\therefore 2x = 19 + 3 = 22 \;\; or \; x = 11$
$\therefore 2x = -19 + 3 = -16 \;\; or \; x = -8$
From eqn. (2), $y = x - 3$
For $x = 11\,, y = 11 - 3 = 8$
For $x = -8\,, y = -8 - 3 = -11$
$\therefore x = 11\,, y = 8; x = -8\,, y = -11$

b) $x^3 + y^3 = 4914$ …………………………………… (1)
$x + y = 18$ ……………………………………………… (2)
From eqn. (2), cubing both sides
$(x + y)^3 = 18^3$
$\Rightarrow x^3 + y^3 + 3x^2y + 3xy^2 = 5832$
$\Rightarrow x^3 + y^3 + 3xy(x + y) = 5832$
But from eqn. (1) and eqn. (2), $x^3 + y^3 = 4914\,, x + y = 18$
$\Rightarrow 4914 + 3xy(18) = 5832$
$\therefore xy = 17$ ……………………………………………… (3)
Squaring eqn. (2) on both sides
$\Rightarrow \; (x + y)^2 = 18^2$
$\Rightarrow x^2 + y^2 + 2xy = 324$ …………………………………… (4)
Substitute eqn. (3) in to eqn. (4)
$\Rightarrow x^2 + y^2 + 34 = 324$

$\therefore x^2 + y^2 = 290$ ………………………………………………… (5)
Also from $(x - y)^2 = x^2 + y^2 - 2xy$
$\Rightarrow (x - y)^2 = 290 - 34 = 256$
$\therefore x - y = \pm 16$ …………………………………………………… (6)
Solving eqn. (2) and eqn. (6) simultaneously
$x + y = 18\,, x - y = \pm 16$
Adding the equations,
$\Rightarrow 2x = \pm 16 + 18$
$\therefore 2x = 16 + 18 = 34 \;\; or\; x = 17$
$\therefore 2x = -16 + 18 = 2 \;\; or\; x = 1$
Also subtracting the equations,
$\Rightarrow 2y = 18 - \;\pm 16$
$\therefore 2y = 18 - 16 = 2 \;\; or\; y = 1$
$\therefore 2y = 18 + 16 = 34 \;\; or\; y = 17$
$\therefore x = 1\,, y = 17; x = 17\,, y = 1$

NOTE:
The equations have been reduced to a solution depending on the quadratic equation.

c) $x^3 - y^3 = 218$ ………………………………………………… (1)
$x - y = 2$ ……………………………………………………… (2)
From eqn. (2), cubing both sides
$(x - y)^3 = 2^3$
$\Rightarrow x^3 - y^3 - 3x^2y + 3xy^2 = 8$
$\Rightarrow x^3 - y^3 - 3xy(x - y) = 8$
But from eqn. (1) and eqn. (2), $x^3 - y^3 = 218\,, x - y = 2$
$\Rightarrow 218 - 3xy(2) = 8$
$\therefore xy = 35$ ……………………………………………………… (3)
Squaring eqn. (2) on both sides
$\Rightarrow \;(x - y)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 2xy = 4$ …………………………………… (4)
Substitute eqn. (3) in to eqn. (4)
$\Rightarrow x^2 + y^2 - 70 = 4$
$\therefore x^2 + y^2 = 74$ ………………………………………………… (5)
Also from $(x + y)^2 = x^2 + y^2 + 2xy$
$\Rightarrow (x + y)^2 = 74 + 70 = 144$
$\therefore x + y = \pm 12$ ………………………………………………… (6)
Solving eqn. (2) and eqn. (6) simultaneously
$x - y = 2\,, x + y = \pm 12$
Adding the equations,
$\Rightarrow 2x = \pm 12 + 2$
$\therefore 2x = 12 + 2 = 14 \;\; or\; x = 7$
$\therefore 2x = -12 + 2 = -10 \;\; or\; x = -5$
Also subtracting the equations,
$\Rightarrow 2y = \pm 12 - 2$
$\therefore 2y = 12 - 2 = 10 \;\; or\; y = 5$
$\therefore 2y = -12 - 2 = -14 \;\; or\; y = -7$
$\therefore x = 7\,, y = 5; x = -5\,, y = -7$

d) $2x + 3y = 5$ .... (1)
$xy = 1$ .... (2)
Re-writing eqn. (2) in terms of terms of eqn. (1)
$\Rightarrow 2x.3y = 6$ (No change in the meaning in eqn. (2))
Now squaring eqn. (1)
$(2x + 3y)^2 = 5^2$
$\Rightarrow 4x^2 + 2.2x.3y + 9y^2 = 25$, but $\Rightarrow 2x.3y = 6$............. (*)
$\therefore 4x^2 + 9y^2 = 25 - 12 = 13$ .... (3)
Also from $(2x - 3y)^2 = 4x^2 + 9y^2 - 2.2x.3y$
But from eqn. (*) and eqn. (3), $2x.3y = 6 , 4x^2 + 9y^2 = 13$
$\Rightarrow (2x - 3y)^2 = 13 - 12 = 1$
$2x - 3y = \pm 1$.. (4)
Solving eqn. (1) and eqn. (4) simultaneously
$2x + 3y = 5 , 2x - 3y = \pm 1$
Adding the equations,
$\Rightarrow 4x = 5 \pm 1$
$\therefore 4x = 5 + 1 = 6 \;\; or \; x = \frac{3}{2}$
$\therefore 4x = 5 - 1 = 4 \;\; or \; x = 1$
Subtracting the equations,
$\Rightarrow 6y = 5 - \;\pm 1$
$\therefore 6y = 5 - 1 = 4 \;\; or \; y = \frac{3}{2}$
$\therefore 6y = 5 - \;-1 = 6 \;\; or \; y = 1$
$\therefore x = \frac{3}{2} , y = \frac{2}{3} ; x = 1 , y = 1$

e) Left as an exercise. **ANSWER**: $\boldsymbol{x = 2 , y = 7}$; $\boldsymbol{x = -\frac{8}{9} , y = -\frac{97}{19}}$

f) Left as an exercise. **ANSWER**: $\boldsymbol{x = 5 , y = 4}$; $\boldsymbol{x = -\frac{8}{3} , y = -\frac{15}{2}}$

Task

Solve: $3x - 5y = 2 , xy = 8$ **ANSWER**: $\boldsymbol{x = 4 , y = 2 ; x = -3\frac{1}{3} , y = -2\frac{2}{5}}$

HOMOGENEOUS EQUATIONS

Homogeneous equations are equations in which all terms have the same degree.
$e.g.\; ax^3 + bx^2y + cy^2x + dy^3 = f$, is a homogeneous of 3^0
$lx^2 + mxy + ny^2 = p$, is a homogeneous of 2^0
If the powers in every term in variables x and y are added, give a uniform degree.

Solving Homogeneous Equations

The equations to be considered are:

$$\left.\begin{aligned} \boldsymbol{lx^2 + mxy + ny^2} &= \boldsymbol{A} \\ \boldsymbol{px^2 + qxy + ry^2} &= \boldsymbol{B} \end{aligned}\right\} \text{Both quadratic}$$

$$\boldsymbol{lx^2 + mxy + ny^2 = A}$$
$$\boldsymbol{qxy = B}$$

, where $l , m , n , p , q , r , A , B$ are constants.

Examples

1) *Solve the pair of simultaneous equations.*

$x^2 + xy + 4y^2 = 6$ .... (1)
$3x^2 + 8y^2 = 14$ .... (2)

Solution

Re-writing the equations by factoring

$\Rightarrow x^2\left[1 + \frac{y}{x} + 4\left(\frac{y}{x}\right)^2\right] = 6$ .... (3)

$\Rightarrow x^2\left[3 + 8\left(\frac{y}{x}\right)^2\right] = 14$ .... (4)

Eqn. (3) ÷ eqn. (4)

$\frac{x^2\left[1+\frac{y}{x}+4\left(\frac{y}{x}\right)^2\right]}{x^2\left[3+8\left(\frac{y}{x}\right)^2\right]} = \frac{6}{14}$; let $m = \frac{y}{x}$

$\Rightarrow \frac{1+m+4m^2}{3+8m^2} = \frac{3}{7}$ or $7 + 7m + 28m^2 = 9 + 24m^2$

$\therefore 4m^2 + 7m - 2 = 0$ or $(4m - 1)(m + 2) = 0$

$\therefore m = \frac{1}{4}, -2$

But $m = \frac{y}{x}$ or $y = mx$

$\Rightarrow y = \frac{1}{4}x$ .... (5)

$\Rightarrow y = -2x$ .... (6)

Using any of the equations (3) or (4)

From eqn. (4) $x^2\left[3 + 8\left(\frac{y}{x}\right)^2\right] = 14$

$\Rightarrow x^2[3 + 8m^2] = 14$

For $m = -2$,

$x^2[3 + 8(-2)^2] = 14$ or $x^2 = \frac{14}{35} = \frac{2}{5}$

$\therefore x = \pm\sqrt{\frac{2}{5}}$

From eqn. (6), $y = -2x = -2\left(\pm\sqrt{\frac{2}{5}}\right)$

$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}$

$\therefore x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}$

Now for $m = \frac{1}{4}$, equation $x^2[3 + 8m^2] = 14$ becomes;

$\Rightarrow x^2\left[3 + 8\left(\frac{1}{4}\right)^2\right] = 14$ or $x^2 = 4$

$\therefore x = \pm 2$

From eqn. (6), $y = -2x = -2(\pm 2)$

$\therefore x = 2, y = -4$

$\therefore x = -2, y = 4$

$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}$; $x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}$; $x = 2, y = -4$; $x = -2, y = 4$

2) *Solve the pair of simultaneous equations.*

$x^2 + 4xy + y^2 = 13$ .... (1)
$2x^2 + 3xy = 8$ .... (2)

Solution

Re-writing the equations by factoring

$\Rightarrow x^2\left[1+4\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right]=13$ …………………… (3)

$\Rightarrow x^2\left[2+3\left(\frac{y}{x}\right)\right]=8$ ………………………………… (4)

Eqn. (3) ÷ eqn. (4)

$\frac{x^2\left[1+4\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right]}{x^2\left[2+3\left(\frac{y}{x}\right)\right]}=\frac{13}{8}$; let $m=\frac{y}{x}$

$\Rightarrow \frac{1+4m+m^2}{2+3m}=\frac{13}{8}$ $or\ 1+4m+m^2=2+3m$

$\therefore 8m^2-7m-18=0$

$\therefore m=\frac{7\pm\sqrt{49+4(8)(18)}}{16}=\frac{7\pm25}{16}$

$\therefore m=2\,,-\frac{9}{8}$

From $m=\frac{y}{x}\Rightarrow y=mx$

$\Rightarrow y=2x$ …………………………………………………………… (5)

$\Rightarrow y=-\frac{9}{8}x$ ……………………………………………………… (6)

From equation (4)

$\Rightarrow x^2\left[2+3\left(\frac{y}{x}\right)\right]=8 \therefore x^2[2+3m]=8$

For $m=2\,,x^2(2+6)=8$

$\therefore x^2=1\ \ or\ x=\pm1$

But $y=2x$

$\therefore$ For $x=1\,,y=2$

$\therefore$ For $x=-1\,,y=-2$

$\therefore x=1\,,y=2\,;x=-1\,,y=-2$

Also for $m=-\frac{9}{8}$

From equation (4)

$\Rightarrow x^2\left[2+3\left(-\frac{9}{8}\right)\right]=8 \therefore x^2=-\frac{64}{11}$, hence roots are complex.

Alternatively

Dealing with question (1)

$x^2+xy+4y^2=6$ …………………………………… (1)

$3x^2+8y^2=14$ ………………………………………… (2)

Eliminating constants 6 and 14

$Eqn.(1)\times14-Eqn.(2)\times6$

$\Rightarrow 14x^2+14xy+56y^2=84$

$\Rightarrow \underline{18x^2+0xy+48y^2=84}$

$-4x^2+14xy+8y^2=0$

$or\ \Rightarrow 2x^2-7xy-4y^2=0$

Factorizing to linear factors gives

$(2x+y)(x-2y)=0$

$\therefore y=-2x\ \ or\ y=\frac{1}{4}x$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$3x^2+8y^2=14\,,y=-2x$, gives

$3x^2+32x^2=14\ \therefore x^2=\frac{2}{5}\ \ or\ x=\pm\sqrt{\frac{2}{5}}$

For $x=\sqrt{\frac{2}{5}}\,,y=-\sqrt{\frac{2}{5}}$

For $x = -\sqrt{\frac{2}{5}}\,, y = \sqrt{\frac{2}{5}}$

Also $3x^2 + 8y^2 = 14\,, y = \frac{1}{4}x$ gives

$3x^2 + 8\left(\frac{1}{4}x\right)^2 = 14 \;\; or\; x = \pm 2$

Now for $x = 2\,, y = \frac{1}{4} \times 2 = \frac{1}{2}$

Also for $x = -2\,, y = -\frac{1}{4} \times 2 = -\frac{1}{2}$

Dealing with question (2)

$x^2 + 4xy + y^2 = 13$ .... (1)

$2x^2 + 3xy = 8$ .... (2)

Eliminating constants 13 and 8

$Eqn.(1) \times 8 - Eqn.(2) \times 13$

$8x^2 + 32xy + 8y^2 = 104$

$\underline{26x^2 + 39xy + 0y^2 = 104}$

$-18x^2 - 7xy + 8y^2 = \; 0 \;\; or\; 18x^2 + 7xy - 8y^2 = 0$

Factoring in to linear factors

$(2x - y)(9x + 8y) = 0$

$\therefore y = 2x \;\; or\; y = -\frac{9}{8}x$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$2x^2 + 3xy = 8\,, y = 2x$

$\Rightarrow 2x^2 + 6x^2 = 8 \;\; \therefore x^2 = 1 \;\; or\; x = \pm 1$

$\therefore$ For $x = 1\,, y = 2 \;\; ; x = -1\,, y = -2$

Also $2x^2 + 3xy = 8\,, y = -\frac{9}{8}x$

$\Rightarrow 2x^2 + \frac{27}{8}x^2 = 8 \;\; \therefore x^2 = -\frac{64}{11} \;\; or\; x = complex$

NOTE:

1. The second alternative gives rise to a quadratic equation in terms of x and y which is factorisable to linear factors after eliminating the constant terms.
2. Substitution method can also be applicable to solve homogeneous simultaneous equations.

Task

Solve the pair of simultaneous equations

a) $2x^2 - xy - y^2 = 8\,, xy = 6$
b) $x^3 + y^3 = 35\,, x^2y + xy^2 = 30$
c) $x^2 - 2xy + 8y^2 = 8\,, 3xy - 2y^2 = 4$

ANSWERS: $\boldsymbol{a)}\;\; \boldsymbol{x = \pm 3\,; y = \pm 1} \quad \boldsymbol{c)}\;\; \boldsymbol{x = \pm 2 \;, y = \pm 1\,; x = \pm 1\,, y = \pm 12}$

Simultaneous Linear Equations with three unknowns

$$\boxed{\begin{aligned} &\boldsymbol{a_1x + b_1y + c_1z = A} \\ &\boldsymbol{a_2x + b_2y + c_2z = B} \\ &\boldsymbol{a_3x + b_3y + c_3z = C} \end{aligned}}$$

Methods to be considered are;

- Elimination method
- Reducing to roe Echelon form

Elimination method of solving linear set of equations

$a_1x + b_1y + c_1z = A$ ………………………………………… (1)
$a_2x + b_2y + c_2z = B$ ………………………………………… (2)
$a_3x + b_3y + c_3z = C$ ………………………………………… (3)

a) Eliminating any variable ($i.e. x \text{ or } y \text{ or } z$) from any pair of equations
Eliminating x from eqn. (1) and eqn. (2)
Multiply $\frac{a_2}{a_1}$ by eqn. (1) and subtract the result from eqn. (2)
The resulting equation is of the form

$\boxed{Py + Qz = R}$ ………………………………………… (4)

b) Eliminating the same variable $i.e. x$ from any other pair of equations
Eliminating x from eqn. (1) and eqn. (3)
Multiply $\frac{a_3}{a_1}$ by eqn. (1) and subtract the result from eqn. (3)
The resulting equation is of the form

$\boxed{my + nz = r}$ ………………………………………… (5)

Now solving eqn. (4) and eqn. (5) as discussed earlier (i.e. solving set of linear equations in two variables), this gives y and z values.
Now substituting y and z values in any of the original equations $.i.e.$ (1) or (2) or (3), the value of x can be obtained.

Examples

Solve the linear set of equations simultaneously

$$x + 4y + 4z = 7$$
$$3x + 2y + 2z = 6$$
$$9x + 6y + 2z = 14$$

Solution

$x + 4y + 4z = 7$ ………………………………………… (1)
$3x + 2y + 2z = 6$ ………………………………………… (2)
$9x + 6y + 2z = 14$ ………………………………………… (3)

Eliminating x from eqn. (1) and eqn. (2)

$Eqn.(1) \times \left(\frac{3}{1}\right) - eqn.(2)$

$3x + 12y + 12z = 21$
$\underline{3x + 2y + 2z = 6}$
$10y + 10z = 15 \text{ or } 2y + 2z = 3$……….. (4)

Eliminating x from eqn. (1) and eqn. (3)

$Eqn.(1) \times \left(\frac{9}{1}\right) - eqn.(3)$

$9x + 36y + 36z = 63$
$\underline{9x + 6y + 2z = 14}$
$30y + 34z = 49$ ………………………………… (5)

Now eliminating y from eqn. (4) and eqn. (5)

$Eqn.(4) \times \left(\frac{30}{2}\right) - eqn.(5)$

$30y + 30z = 45$
$\underline{30y + 34z = 49}$
$-4z = -4 \therefore z = 1$

Put $z = 1$ in to eqn. (4) or (5)

$2y + 2z = 3\,, z = 1$

$\Rightarrow 2y + 2 = 3 \therefore y = \frac{1}{2}$

Now substitute $z = 1\,, y = \frac{1}{2}$ in to eqn. (1) or eqn. (2) or eqn. (3) the original equations

Using eqn. (1)

$x + 4y + 4z = 7 \;\; \therefore x + 2 + 4 = 7 \;\; or\; x = 1$

$\therefore x = 1\,, y = \frac{1}{2}\,, z = 1$

Task

Solve the simultaneous equations

a) $x + 2y - 3z = 3$
$2x - y - z = 11$ **ANSWER**: $\boldsymbol{x = 2\,, y = -4\,, z = -3}$
$3x + 2y + z = -5$

b) $3x + 2y - z = 19$
$4x - y + 2z = 4$ **ANSWER**: $\boldsymbol{x = 3\,, y = 4\,, z = -2}$
$2x + 4y - 5z = 32$

Solving Linear set of equations in 3 unknowns by reducing to Row echelon form

Consider the equations to be solved

$a_1x + b_1y + c_1z = A$ .... (1)

$a_2x + b_2y + c_2z = B$ .... (2)

$a_3x + b_3y + c_3z = C$ .... (3)

Writing the equations in matrix form

$$\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{\textbf{Coefficient}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\textbf{Variable matrix}} = \underbrace{\begin{pmatrix} A \\ B \\ C \end{pmatrix}}_{\textbf{Constant matrix}}$$

, which may be written in short form $\mathbf{P.V = K}$

Writing elements of K in P matrix, the augmented matrix is formulated as;

$$Q = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array}\right)$$

Eliminating elements beow the major diagonal in the coefficient matrix gives rise to a triangular matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & A \\ \boldsymbol{a_2} & b_2 & c_2 & B \\ \boldsymbol{a_3} & \boldsymbol{b_3} & c_3 & C \end{pmatrix}$$

, elements to be zeroed/ eliminated are $a_1\,, a_2\,, a_3$

To eliminate a_2, subtract $\frac{\boldsymbol{a_2}}{\boldsymbol{a_1}}$ times first row from the 2^{nd} row, and to eliminate a_3, subtract $\frac{\boldsymbol{a_3}}{\boldsymbol{a_1}}$ times first row from 3^{rd} row $.i.e.$

$$\begin{pmatrix} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{pmatrix} \begin{matrix} Row\ 1 = R_1 \\ Row\ 2 = R_2 \\ Row\ 3 = R_3 \end{matrix}$$

Now eliminating using the formulae

$$\left(\begin{array}{ccc:c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array}\right) \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 - a_2/a_1 R_1 \\ R_3 \rightarrow R_3 - a_3/a_1 R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc:c} a_1 & b_1 & c_1 & A \\ 0 & d & e & D \\ 0 & f & g & E \end{array}\right)$$

Now eliminating **f**, subtract $\frac{f}{d}$ times 2^{nd} row from the 3^{rd} row

$$\left(\begin{array}{ccc:c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & D \\ a_3 & b_3 & c_3 & H \end{array}\right) \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 \\ R_3 \rightarrow R_3 \end{array}$$

, which is a triangular matrix

Finally detach the right hand column back to its original position

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d & e \\ 0 & 0 & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ D \\ H \end{pmatrix}$$

Multiplying the matrices, the result will be;

$a_1 x + b_1 y + c_1 z = A$ .... (i)

$dy + ez = D$ .... (ii)

$hz = H$ .... (iii)

From equation (iii) and by back substitution, values of x, y and z can be evaluated.

Examples

Solve the linear set of equations by reducing to row echelon form

$x - 2y - 3z = 0$

$3x + 5y + 2z = 0$

$2x + 3y - z = 2$

Solution

$x - 2y - 3z = 0$

$3x + 5y + 2z = 0$

$2x + 3y - z = 2$

Writing the equations in matrix form

$$\begin{pmatrix} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Formulating the augmented matrix

$$\left(\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Now eliminating 3 and 2 from column 1

$$\left(\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 - 3/1 R_1 \\ R_3 \rightarrow R_3 - 2/1 R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 7 & 5 & 2 \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_3 \rightarrow R_3 - 7/11 R_2 \end{array}$$

Now eliminating 7 from 3^{rd} row

$$\left(\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

Detaching the matrices back to their original positions

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 11 & 11 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$x - 2y - 3z = 0$ ……………………. (i)

$11y + 11z = 0$ ………… (ii)

$-2z = 2$ ………….. (iii)

From eqn. (iii), $z = 1$

Put $z = 1$ in to eqn. (ii)

$\Rightarrow 11y - 11 = 0 \;\therefore y = 1$

Put $z = 1, y = 1$ in to eqn. (i)

$x - 2 + 3 = 0 \;\therefore x = -1$

$\therefore x = -1, y = 1, z = -1$

Task

1. *Solve the equations simultaneously using row- echelon method*

   a) $2x + 3y + 4z = -4$
   $4x + 2y + 3z = -11$ **ANSWER**: $\boldsymbol{x = -3, y = 2, z = -1}$
   $3x + 4y + 2z = -3$

   b) $2a + b + 3c = 11$
   $a + 2b - 2c = 3$ **ANSWER**: $\boldsymbol{a = 1, b = 3, c = 2}$
   $4a + 3b + c = 15$

   c) $2p + 3q + 4r = 8$
   $3p - 2q - 3r = -2$ **ANSWER**: $\boldsymbol{p = 1, q = -2, r = 3}$
   $5p + 4q + 2r = 3$

2. *Simplify and solve the following set of simultaneous equations*

   a) $4(x + 3y) - 2(4x + 3z) - 3(x - 2y - 4z = 17)$
   $2(4x - 3y) + 5(x - 4z) + 4(x - 3y + 2z) = 23$
   $3(y + 4z) + 4(2x - y - z) + 2(x + 3y - 2z) = 5$
   ANSWER: $\boldsymbol{x = 1, y = 3, z = -5}$

   b) $5(x + 2y) - 4(3x + 4z) - 2(x + 3y - 5z = 16)$
   $2(3x - y) + 3(x - 4z) + 4(2x - 3y + z) = -16$
   $4(4 - 2z) + 2(2x - 4y - 3) - 3(x + 4y - 2z) = -62$
   ANSWER: $\boldsymbol{x = 2, y = 4, z = -3}$

Other 3 − unknown simultaneous equations

Solve the equations

i). $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2}$, $11x - 3y + 7z = 8$

ii). $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2}$, $6x + 6y + 2z = 6$ (***Uneb* 2005**)

Solution

i). Let $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2} = \lambda$

$$\left.\begin{array}{l} \Rightarrow x + 3 = \lambda \;\therefore x = \lambda - 3 \\ \Rightarrow \frac{y-1}{2} = \lambda \;\therefore y = 2\lambda + 1 \\ \Rightarrow \frac{z-5}{-2} = \lambda \;\therefore z = -2\lambda + 5 \end{array}\right\} \quad \text{………………………………………. (1)}$$

Substituting eqn. (1) in to the 2^{nd} equation $11x - 3y + 7z = 8$

$\Rightarrow 11(\lambda - 3) - 3(2\lambda + 1) + 7(-2\lambda + 5) = 8$

$\Rightarrow 11\lambda - 6\lambda - 14\lambda - 33 - 3 + 35 = 8$

$\Rightarrow -9\lambda = 9 \therefore \lambda = -1$

Substituting $\lambda = -1$ I to eqn. (1) gives

$x = -4\,, y = -1\,, z = 7$

ii). Let $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2} = \lambda$

$\Rightarrow \frac{4x-3y}{4} = \lambda \therefore 4x - 3y = 4\lambda$ ………………………………. (1)

$\Rightarrow \frac{2y-x}{3} = \lambda \therefore 2y - x = 3\lambda$ …………………………………. (2)

$\Rightarrow \frac{z+4y}{2} = \lambda \therefore z + 4y = 2\lambda$ …………………………………. (3)

Expressing x, y, z in terms of λ

Eliminating x from eqn. (1) and (2)

$eqn.(1) + eqn.(2) \times 4$

$\Rightarrow 4x - 3y + 8y - 4x = 4\lambda + 12\lambda$

$\therefore 5y = 16\lambda \Rightarrow y = \frac{16\lambda}{5}$ ………………………………………… (4)

Eqn. (4) in to (2) to find x in terms of λ

$\Rightarrow 2y - x = 3\lambda \therefore x = 2y - 3\lambda \ or \ x = \frac{32\lambda}{5} - 3\lambda$

$\Rightarrow x = \frac{17\lambda}{5}$ ……………………………………………………………… (5)

Eqn. (4) in to (3) to find z in terms of λ

$z + 4y = 2\lambda \Rightarrow \text{z} = 2\lambda - 4y$

$\Rightarrow z = 2\lambda - \frac{64\lambda}{5} = -\frac{54\lambda}{5}$ ………………………………………. (6)

Substituting eqn. (4), (5), (6) in to $6x + 6y + 2z = 6$

$\Rightarrow 6\left(\frac{17\lambda}{5}\right) + 6\left(\frac{16\lambda}{5}\right) + 2\left(-\frac{54\lambda}{5}\right) = 6$

$\Rightarrow 9\lambda = 3 \therefore \lambda = \frac{1}{3}$

Substituting $\lambda = \frac{1}{3}$ in to eqn. (4), (5) and (6) gives

$x = \frac{17}{5} \times \frac{1}{3} = \frac{17}{15}$

$y = \frac{16}{5} \times \frac{1}{3} = \frac{16}{15}$

$x = -\frac{54}{5} \times \frac{1}{3} = -\frac{18}{5}$

$\therefore x = \frac{17}{15}\,, y = \frac{16}{15}\,, z = -\frac{18}{5}$

Task

Solve the simultaneous equations

a) $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ and $3x + 4y + 2z - 25 = 0$ **ANSWER**: $\boldsymbol{x = 5\,, y = 0\,, z = 5}$

b) $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}$ and $x + y + z = 2$ **ANSWER**: $\boldsymbol{x = 1\,, y = -2\,, z = 3}$

c) $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}$ and $4x + 2y + 5z = 15$ **ANSWER**: $\boldsymbol{x = 0\,, y = 5\,, z = 1}$

d) $\frac{x-y}{4} = \frac{z-y}{3} = 2z - x$ and $x + 3y + 2z = 4$ **ANSWER**: $\boldsymbol{x = 3, y = -1\,, z = 2}$

SURDS AND INDICES

Indices:

When a quantity $\boldsymbol{a}$ multiplies it self $n-$ times $.i.e. a \times a \times a \times \ldots\ldots\ldots\ldots..n-$ times, the product is called a *power of* $\boldsymbol{a}.i.e.a^n$

Definition:

Let $\boldsymbol{a}, x$ and y be real numbers such that $\boldsymbol{a}^x = y$, then y is a power of $\boldsymbol{a}, x$ is called index and $\boldsymbol{a}$ is the base.

For example in $2^4, 3^{\frac{1}{2}}, 5^{\sqrt{2}}$, $2, 3$ and 5 are the bases and $4, \frac{1}{2}$ and $\sqrt{2}$ are indices.

Laws of indices:

If m and n are integers then

i) $a^m \times a^n = a^{m+n}$

ii) $(a^m)^n = a^{mn}$

iii) $\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$

iv) $(a.b)^m = a^m.b^m$

v) $a^{-n} = \frac{1}{a^n}$

vi) $a^0 = 1$

vii) If $a^{\frac{p}{q}} = b$, then $a^p = b^q$ for $q > 0$

viii) If $a^m = a^n$ then $m = n$

Proof of the laws:

i) $\boxed{a^m \times a^n = a^{m+n}}$

Several cases arise,

Case I: $m > 0$ and $n > 0$ $.i.e.$ both m and n are positive.

By definition,

$a^m = a \times a \times a \times \ldots\ldots\ldots\ldots..m-$ times and

$a^n = a \times a \times a \times \ldots\ldots\ldots\ldots..n-$ times

$$\Rightarrow a^m \times a^n = (a \times a \times a \times \ldots\ldots\ldots\ldots..m-times) \times (a \times a \times a \times \ldots\ldots\ldots\ldots..n-times)$$

$$= a \times a \times a \times \ldots\ldots\ldots\ldots..(m+n)-times$$

$$= a^{m+n}$$

$\therefore a^m \times a^n = a^{m+n}$

Case II: $m < 0$ and $n < 0$ $.i.e.$ both m and n are negative

Let $m = -k$ and $n = -l$ where $k, n > 0$

$$\Rightarrow a^m \times a^n = a^{-k} \times a^{-l}$$

$$= \frac{1}{a^k} \times \frac{1}{a^l}$$

$$= \frac{1}{a^{k+l}}, \text{ by first case}$$

$$\therefore a^{-k} \times a^{-l} = a^{-(k+l)} = a^{-k-l} = \frac{1}{a^{k+l}}$$

Case III: $m > 0$ and $n < 0$ $.i.e.$ both $m-$ positive and $n-$ negative

Let $n = -l$, $l > 0$

$$\Rightarrow a^m \times a^n = a^m \times a^{-l}$$

$$= a^m \times \frac{1}{a^l}$$

$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l}$

Now if $m = l$, then

$a^m \times a^{-l} = a^l \times a^{-l} = a^l \times \frac{1}{a^l} = 1$

But by first case,

$a^l \times a^{-l} = a^0 = 1$

$$\boxed{\therefore a^0 = 1}$$

Also from $a^l \times a^{-l} = 1$, then

$\boxed{a^{-l} = \frac{1}{a^l}}$, showing that a^{-n} is the reciprocal of a^n

Further simplification of $a^m \times a^{-l} = a^m \times \frac{1}{a^l}$, for $m > l$

$\Rightarrow a^m \times a^{-l} = \left(\left(a \times \frac{1}{a}\right) \times \left(a \times \frac{1}{a}\right) \times \ldots\ldots\ldots\ldots.l - times\right) \times a \times a \times \ldots\ldots\ldots\ldots.(m-l) - times$

$= (1 \times 1 \times \ldots\ldots\ldots\ldots l - times) \times a^{m-l} = a^{m-l}$

$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l} = a^{m-l}$

Now again for $m < l$, then

$a^m \times a^{-l} = \left(\left(a \times \frac{1}{a}\right) \times \left(a \times \frac{1}{a}\right) \times \ldots\ldots\ldots\ldots.m - times\right) \times \left(\frac{1}{a} \times \frac{1}{a} \times \ldots\ldots\ldots\ldots.(l-m) - times\right)$

$= (1 \times 1 \times \ldots\ldots\ldots\ldots\ldots\ldots m - times) \times \frac{1}{a^{l-m}} = a^{-(l-m)} = a^{m-l}$

$\therefore$ For any integer m and n, $a^m \times a^n = a^{m+n}$ holds.

ii) $\boxed{(a^m)^n = a^{m.n}}$

Case I: let $m, n > o$ .i.e. m and n are positive.

$(a^m)^n = a^m \times a^m \times \ldots\ldots\ldots\ldots\ldots\ldots.\times\ n - times$

$= (a \times a \times a \times \ldots\ldots\ldots.m - times) \times (a \times a \times a \times \ldots\ldots.m - times) \times \ldots\ldots.n - times$

$= a \times a \times a \times \ldots\ldots\ldots\ldots\ldots.\ldots.mn - times$

$= a^{m.n}$

Now if $m > 0, n < 0$, put $n = -l$ where $l > 0$

$\Rightarrow (a^m)^n = (a^m)^{-l} = \frac{1}{(a^m)^l} = \frac{1}{a^{ml}} = a^{-ml}$

$\therefore (a^m)^{-l} = a^{m(-l)} = a^{m.n}$,for $n = -l$

Also if $m > o$ and $n = 0$, then

$(a^m)^n = (a^m)^0 = a^0 = 1$

iii) $\boxed{\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}}$

$\frac{a^m}{a^n} = \frac{a \times a \times a \times \ldots\ldots\ldots\ldots m - times}{a \times a \times a \times \ldots\ldots\ldots\ldots n - times}$

$= a.a.a \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(m-n) - times$

$= a^{m-n} \quad \therefore \frac{a^m}{a^n} = a^{m-n}$

iv) $\boxed{(a.b)^m = a^m.b^m}$

$(a.b)^m = ab \times ab \times ab \ldots\ldots\ldots\ldots\ldots\ldots\ldots.m - times$

$= (a \times a \times a \times \ldots\ldots\ldots\ldots\ldots.m - times) \times (b \times b \times b \times \ldots\ldots\ldots\ldots\ldots.m - times)$

$\therefore (a.b)^m = a^m.b^m$

NOTE:

The above laws hold for rational/ fractional indices.

Consider a^r and a^s where $r = \frac{p}{q}$ and $s = \frac{m}{n}$ with p, m, q, n are integers and $p, q > 0$

Let $a^r = b$ and $a^s = c$

$\Rightarrow a^{\frac{p}{q}} = b \quad \therefore a^p = b^q$

$\Rightarrow a^{\frac{m}{n}} = c \quad \therefore a^m = c^n$

Making powers of b and c the same

$\Rightarrow (a^p)^n = (b^q)^n \quad \therefore a^{np} = b^{qn}$

Also $(a^m)^q = (c^n)^q \quad \therefore a^{mq} = c^{nq}$

Now multiplying

$\Rightarrow b^{qn}.c^{nq} = a^{np}.a^{mq}$

$\Rightarrow (b.c)^{nq} = a^{np+mq}$

$\therefore b.c = a^{\frac{np+mq}{nq}} = a^{\frac{m}{n}+\frac{p}{q}}$

$\Rightarrow a^r.a^s = a^{\frac{m}{n}+\frac{p}{q}} = a^{r+s}$

Illustrative examples

1. *Simplify*

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}}$

b) $b^{-3}.x^{-2} \div 4b^2x$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n}$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1}$

e) $\left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5}$

Solution

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}} = x^{m+2n+(m-8n)-(5m-6n)}$

$= x^{(m+3m-5m)+(2n-8n+6n)}$

$= x^{-m+0} = x^{-m} = \frac{1}{x^m}$

b) $b^{-3}.x^{-2} \div 4b^2x = \frac{b^{-3}.x^{-2}}{4b^2x}$

$= \frac{b^{-3}}{b^2}.\frac{x^{-2}}{x}.\frac{1}{4} = \frac{b^{-3-2}.x^{-2-1}}{4}$

$= \frac{b^{-5}.x^{-3}}{4} = \frac{1}{4b^5.x^3}$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = \frac{3(2^n.2^1)-4(2^n.2^{-1})}{(2^n.2^1)-2^n}$, let $2^n = x$

$= \frac{3(2x)-4\left(\frac{x}{2}\right)}{2x-x} = \frac{6x-2x}{x} = \frac{4x}{x} = 4$

$\therefore \frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = 4$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1} = [3^2]^{-\frac{1}{2}n} \times 3^{n+2} \times (3^3)^{n+1}$

$= 3^{-2\times\frac{1}{2}n+n+2+3n+3} = 3^{3n+5}$

e) $\left(\frac{a^{-1}b^{2}}{a^{2}b^{-4}}\right)^{7} \div \left(\frac{a^{3}b^{-5}}{a^{-2}b^{3}}\right)^{-5} = \frac{a^{-7}b^{14}}{a^{14}b^{-28}} \div \frac{a^{-15}b^{25}}{a^{10}b^{-15}}$

$= \left(\frac{a^{-7}}{a^{14}}\right).\left(\frac{b^{14}}{b^{-28}}\right) \div \left(\frac{a^{-15}}{a^{10}}\right).\left(\frac{b^{25}}{b^{-15}}\right)$

$= a^{-7-14}.b^{14-(-28)} \div a^{-15-10}.b^{25-(-15)}$

$= \frac{a^{-21}.b^{42}}{a^{-25}.b^{40}} = a^{-21-(-25)}.b^{42-40} = a^{4}.b^{2}$

2. *Simplify*

a) $\frac{x^{2}(x^{2}+1)^{-\frac{1}{2}}-(x^{2}+1)^{\frac{1}{2}}}{x^{2}}$

b) $\frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}$

c) $\frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x}$

d) $\frac{x^{\frac{3}{2}}+xy}{xy-y^{3}} - \frac{\sqrt{x}}{\sqrt{x}-y}$

e) $\frac{(2^{2n}-3.2^{2n-3})(3^{n}-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})}$

Solution

a) $\frac{x^{2}(x^{2}+1)^{-\frac{1}{2}}-(x^{2}+1)^{\frac{1}{2}}}{x^{2}}$, let $x^{2}+1=a$

$\Rightarrow Expression = \frac{x^{2}.a^{-\frac{1}{2}}-a^{\frac{1}{2}}}{x^{2}} = \frac{a^{-\frac{1}{2}}(x^{2}-a^{1})}{x^{2}}$ (Factoring term with smallest power)

$= \frac{a^{-\frac{1}{2}}[x^{2}-(x^{2}+1)]}{x^{2}} = \frac{(x^{2}+1)^{-\frac{1}{2}}.(-1)}{x^{2}}$

$\therefore \frac{x^{2}(x^{2}+1)^{-\frac{1}{2}}-(x^{2}+1)^{\frac{1}{2}}}{x^{2}} = -\frac{1}{x^{2}(x^{2}+1)^{-\frac{1}{2}}}$

b) $\frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}$, let $b=1+x$

$Expression = \frac{b^{\frac{1}{3}}-\frac{1}{3}xb^{-\frac{2}{3}}}{b^{\frac{2}{3}}} = \frac{b^{-\frac{2}{3}}\left(b-\frac{1}{3}x\right)}{b^{\frac{2}{3}}}$ (Factoring term with smallest power)

$= b^{-\frac{2}{3}-\frac{2}{3}}.\left(b-\frac{1}{3}x\right) = b^{-\frac{4}{3}}.\left(b-\frac{1}{3}x\right)$

$= (1+x)^{-\frac{4}{3}}.\left(1+x-\frac{1}{3}x\right) = (1+x)^{-\frac{4}{3}}.\left(1+\frac{2}{3}x\right)$

$\therefore \frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}} = \frac{3+2x}{3(1+x)^{\frac{4}{3}}}$

c) $\frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x}$, let $1-x=a\,, b=1+x$

$Expression = \frac{\sqrt{a}.\frac{1}{2}b^{-\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}.\sqrt{b}}{a} = \frac{\frac{1}{2}.a^{\frac{1}{2}}.b^{-\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}.b^{\frac{1}{2}}}{a}$

$= \frac{a^{-\frac{1}{2}}.b^{-\frac{1}{2}}}{2}.\frac{(1+b)}{a} = a^{-\frac{1}{2}-1}.b^{-\frac{1}{2}}.(1+b)$

$= a^{-\frac{3}{2}}.b^{-\frac{1}{2}}.(1+b)$

$= (1-x)^{-\frac{3}{2}}.(1+x)^{-\frac{1}{2}}.(1+1+x)$

$= (1-x)^{-\frac{3}{2}}.(1+x)^{-\frac{1}{2}}.(2+x) = \frac{2+x}{(1-x)^{\frac{3}{2}}.(1+x)^{\frac{1}{2}}}$

$= \frac{2+x}{(1-x)(1-x)^{\frac{1}{2}}.(1+x)^{\frac{1}{2}}} = \frac{2+x}{(1-x)(1-x^{2})^{\frac{1}{2}}}$

$$\therefore \frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x}=\frac{2+x}{(1-x)\sqrt{1-x^2}}$$

d) $\frac{x^{\frac{3}{2}}+xy}{xy-y^3}-\frac{\sqrt{x}}{\sqrt{x}-y}=\frac{x^{\frac{3}{2}}+xy}{xy-y^3}-\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-y}$, let $x^{\frac{1}{2}}=a\,, x=a^2$

$$Expression=\frac{a^3+a^2y}{a^2y-y^3}-\frac{a}{a-y}=\frac{a^2(a+y)}{y(a^2-y^2)}-\frac{a}{a-y}$$
$$=\frac{a^2(a+y)}{y(a+y)(a-y)}-\frac{a}{a-y}$$
$$=\frac{a^2}{y(a-y)}-\frac{a}{a-y}=\frac{a}{a-y}\left(\frac{a}{y}-1\right)$$
$$=\frac{a}{a-y}\frac{(a-y)}{y}=\frac{a}{y}$$
$$=\frac{x^{\frac{1}{2}}}{y}=\frac{\sqrt{x}}{y}$$

$$\therefore \frac{x^{\frac{3}{2}}+xy}{xy-y^3}-\frac{\sqrt{x}}{\sqrt{x}-y}=\frac{\sqrt{x}}{y}$$

e) $\frac{(2^{2n}-3.2^{2n-3})(3^n-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})}=\frac{[(2^n)^2-3((2^n)^2.2^{-3})][3^n-2.3^n.3^{-2}]}{3^n.3^{-4}.((2^2)^{n+2}-(2^n)^2)}$ let $2^n=a\,,\ 3^n=b$

$$=\frac{\left[a^2-3\left(a^2.\frac{1}{8}\right)\right]\left[b-2.b.\frac{1}{9}\right]}{b.\frac{1}{81}.((2^2)^{n+2}-a^2)}=\frac{\left(\frac{8a^2-3a^2}{8}\right).\left(\frac{9b-2b}{9}\right)}{\frac{b}{81}.((2^n)^2.2^4-a^2)}$$
$$=\frac{\frac{5a^2}{8}.\frac{7b}{9}}{\frac{b}{81}.(a^2.16-a^2)}=\frac{\frac{5a^2}{8}.\frac{7b}{9}}{\frac{b}{81}.15a^2}$$
$$=\frac{5a^2}{8}.\frac{7b}{9}.\frac{81}{15a^2b}=\frac{5\times7\times81}{8\times9\times15}=\frac{21}{8}$$

$$\therefore \frac{(2^{2n}-3.2^{2n-3})(3^n-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})}=\frac{21}{8}$$

3. *Simplify*

a) $\left(\frac{81}{256}\right)^{-\frac{5}{4}}$

b) $\frac{(0.3)^{\frac{1}{3}}.\left(\frac{1}{27}\right)^{\frac{1}{4}}.(9)^{\frac{1}{6}}.(0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.\left(\frac{1}{3}\right)^{-2}.(243)^{-\frac{1}{4}}}$

c) $\frac{3^{-2}\times6^2\times\sqrt{48}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$

d) $\frac{8^{\frac{2}{3}}+4^{\frac{3}{2}}}{16^{\frac{3}{4}}}$

e) $\frac{(8)^{\frac{1}{6}}\times(4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}}\times(16)^{\frac{1}{2}}}$

Solution

a) $\left(\frac{81}{256}\right)^{-\frac{5}{4}}=\left[\frac{3^4}{4^4}\right]^{-\frac{5}{4}}=\frac{3^{4\times-\frac{5}{4}}}{4^{4\times-\frac{5}{4}}}=\frac{3^{-5}}{4^{-5}}=\frac{4^5}{3^5}=\frac{1024}{243}$

b) $\frac{(0.3)^{\frac{1}{2}}.\left(\frac{1}{27}\right)^{\frac{1}{4}}.(9)^{\frac{1}{6}}.(0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.\left(\frac{1}{3}\right)^{-2}.(243)^{-\frac{1}{4}}}=\frac{\left(\frac{3}{10}\right)^{\frac{1}{3}}.\left(\frac{1}{3^3}\right)^{\frac{1}{4}}.(3^2)^{\frac{1}{6}}.\left(\frac{81}{100}\right)^{\frac{2}{3}}}{\left(\frac{9}{10}\right)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.(3^{-1})^{-2}.(3^5)^{-\frac{1}{4}}}$

$$=\frac{(3)^{\frac{1}{3}}.(10)^{-\frac{1}{3}}.(3)^{-\frac{3}{4}}.(3)^{\frac{1}{3}}.\left(\frac{3^4}{10^2}\right)^{\frac{2}{3}}}{(3^2)^{\frac{2}{3}}.(10^{-1})^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.(3^2).(3)^{-\frac{5}{4}}}=\frac{3^{\frac{1}{3}+-\frac{3}{4}+\frac{1}{3}+\frac{8}{3}}.10^{-\frac{1}{3}+-\frac{4}{3}}}{3^{\frac{4}{3}+-\frac{1}{2}+2+-\frac{5}{4}}.10^{-\frac{2}{3}}}$$

$$=\frac{3^{\frac{31}{12}}}{3^{\frac{19}{12}}}\times\frac{10^{-\frac{5}{3}}}{10^{-\frac{2}{3}}}=3^{\frac{31}{12}-\frac{19}{12}}\times10^{-\frac{5}{3}+\frac{2}{3}}$$

$$=3^1\times10^{-1}=\frac{3}{10}$$

c) $\frac{3^{-2}\times 6^2\times\sqrt{48}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}} = \frac{3^{-2}\times(3\times 2)^2\times(3\times 16)^{\frac{1}{2}}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$

$$= \frac{3^{-2}\times(3\times 2)^2\times(3\times 16)^{\frac{1}{2}}}{5^2\times(5^{-2})^{\frac{1}{3}}\times(3\times 5)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}} = \frac{3^{-2}\times 3^2\times 2^2\times 3^{\frac{1}{2}}\times(2^4)^{\frac{1}{2}}}{5^2\times 5^{-\frac{2}{3}}\times 3^{-\frac{4}{3}}\times 5^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$$

$$= \frac{3^{-2+2+\frac{1}{2}}\times 2^{2+2}}{5^{2-\frac{2}{3}-\frac{4}{3}}\times 3^{-\frac{4}{3}+\frac{1}{3}}} = \frac{3^{\frac{1}{2}}}{3^{-1}}\times 2^4\times\frac{1}{5^0}$$

$$= 3^{\frac{1}{2}+1}\times 16 = 16\sqrt{27}$$

d) $\frac{8^{\frac{2}{3}}+4^{\frac{3}{2}}}{16^{\frac{3}{4}}} = \frac{(2^3)^{\frac{2}{3}}+(2^2)^{\frac{3}{2}}}{(2^4)^{\frac{3}{4}}} = \frac{2^2+2^3}{2^3} = \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2}$

e) $\frac{(8)^{\frac{1}{6}}\times(4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}}\times(16)^{\frac{1}{2}}} = \frac{(2^3)^{\frac{1}{6}}\times(2^2)^{\frac{1}{3}}}{(2^5)^{\frac{1}{6}}\times(2^4)^{\frac{1}{2}}} = \frac{2^{\frac{3}{6}}\times 2^{\frac{2}{3}}}{2^{\frac{5}{6}}\times 2^{\frac{1}{3}}}$

$$= 2^{\frac{1}{2}+\frac{2}{3}-\frac{5}{6}-\frac{1}{3}} = 2^0 = 1$$

PROOFS IN INDICES

1. *If* $a = 2, b = 3$ *, show that* $(a^3b^2c^3)^2\sqrt{a^{-3}b^{-4}c^6} = 144\sqrt{2}.c^9$

Solution

$(a^3b^2c^3)^2\sqrt{a^{-3}b^{-4}c^6} = a^6b^4c^6.(a^{-3}b^{-4}c^6)^{\frac{1}{2}}$

$= a^6b^4c^6.a^{-\frac{3}{2}}b^{-2}c^3$

$= a^{6-\frac{3}{2}}.b^{4-2}.c^{3+6}$

$= a^{\frac{9}{2}}.b^2.c^9$

$\Rightarrow a^{\frac{9}{2}}.b^2.c^9 = (2)^{\frac{9}{2}}.(3)^2.c^9 = (2^9)^{\frac{1}{2}}.9c^9$

$$= (2^8\times 2)^{\frac{1}{2}}.9c^9 = 2^4\times 2^{\frac{1}{2}}.9c^9 = 16\times 9\sqrt{2}c^9 = 144\sqrt{2}.c^9$$

2. *If* $x = \left[a+\sqrt{a^2+b^3}\right]^{\frac{1}{3}} + \left[a-\sqrt{a^2+b^3}\right]^{\frac{1}{3}}$*, show that* $x^3+3bx-2a = 0$

Solution

Let $\sqrt{a^2+b^3} = B$

$\Rightarrow x = [a+B]^{\frac{1}{3}} + [a-B]^{\frac{1}{3}}$

Cubing both sides

$x^3 = \left[(a+B)^{\frac{1}{3}} + (a-B)^{\frac{1}{3}}\right]^3$

$= \left[(a+B)^{\frac{1}{3}}\right]^3 + 3\left[(a+B)^{\frac{1}{3}}\right]^2(a-B)^{\frac{1}{3}} + 3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{2}{3}} + \left[(a-B)^{\frac{1}{3}}\right]^3$

$= a+B+3(a+B)^{\frac{2}{3}}.(a-B)^{\frac{1}{3}} + 3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{2}{3}} + a - B$

$= 2a+3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{1}{3}}\left[(a+B)^{\frac{1}{3}} + (a-B)^{\frac{1}{3}}\right]$

$= 2a+3[(a+B)(a-B)]^{\frac{1}{3}}.x$

$= 2a+3(a^2-B^2)^{\frac{1}{3}}.x$

$= 2a+3\left(a^2-(a^2+b^3)\right)^{\frac{1}{3}}.x \quad [\because \sqrt{a^2+b^3} = B]$

$= 2a+3\left(-b^{3\times\frac{1}{3}}\right).x$

$= 2a-3bx$

$\therefore x^3-2a+3bx = 0$

3. *If* $x = 3^{\frac{2}{3}}+3^{-\frac{2}{3}}$*, show that* $9x^3-27x = 82$

Solution

$x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, cubing both sides,

$\Rightarrow x^3 = \left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)^3 = \left(3^{\frac{2}{3}}\right)^3 + \left(3^{-\frac{2}{3}}\right)^3 + 3.3^{\frac{2}{3}}.3^{-\frac{2}{3}}\left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)$

$[\because (\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)]$

$\Rightarrow x^3 = 3^2 + 3^{-2} + 3x = 9 + \frac{1}{9} + 3x$

$\therefore x^3 = \frac{82}{9} + 3x \;\; or \;\; 9x^3 - 27x = 82$

4. *If* $2^x = 4^y = 8^z$ *and* $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$, *show that* $x = \frac{7}{16}$, $y = \frac{7}{32}$ *and* $z = \frac{7}{48}$

Solution

$2^x = 4^y \Rightarrow 2^x = 2^{2y} \;\therefore x = 2y$ ……………………………….. (1)

Also $2^x = 8^z \Rightarrow 2^x = 2^{3z} \;\therefore x = 3z$ ………………………… (2)

$Eqn. (1) = eqn. (2)$

$\Rightarrow 2y = 3z \;\; or \;\; z = \frac{2y}{3}$ ……………………………………………. (3)

From $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2y}{3}\right)} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2}{3}\times\frac{x}{2}\right)} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{2x} + \frac{3}{8x} = \frac{22}{7}$

$\Rightarrow \frac{11}{8x} = \frac{22}{7} \;\therefore x = \frac{7}{22} \times \frac{11}{8} = \frac{7}{16}$

From eqn. (1), $x = 2y$

$\Rightarrow y = \frac{x}{2} = \frac{7}{16} \times \frac{1}{2} = \frac{7}{32}$

From eqn. (2), $x = 3z$

$\Rightarrow z = \frac{x}{3} = \frac{7}{16} \times \frac{1}{3} = \frac{7}{48}$

$\therefore x = \frac{7}{16}$, $y = \frac{7}{32}$, $z = \frac{7}{48}$

5. *If* $\frac{x}{y} = \frac{z}{w}$, *show that* $\frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = (wxyz)^{\frac{m}{2}}$

Solution

Let $\frac{x}{y} = \frac{z}{w} = k$

$\Rightarrow x = yk\,, z = wk$

$\Rightarrow \frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = \frac{y^m k^m + y^m + w^m k^m + w^m}{y^{-m} k^{-m} + y^{-m} + w^{-m} k^{-m} + w^{-m}}$

$= \frac{y^m (k^m + 1) + w^m (k^m + 1)}{y^{-m}(k^{-m} + 1) + w^{-m}(k^{-m} + 1)}$

$= \frac{(k^m + 1)(y^m + w^m)}{(k^{-m} + 1)(y^{-m} + w^{-m})}$

$= \frac{(k^m + 1)}{\left(\frac{1}{k^m} + 1\right)} \cdot \frac{(y^m + w^m)}{\left(\frac{1}{y^m} + \frac{1}{w^m}\right)}$

$= \frac{k^m (k^m + 1)}{(1 + k^m)} \cdot \frac{(y^m + w^m)(y^m . w^m)}{(y^m + w^m)}$

$= k^m . y^m . w^m = (k.y.w)^m$

But from $\frac{x}{y} = \frac{z}{w} = k \Rightarrow \frac{x}{y} = k$, $\frac{z}{w} = k$, multiplying the equations

$\Rightarrow k^2 = \frac{xz}{yw} \;\; or \; k = \left(\frac{xz}{yw}\right)^{\frac{1}{2}}$

$$\therefore (k.y.w)^m = \left(\left(\frac{xz}{yw}\right)^{\frac{1}{2}}.y.w\right)^m = \left((x.z.y^{-1}.w^{-1})^{\frac{1}{2}}.y.w\right)^m$$

$$= \left(x^{\frac{1}{2}}.z^{\frac{1}{2}}.y^{1-\frac{1}{2}}.w^{1-\frac{1}{2}}\right)^m = \left(x^{\frac{1}{2}}.z^{\frac{1}{2}}.y^{\frac{1}{2}}.w^{\frac{1}{2}}\right)^m = (wxyz)^{\frac{m}{2}}$$

6. *If* $a^x = b^y = c^z = d^w$ *and* $ab = cd$ *, show that* $\frac{1}{x} + \frac{1}{y} = \frac{1}{z} + \frac{1}{w}$

Solution

Let $a^x = b^y = c^z = d^w = k$

$\Rightarrow a^x = k \;\; or \;\; a = k^{\frac{1}{x}}$

$\Rightarrow b^y = k \;\; or \;\; b = k^{\frac{1}{y}}$

$\Rightarrow c^z = k \;\; or \;\; c = k^{\frac{1}{z}}$

$\Rightarrow d^w = k \;\; or \;\; d = k^{\frac{1}{w}}$

From $a.b = c.d$

$\Rightarrow k^{\frac{1}{x}}.k^{\frac{1}{y}} = k^{\frac{1}{z}}.k^{\frac{1}{w}} \;\; or \;\; k^{\frac{1}{x}+\frac{1}{y}} = k^{\frac{1}{w}+\frac{1}{z}}$

$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{z} + \frac{1}{w}$

7. *If* $3^x = 5^y = 75^z$*, show that* $xz = z(2x + y)$

Solution

Let $3^x = 5^y = 75^z = k$

$\Rightarrow 3^x = k \;\; or \;\; 3 = k^{\frac{1}{x}}$

$\Rightarrow 5^y = k \;\; or \;\; 5 = k^{\frac{1}{y}}$

$\Rightarrow 75^z = k \;\; or \;\; 75 = k^{\frac{1}{z}}$

But $75 = 25 \times 3 = 5^2 \times 3$

$\Rightarrow k^{\frac{1}{z}} = \left(k^{\frac{1}{y}}\right)^2 \times k^{\frac{1}{x}}$

$\Rightarrow k^{\frac{1}{z}} = k^{\frac{1}{x}+\frac{2}{y}}$

$\therefore \frac{1}{z} = \frac{1}{x} + \frac{2}{y} \;\; or \;\; \frac{1}{z} = \frac{y+2}{xy}$

$\therefore xz = z(2x + y)$

8. *If* $\frac{x^{n+1}}{a^n} = \frac{y^{n+1}}{b^n} = \frac{z^{n+1}}{c^n} = a + b + c$*, prove that* $\left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = a + b + c$

Solution

Let $a + b + c = k$

$\Rightarrow \frac{x^{n+1}}{a^n} = k \;\; or \;\; a^n = \frac{x^{n+1}}{k}$

$\therefore a = \left(\frac{x^{n+1}}{k}\right)^{\frac{1}{n}} = \frac{x^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$

Similarly

$\therefore b = \frac{y^{\frac{n+1}{n}}}{k^{\frac{1}{n}}} \;\; , \;\; c = \frac{z^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$

$\Rightarrow \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = \left(a.k^{\frac{1}{n}} + b.k^{\frac{1}{n}} + c.k^{\frac{1}{n}}\right)^{\frac{n}{n+1}}$

$= \left[(a + b + c).k^{\frac{1}{n}}\right]^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}}.k^{\frac{1}{n+1}}$

But $a + b + c = k$

$\therefore \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}}.(a + b + c)^{\frac{1}{n+1}}$

$= (a+b+c)^{\frac{n}{n+1}+\frac{1}{n+1}} = a+b+c$

9. *If* $a = xy^{p-1}, b = xy^{q-1}, c = xy^{r-1}$, *prove that* $a^{q-r} \times b^{r-p} \times c^{p-q} = 1$

Solution

$a^{q-r} \times b^{r-p} \times c^{p-q} = (xy^{p-1})^{q-r} \times (xy^{q-1})^{r-p} \times (xy^{r-1})^{p-q}$

$= x^{(q-r)}.y^{(p-1)(q-r)} \times x^{(r-p)}.y^{(q-1)(r-p)} \times x^{(p-q)}.y^{(r-1)(p-q)}$

$= x^{q-r+r-p+p-q} \times y^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= x^0.y^{pq-pr-q+r+qr-qp-r+p+rp-rq-p+q}$

$= x^0.y^0 = 1$

Solving equations involving indices

1. *Find* x *if* $x^{x\sqrt{x}} = (x\sqrt{x})^x$, *where* x *is a positive integer.*

Solution

$x^{x\sqrt{x}} = (x\sqrt{x})^x \Rightarrow x^{x.x^{\frac{1}{2}}} = \left(x.x^{\frac{1}{2}}\right)^x$

$\Rightarrow x^{x^{\frac{3}{2}}} = \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x}$

$\Rightarrow x^{\frac{3}{2}} = \frac{3}{2}x$, squaring both sides

$\Rightarrow x^3 = \frac{9}{4}x^2$ *or* $4x^3 - 9x^2 = 0$

$\Rightarrow x^2(4x-9) = 0 \;\therefore x = 0\;, \frac{9}{4}$

But $x \neq 0 \;\therefore x = \frac{9}{4}$

2. *Solve* $4^x.2^y = 128$ *and* $3^{2x+2y} = 9^{xy}$

Solution

$4^x.2^y = 128 \Rightarrow 2^{2x+y} = 2^7$

$\therefore 2x + y = 7$……………………………………………………. (1)

Also $3^{2x+2y} = 9^{xy} \Rightarrow 3^{2x+2y} = 3^{2xy}$

$\therefore 2x + 2y = 2xy$ *or* $x + y = xy$…………………. (2)

From eqn. (1), $y = 7 - 2x$…………………………………. (3)

Eqn. (3) in to eqn. (2)

$\Rightarrow x + 7 - 2x = x(7-2x)$

$\Rightarrow 7 - x = 7x - 2x^2$ *or* $2x^2 - 8x + 7 = 0$

$\therefore x = \frac{8 \pm \sqrt{(-8)^2 - 4\times2\times7}}{2\times2}$

$= \frac{8\pm\sqrt{8}}{4} = \frac{8\pm2\sqrt{2}}{4} = \frac{4\pm\sqrt{2}}{2}$

From eqn. (3) $y = 7 - 2x$

For $x = \frac{4\pm\sqrt{2}}{2}, y = 7 - 2\left(\frac{4\pm\sqrt{2}}{2}\right) = 7 - \left(4 \pm \sqrt{2}\right)$

$\therefore x = \frac{4+\sqrt{2}}{2}, y = 3 - \sqrt{2}\,; x = \frac{4-\sqrt{2}}{2}, y = 11 + \sqrt{2}$

3. *Solve*

i) $a^{x-3}.a^{y+2} = a^2.a^x, a^x.a^y = a$, *where* $a \neq 0, 1$

ii) $(\sqrt{a})^{x+y} = (\sqrt[3]{a})^{y+z-1}, (\sqrt{b})^{x+z-2} = (\sqrt[5]{b})^{y+z}, (\sqrt[4]{c})^y = (\sqrt[7]{c})^{x+y+z}$

iii) $2^{x+y+z} = 8^{x+z-y}, 5^{3y+2} = 25^{x+z}, 3^{2x+y+2z} = 9^{3x+y}$

Solution

i) $a^{x-3}.a^{y+2} = a^2.a^x$

$\Rightarrow a^{x-3+y+2} = a^{2+x}$

$\Rightarrow x + y - 1 = 2 + x \;\therefore y = 3$

Also $a^x.a^y = a \Rightarrow a^{x+y} = a$
$\Rightarrow x + y = 1$, but $y = 3$
$\Rightarrow x = 1 - 3 = -2$
$\therefore x = -2\,, y = 3$
ii) $\left(\sqrt{a}\right)^{x+y} = \left(\sqrt[3]{a}\right)^{y+z-1}$
$\Rightarrow a^{\frac{x+y}{2}} = a^{\frac{y+z-1}{3}}$
$\Rightarrow \frac{x+y}{2} = \frac{y+z-1}{3}$ $\; or \;$ $3x + 3y = 2y + 2z - 2$
$\therefore 3x + y - 2z = -2$ .... (1)
$\left(\sqrt{b}\right)^{x+z-2} = \left(\sqrt[5]{b}\right)^{y+z}$
$\Rightarrow b^{\frac{x+z-2}{2}} = b^{\frac{y+z}{5}}$
$\Rightarrow \frac{x+z-2}{2} = \frac{y+z}{5}$ $\; or \;$ $5x + 5z - 10 = 2y + 2z$
$\therefore 5x - 2y + 3z = 10$ .... (2)
$\left(\sqrt[4]{c}\right)^{y} = \left(\sqrt[5]{c}\right)^{x+y+z}$
$\Rightarrow c^{\frac{y}{4}} = c^{\frac{x+y+z}{7}}$
$\Rightarrow \frac{y}{4} = \frac{x+y+z}{7}$ $\; or \;$ $7y = 4x + 4y + 4z$
$\therefore 4x - 3y + 4z = 0$ .... (3)
Eliminating y from eqn. (1) and eqn. (2)
$2 \times eqn, (1) + eqn. (2)$
$\Rightarrow 6x + 2y - 4z + 5x - 2y + 3z = -4 + 10$
$\therefore 11x - z = 6$ .... (4)
Eliminating y from eqn. (1) and eqn. (3)
$3 \times eqn, (1) + eqn. (3)$
$\Rightarrow 9x + 3y - 6z + 4x - 3y + 4z = -6 + 0$
$\therefore 13x - 2z = -6$ .... (5)
Eliminating z from eqn. (4) and eqn. (5)
$2 \times eqn, (4) - eqn. (5)$
$\Rightarrow 22x - 2z - (13x - 2z) = 12 - \; -6$
$\Rightarrow 9x = 18 \;\; \therefore x = 2$
From eqn. (4), $z = 11x - 6$
$\Rightarrow z = 22 - 6 = 16$
From eqn. (1), $y = -2 + 2z - 3x$
$\Rightarrow y = -2 + 32 - 6 = 24$
$\therefore x = 2\,, y = 24\,, z = 16$
iii) $2^{x+y+z} = 8^{x+z-y}$
$\Rightarrow 2^{x+y+z} = 2^{3x+3z-3y}$ $\; or \;$ $x + y + z = 3x + 3z - 3y$
$\Rightarrow 2x - 4y + 2z = 0$
$\therefore x - 2y + z = 0$ .... (1)
$5^{3y+2} = 25^{x+z}$
$\Rightarrow 5^{3y+2} = 5^{2x+2z}$ $\; or \;$ $3y + 2 = 2x + 2z$
$\therefore 2x - 3y + 2z = 2$ .... (2)
$3^{2x+y+2z} = 9^{3x+y}$
$\Rightarrow 3^{2x+y+2z} = 3^{6x+2y}$ $\; or \;$ $2x + y + 2z = 6x + 2y$
$\therefore 4x + y - 2z = 0$ .... (3)
Eliminating z from eqn. (1) and eqn. (2)
$2 \times eqn, (1) - eqn. (2)$

$\Rightarrow 2x - 4y + 2z - (2x - 3y + 2z) = 0 - 2$
$\Rightarrow -y = -2 \;\; \therefore y = 2$
Eliminating z from eqn. (2) and eqn. (3)
$eqn, (2) + eqn. (3)$
$\Rightarrow 6x - 2y = 2$, but $y = 2$
$\Rightarrow 6x - 4 = 2 \;\; \therefore x = 1$
From eqn. (1), $z = 2y - x$
$\Rightarrow z = 4 - 1 = 3$
$\therefore x = 1 , y = 2 , z = 3$

Task

1. *Show that*

i) $\frac{\sqrt{x^3}+xy}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \sqrt{\left(\frac{x}{y}\right)}$

ii) $If\ x = \sqrt[3]{p+q} + \sqrt[3]{p-q}\ and\ p^2 - q^2 = r,\ then\ x^3 - 3rx - 2p = 0$

iii) $If\ (2.381)^x = (0.2381)^y = 10^z,\ then\ \frac{1}{x} = \frac{1}{y} + \frac{1}{z}$

iv) $If\ x = 3^{\frac{1}{4}} + 3^{-\frac{1}{4}}, y = 3^{\frac{1}{4}} - 3^{-\frac{1}{4}},\ then\ 3(x^2 + y^2)^2 = 64$

v) $If\ 2^x = 3^y = 6^{-z},\ then\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

2. *Prove that*

a) $\frac{16(32)^m - 2^{3m-2}.4^{m+1}}{15.(2^{m-1})(16)^m} - \frac{5^m}{\sqrt{5^{2m}}} = 1$

b) $\frac{\left(9^{n+\frac{1}{4}}\right).\sqrt{3\times 3^n}}{3\sqrt{3^{-n}}} = 27$

c) $If\ pqr = 1,\ then\ \frac{1}{1+p+q^{-1}} + \frac{1}{1+q+r^{-1}} + \frac{1}{1+r+p^{-1}} = 1$

d) $\frac{(81)^n.3^5 - 3^{4n-1}.(243)}{9^{2n}.3^2} - \frac{4.3^n}{3^{n+1} - 3^n} = 4$

e) $If\ x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}},\ then\ 9x^3 - 27x = 82$

3. *Simplify*

a) $\sqrt[3]{a^6 b^{-2} c^{-4}} \times \sqrt[4]{a^{-6} b^4 c^3}$ **ANSWER**: $a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{3}{2}}$

b) $\left(\frac{5^{-1}.7^2}{5^2.7^{-4}}\right)^{\frac{7}{2}} \div \left(\frac{5^3.7^{-5}}{5^{-2}.7^3}\right)^{\frac{7}{2}}$ **ANSWER**: 175

LOGARITHMS

Definition:

The logarithm of a *positive* quantity N to any given base a is defined as the index of the power to which the base a must be raised to make it equal to the given quantity $N. i.e.$ if $\boxed{\boldsymbol{a^x = N}}$, then x is called the logarithm of N to the base a and is written as

$$\boxed{\boldsymbol{\log_a N = x}}$$

NOTE:

1. We notice that a logarithm is just a power, thus it can also be called an *index* or *an exponent*. This implies that in the derivation of laws of logarithms, laws of indices come in to use.
2. Logarithms are defined for only *positive numbers*.

PROPERTIES/LAWS/THEOREMS OF LOGARITHMS

1. *The logarithm of unity to any base is zero.*

$\boxed{\boldsymbol{\log_a 1 = 0}}$

Proof:

From $a^0 = 1$ $[Seting\ x = 0\ for\ a^x = N]$, then
$0 = \log_a 1$ $\quad [From\ defination]$
$\therefore \log_a 1 = 0$

2. *Logarithm of any number to the same base is 1*

$\boxed{\boldsymbol{\log_a a = 1}}$

Proof:

From $a^1 = a$ $[Seting\ x = 1\ for\ a^x = N]$, then
$\log_a a = 1$ $\quad [From\ defination\ \ x = \log_a N]$
$\therefore \log_a a = 1$

3. *A number a power logarithm of another number N to base of a number a is N*

$\boxed{\boldsymbol{a^{\log_a N} = N}}$

Proof:

Let $\log_a N = x \Rightarrow a^x = N$
$\therefore a^{\log_a N} = N$
In general,

$\boxed{\boldsymbol{1.\, a^{\log_a b \, \log_b N} = N}}$

Proof:

Let $a^{\log_a b} = y \Rightarrow y = b$
Now $y^{\log_b N} = b^{\log_b N} = N$
$1.\, a^{\log_a b \, \log_b N} = N$

4. *The logarithm of a product is equal to the sum of logarithms of its factors. i. e.*

$\boxed{\boldsymbol{\log_a(M.N) = \log_a M + \log_a N}}$

Proof:
From **LHS** = $\log_a(M.N)$
Let $\log_a(M.N) = b \Rightarrow a^b = M.N$.......................... (1)
Let $\log_a M = x \Rightarrow a^x = M$...................................... (2)
Let $\log_a M = y \Rightarrow a^y = N$....................................... (3)
Eqn. (2) , (3) in to eqn. (1)
$\Rightarrow a^b = a^x.a^y = a^{x+y}$
$\Rightarrow b = x + y$
$\therefore \log_a(M.N) = \log_a M + \log_a N$

NOTE:

- $\log_a(M + N) \neq \log_a M + \log_a N$
- $\log_a(M.N) = \log_a M \times \log_a N$

5. *The logarithm of a fraction is equal to the logarithm of numerator diminished by the logarithm of the denominator .i.e.*

$$\boxed{\mathbf{\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N}}$$

Let $\log_a(M.N) = b \Rightarrow a^b = \frac{M}{N}$............................... (1)
Let $\log_a M = x \Rightarrow a^x = M$...................................... (2)
Let $\log_a M = y \Rightarrow a^y = N$.. (3)
Eqn. (2), (3) in to eqn. (1)
$\Rightarrow a^b = a^x \div a^y = a^{x-y}$
$\Rightarrow b = x - y$
$\therefore \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

NOTE:

- $\log_a\left(\frac{M}{N}\right) \neq \log_a M \div \log_a N$
- $\log_a(M - N) = \log_a M - \log_a N$

6. *The logarithm of the p^{th} power of a positive number is p times the logarithm of that number.i.e.*

$$\boxed{\mathbf{\log_a M^p = p \log_a M}}$$

Proof:
Let $\log_a M = x \Rightarrow a^x = M$
Now powering by p on both sides
$(a^x)^p = M^p \Rightarrow a^{xp} = M^p$
$\Rightarrow \log_a M^p = xp = p\log_a M$
$\therefore \log_a M^p = p\log_a M$
Similarly if $p = \frac{1}{r}$, then $\boxed{\mathbf{\log_a M^{\frac{1}{r}} = \frac{1}{r}\log_a M}}$

7. *To transform a system of logarithms having a given base to a new system with a different base. i. e.* $\boxed{\boldsymbol{\log_{mn} N = \frac{\log_n N}{1+\log_n M}}}$

Proof:

Let $\log_a N = x$ be given and required $\log_b N = y \Rightarrow b^y = N$

$\Rightarrow \log_a N = x$, becomes

$\log_a b^y = x = \log_a N$

$\Rightarrow y \log_a b = \log_a N \quad or \quad y = \frac{1}{\log_a b} \times \log_a N$

$\therefore \log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$

NOTE:

1. To transform a logarithm from base a to base b we have only to multiply it by $\frac{\mathbf{1}}{\boldsymbol{\log_a b}}$. This is known as *Base changing formula*
2. From the above base change of bases, if $N = a$,

$\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$, becomes

$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b}$

$$\boxed{\boldsymbol{\log_b a = \frac{1}{\log_a b}}}$$

Proof:

Let $\log_b a = x \Rightarrow b^x = a$

$\Rightarrow b = a^{\frac{1}{x}}$, introducing logarithm to base a on both sides

$\Rightarrow \log_a b = \log_a a^{\frac{1}{x}} = \frac{1}{x} \log_a a = \frac{1}{x}$

$\therefore \log_b a = \frac{1}{\log_a b} \quad or \quad \log_a b = \frac{1}{\log_b a}$

8. *A base and its reciprocal inverse as a base*

$$\boxed{\boldsymbol{\log_b N = -\log_{\frac{1}{b}} N}}$$

Proof:

Let $\log_b N = x \Rightarrow b^x = N \quad or \quad b = N^{\frac{1}{x}}$

Introducing logarithm to base $\frac{1}{b}$

$\Rightarrow \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N^{\frac{1}{x}} = \frac{1}{x} \log_{\frac{1}{b}} N$

$\Rightarrow x \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N \quad or \quad x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b}$

From change of base formula

$\Rightarrow \log_{\frac{1}{b}} b = \frac{\log_b b}{\log_b \frac{1}{b}} = \frac{1}{\log_b b^{-1}} = \frac{1}{-\log_b b} = -1$

$\Rightarrow x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b} = \frac{\log_{\frac{1}{b}} N}{-1} = -\log_{\frac{1}{b}} N$

$\therefore \log_b N = -\log_{\frac{1}{b}} N$

9. *Logarithm of a number with a composite base*

$$\boldsymbol{\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N}$$

Proof:

Let $\log_{mn} N = x \Rightarrow (mn)^x = N \;\; or \;\; mn = N^{\frac{1}{x}}$

Introducing logarithm to base n on both sides

$\Rightarrow \log_n(mn) = \log_n N^{\frac{1}{x}} = \frac{1}{x}\log_n N$

$\Rightarrow x = \frac{\log_n N}{\log_n (mn)} = \frac{\log_n N}{\log_n m + \log_n n} = \frac{\log_n N}{\log_n m + 1}$

$\therefore \log_{mn} N = \frac{\log_n N}{1+\log_n M}$

Application of laws of logarithms

A. Solving equations

1. *Solve the equations:*
   i) $\log(x^2 + 2x) = 0.9031$
   ii) $\log_a(5x-6) + \log_a(2x+3) = \log_a(10x^2 - 3x - 6)$
   iii) $2\log_a(x-2) = \log_a(2x-5)$
   iv) $\log_x 3 + \log_x 9 + 729 = 9$

Solution

i) $\log(x^2 + 2x) = 0.9031$

$\Rightarrow \log_{10}(x^2 + 2x) = 0.9031$

$\Rightarrow 10^{0.9031} = x^2 + 2x$ (From the definition)

$\Rightarrow x^2 + 2x - 8 = 0 \;\; or \;\; (x+4)(x-2) = 0$

$\therefore x = -4, 2$

ii) $\log_a(5x-6) + \log_a(2x+3) = \log_a(10x^2 - 3x - 6)$

$\Rightarrow \log_a(5x-6)(2x+3) = \log_a(10x^2 - 3x - 6)$

Since the bases are the same, then also the numbers must be equal.

$\Rightarrow (5x-6)(2x+3) = 10x^2 - 3x - 6$

$\Rightarrow 10x^2 + 3x - 18 = 10x^2 - 3x - 6$

$\Rightarrow 6x = 12 \;\; \therefore x = 2$

iii) $\log_a(x-2) = \log_a(2x-5)$

$\Rightarrow \log_a(x-2)^2 = \log_a(2x-5)$

$\Rightarrow (x-2)^2 = 2x - 5$

$\Rightarrow x^2 - 4x + 3 = 2x - 5$

$\Rightarrow x^2 - 6x + 9 = 0 \;\; or \;\; (x-3)^2 = 0$

$\therefore x = 3$

iv) $\log_x 3 + \log_x 9 + 729 = 9$

$\Rightarrow \log_x(3 \times 9 \times 729) = 9$

$\Rightarrow x^9 = 19683$

$\Rightarrow (x^9)^{\frac{1}{9}} = (19683)^{\frac{1}{9}}$

$\therefore x = 3$

Task

1. *Solve the equations*
   i) $\ln(x^2 - 1) - \ln(x-1) = \ln 5 \quad (x > 0)$
   ii) $\ln 4 + 2\ln x - \ln(x^2 + x) = \ln 2 \quad (x > 0)$ **ANSWER**: i) $\frac{3}{2}$ ii) $\frac{3}{2}$

NOTE: $\log_e N = ln\, N$

2. *Solve that equations*

i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$

ii) $11^{\log_x 7} = 7$

iii) $\log_{10} x - 2 + \log_x 10 = 0$

iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

v) $\log_2 x - \log_x 8 = 2$ $\quad(\boldsymbol{Uneb\ 2007})$

vi) $\log_4(6-x) = \log_2 x$ $\quad(\boldsymbol{Uneb\ 2003})$

vii) $\log_n 4 + \log_4 n^2 = 3$

viii) $\log_2 x + \log_x 64 = 5$

ix) $\log_5 x - \log_x 5 = \frac{3}{2}$

Solution

i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$

Using $a^{\log_a x} = x$,

$\Rightarrow 2 + 7 = x \;\therefore x = 9$

ii) $11^{\log_x 7} = 7$

Let $\log_x 7 = y \;\Rightarrow 11^y = 7$

Introducing logarithm to base 11

$\Rightarrow \log_{11} 11^y = \log_{11} 7 \;\; or\; y = \log_{11} 7$

But $\log_x 7 = y \;\;\therefore \log_x 7 = \log_{11} 7$

Since numbers are the same, then $x = 11$

Alternatively

From $a^{\log_a x} = x$, since $11^{\log_x 7} = 7$, then $x = 11$

$\Rightarrow a^{\log_a 7} = 7$, also a must be 11

$\therefore 11^{\log_{11} 7} = 7 \Rightarrow x = 11$

iii) $\log_{10} x - 2 + \log_x 10 = 0$

$\Rightarrow \log_{10} x + \log_x 10 = 2$

Changing to base 10

$\Rightarrow \log_{10} x + \frac{\log_{10} 10}{\log_{10} x} = 2$

$\Rightarrow \log_{10} x + \frac{1}{\log_{10} x} = 2$; let $\log_{10} x = y$

$\Rightarrow y + \frac{1}{y} = 2 \;\; or\; y^2 - 2y + 1 = 0 \;\;\therefore y = 1$

$\therefore \log_{10} x = 1 \;\; or \;\; x = 10$

iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

Since 4 and 16 are multiples of 2, changing the base to 2

$\Rightarrow \log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 16} = \frac{21}{4}$

$\Rightarrow \log_2 x + \frac{\log_2 x}{\log_2 2^2} + \frac{\log_2 x}{\log_2 2^4} = \frac{21}{4}$

$\Rightarrow \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{4} = \frac{21}{4}$

$\Rightarrow \left(1 + \frac{1}{2} + \frac{1}{4}\right)\log_2 x = \frac{21}{16}$

$\Rightarrow \frac{7}{4}\log_2 x = \frac{21}{4} \;\; or \;\; \log_2 x = 3$

$\therefore x = 2^3 = 8$

v) $\log_2 x - \log_x 8 = 2$, changing to base 2

$\Rightarrow \log_2 x - \frac{\log_2 8}{\log_2 x} = 2$

$\Rightarrow \log_2 x - \frac{3\log_2 2}{\log_2 x} = 2$

$\Rightarrow \log_2 x - \frac{3}{\log_2 x} = 2$, let $\log_2 x = y$

$\Rightarrow y - \frac{3}{y} = 2$ *or* $y^2 - 2y - 3 = 0$

$\Rightarrow (y+1)(y-3) = 0$ *or* $y = -1, 3$

$\Rightarrow \log_2 x = -1$ *or* $2^{-1} = \frac{1}{2} = x$

$\Rightarrow \log_2 x = 3$ *or* $2^3 = 8 = x$

$\therefore x = \frac{1}{2}, 3$

vi) Left as an exercise (**Hint:** Change to base 2)

ANSWER: $x = 2, -3$

vii) $\log_n 4 + \log_4 n^2 = 3$

$\Rightarrow \log_n 4 + 2\log_4 n = 3$

$\Rightarrow \frac{\log_4 4}{\log_4 n} + 2\log_4 n = 3$, let $\log_4 n = x$

$\Rightarrow \frac{1}{x} + 2x = 3$ *or* $2x^2 - 3x + 1 = 0$

$\Rightarrow (2x-1)(x-1) = 0 \;\therefore x = \frac{1}{2}, 1$

$\Rightarrow \log_4 n = \frac{1}{2}$ *or* $4^{1/2} = 2 = n$

$\Rightarrow \log_4 n = 1$ *or* $4^1 = 4 = n$

$\therefore n = 2, 4$

viii) Left as an exercise. **ANSWER**: $4, 8$

ix) Left as an exercise. **ANSWER**: $x = \frac{\sqrt{5}}{5}, 25$

Task

a) Find x if $\log_x 8 - \log_{x^2} 16 = 1$ **ANSWER**: $x = 2$ $[\boldsymbol{HINT}: Change\ to\ base\ 2\ or\ x]$

b) Find x if $\log_x 3 + \log_3 x = 2.5$ **ANSWER**: $x = \sqrt{3}, 9$

c) Find x if $\log_x 3 + \log_3 x = \frac{10}{3}$ **ANSWER**: $x = 27, \sqrt[3]{3}$

d) Solve for x: $\log_{10}\left(\frac{x^2+24}{x}\right) = 1$ **ANSWER**: $x = 4, 6$

B. In proof numbers.

1. Show that $\log_{10}\frac{1}{n}$ is negative where $n \geq 2$

Solution

$\log_{10}\frac{1}{n} = \log_{10} 1 - \log_{10} n = -\log_{10} n$, which is negative for $n \geq 2$

2. *If* $a^2 + b^2 = 7ab$, *prove that* $\log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b)$

Solution

$a^2 + b^2 = 7ab \Rightarrow (a+b)^2 - 2ab = 7ab$

$\therefore (a+b)^2 = 9ab$ *or* $\left(\frac{a+b}{3}\right)^2 = ab$

$\Rightarrow \log\left[\frac{1}{3}(a+b)\right]^2 = \log a.b = \log a + \log b$

$\Rightarrow 2\log\left[\frac{1}{3}(a+b)\right] = \log a + \log b$

$\therefore \log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b)$

3. *Prove that* $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$ *(Assuming that the base of each logarithm is 10)*

Solution

L.H.S:

$\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = \log 2 + 16(\log 16 - \log 15) + 12(\log 25 - \log 24)$

$+7(\log 81 - \log 80)$

$= \log 2 + 16\log 2^4 - 16\log(3\times 5) + 12\log 5^2 - 12\log(3\times 2^3) + 7\log 3^4 - 7\log(2^4\times 5)$

$= \log 2 + 64\log 2 - 16\log 3 - 16\log 5 + 24\log 5 - 12\log 3 - 36\log 2 + 28\log 3 - 28\log 2 - 7\log 5$

$= (1 + 64 - 36 - 28)\log 2 + (-16 - 12 + 28)\log 3 + (-16 + 24 - 7)\log 5$

$= \log 2 + \log 5$

$= \log 2 + \log 5 = \log(2\times 5)$

$= \log 10 = 1$

4. *Prove that* $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

Solution

Let $\log_a(ab) = x \Rightarrow ab = a^x \quad or \quad a = (ab)^{1/x}$

$\log_b(ab) = y \Rightarrow ab = a^y \quad or \quad b = (ab)^{1/y}$

Now $(ab)^1 = (ab)^{1/x}.(ab)^{1/y} = (ab)^{\frac{1}{x}+\frac{1}{y}}$

$\therefore 1 = \frac{1}{x} + \frac{1}{y}$

$\Rightarrow \frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

5. *Prove that* $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_b(abc)} = 1$

Solution

Proceeding as in the previous example,

Let $\log_a(abc) = x \Rightarrow abc = a^x \quad or \quad a = (abc)^{1/x}$

$\log_b(abc) = y \Rightarrow abc = a^y \quad or \quad b = (abc)^{1/y}$

$\log_b(abc) = z \Rightarrow abc = a^z \quad or\ c = (abc)^{1/z}$

Now $(abc)^1 = (abc)^{1/x}.(ab)^{1/y}.(abc)^{1/z} = (ab)^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$\therefore 1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$\Rightarrow \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_b(abc)} = 1$

6. *Prove that*

   i) $\log_a b \times \log_b c \times \log_c d = \log_a d$

   ii) $\log_a b \times \log_b c \times \log_c d \times \log_d a = 1$

   iii) $\log_a b \times \log_b a = 1$

Solution

Changing to another base, say k

i) **L.H.S:** $\log_a b \times \log_b c \times \log_c d = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} = \frac{\log_k d}{\log_k a}$

Now changing to base a

$\Rightarrow \frac{\log_k d}{\log_k a} = \left(\frac{\log_a d}{\log_a k}\right) \times \left(\frac{\log_a a}{\log_a k}\right) = \log_a d$

$\therefore \log_a b \times \log_b c \times \log_c d = \log_a d$

ii) **L.H.S:** $\log_a b \times \log_b c \times \log_c d \times \log_d a = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} \times \frac{\log_k a}{\log_k d} = 1$

iii) **L.H.S:** $\log_a b \times \log_b a = \frac{\log_k b}{\log_k a} \times \frac{\log_k a}{\log_k b} = 1$

Task

Prove that:

a) $\log_a x \times \log_b y = \log_b x \times \log_a y$

b) $\log_b a \times \log_a b \times \log_a c = \frac{1}{\log_c a}$

c) $\log_2 3 \times \log_3 8 = 1$

More conditional proofs

a) *If* $\log(x^3.y^3) = a$ *and* $\log\frac{x}{y} = b$, *show that* $\log x = \frac{1}{6}(3b + a)$ *and* $\log y = \frac{1}{6}(a - 3b)$

b) *If* $a^3 + b^3 = 0$, *prove that* $\log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$ *given that* $a + b \neq 0$.

c) *If* $a^x = b^y = c^z$ *and* $b^2 = ac$, *prove that* $y = \frac{2xz}{x+z}$

d) *If* $\log_{2a} a = x$, $\log_{3a} 2a = y$, $\log_{4a} 3a = z$, *show that* $xyz + 1 = 2yz$

e) *If* $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, *prove that* $1 + abc = 2bc$

f) *If* $z = \log_r(mn)$, $x = \log_m(rn)$ *and* $y = \log_n(mr)$, *prove that* $x + y + z = xyz - 2$

g) *If* $\log_a N = x$ *and* $\log_b N = y$, *prove that* $\log_{ab} N = \frac{xy}{x+y}$

Solution

a) $\log(x^3.y^3) = a$, $\log\frac{x}{y} = b$, to show that: $\log x = \frac{1}{6}(3b + a)$ *and* $\log y = \frac{1}{6}(a - 3b)$

From $\log(x^3.y^3) = a \Rightarrow \log(xy)^3 = a \;\therefore \log(xy) = \frac{a}{3}$

$\Rightarrow \log x + \log y = \frac{a}{3}$ .... (1)

Also $\log\frac{x}{y} = b$ gives

$\Rightarrow \log x - \log y = b$ .... (2)

$eqn.(1) + eqn.(2)$

$\Rightarrow 2\log x = \frac{a}{3} + b = \frac{a+3b}{3}$

$\therefore \log x = \frac{1}{6}(3b + a)$

$eqn.(1) - eqn.(2)$

$\Rightarrow 2\log y = \frac{a}{3} - b = \frac{a-3b}{3}$

$\therefore \log y = \frac{1}{6}(a - 3b)$

b) $a^3 + b^3 = 0$

But $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$\Rightarrow a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

$\Rightarrow (a + b)^3 - 3ab(a + b) = 0$

$\Rightarrow (a + b)[(a + b)^2 - 3ab] = 0$

$\therefore (a + b)^2 - 3ab = 0$ *or* $(a + b)^2 = 3ab$

Introducing logarithms

$\Rightarrow \log(a + b)^2 = \log(3ab)$

$\Rightarrow 2\log(a + b) = \log 3 + \log a + \log b$

$\therefore \log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$

c) Let $a^x = b^y = c^z = k$

$\Rightarrow a^x = k$ *or* $a = k^{1/x}$

$\Rightarrow b^y = k$ *or* $b = k^{1/y}$

$\Rightarrow c^z = k \;\; or \;\; c = k^{1/z}$

Given: $b^2 = ac$

$\Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}}.\, k^{\frac{1}{z}}$

$\Rightarrow k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$

$\therefore \frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}$

$\therefore y = \frac{2xz}{x+z}$

Note: the number portrays the use of laws of indices.

d) $\log_{2a} a = x \;\Rightarrow (2a)^x = a \;\; or \;\; 2a = a^{\frac{1}{x}}$

$\log_{3a} 2a = y \;\Rightarrow (3a)^y = 2a \;\; or \;\; 3a = (2a)^{\frac{1}{y}}$

$\log_{4a} 3a = z \Rightarrow (4a)^z = 3a \;\; or \;\; 4a = (3a)^{\frac{1}{z}}$

From $2a = a^{\frac{1}{x}} \Rightarrow 2 = a^{\frac{1}{x}-1}$ .... (1)

From $3a = (2a)^{\frac{1}{y}} \Rightarrow 3a = \left(a^{\frac{1}{x}}\right)^{\frac{1}{y}} = a^{\frac{1}{xy}}$

$\therefore 3 = a^{\frac{1}{xy}-1}$ .... (2)

From $4a = (3a)^{\frac{1}{z}} \Rightarrow 2^2.a = \left(a^{\frac{1}{xy}}\right)^{\frac{1}{z}} = a^{\frac{1}{xyz}}$

$\therefore \left(a^{\frac{1}{x}-1}\right)^2 .a = a^{\frac{1}{xyz}}$

$\Rightarrow a^{\left(\frac{2}{x}-2+1\right)} = a^{\frac{1}{xyz}}$

$\therefore \frac{2}{x} - 1 = \frac{1}{xyz} \;\; or \;\; 2yz - xyz = 1$

$\therefore xyz + 1 = 2yz$

e) $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, to prove that $1 + abc = 2bc$

$a = \log_{24} 12 \Rightarrow (24)^a = 12 \;\; or \;\; 24 = (12)^{\frac{1}{a}}$ .... (1)

$b = \log_{36} 24 \Rightarrow (36)^b = 24 \;\; or \;\; 36 = (24)^{\frac{1}{b}}$ .... (2)

$c = \log_{48} 36 \Rightarrow (48)^c = 36 \;\; or \;\; 48 = (36)^{\frac{1}{c}}$ .... (3)

From eqn. (1)

$24 = (12)^{\frac{1}{a}} \Rightarrow 2 = \frac{(12)^{\frac{1}{a}}}{12} = (12)^{\frac{1}{a}-1}$ .... (4)

From eqn. (2)

$36 = (24)^{\frac{1}{b}} \Rightarrow 3 = \frac{(24)^{\frac{1}{b}}}{12} = \frac{(12)^{\frac{1}{b}}.(2)^{\frac{1}{b}}}{12} = (12)^{\frac{1}{b}-1}.(2)^{\frac{1}{b}}$ .... (5)

$eqn. (4) in\ to\ eqn. (5)$

$\Rightarrow 3 = (12)^{\frac{1}{b}-1}.\left((12)^{\frac{1}{a}-1}\right)^{\frac{1}{b}}$

$\Rightarrow 3 = (12)^{\frac{1}{ab}-\frac{1}{b}}.(12)^{\frac{1}{b}-1} = (12)^{\frac{1}{ab}-\frac{1}{b}+\frac{1}{b}-1}$

$\therefore 3 = (12)^{\frac{1}{ab}-1}$ .... (6)

$eqn. (4), (6) in\ to\ eqn. (3)$

$48 = (36)^{\frac{1}{c}} \Rightarrow 2^2 \times 12 = [3(12)]^{\frac{1}{c}}$

$\Rightarrow \left[(12)^{\frac{1}{a}-1}\right]^2 .(12) = \left[(12)^{\frac{1}{ab}-1}.(12)\right]^{\frac{1}{c}}$

$\Rightarrow (12)^{\frac{2}{a}-2+1} = \left[(12)^{\frac{1}{ab}-1+1}\right]^{\frac{1}{c}}$

$\therefore \frac{2}{a} - 1 = \frac{1}{abc} \quad or \;\; 2bc - abc = 1$

$\therefore 1 + abc = 2bc$

f) Given: $z = \log_r(mn)$, $x = \log_m(rn)$, $y = \log_n(mr)$ to prove: $x + y + z = xyz - 2$

$z = \log_r(mn) \Rightarrow r^z = mn \;\; or \;\; r = (mn)^{\frac{1}{z}}$ .... (1)

$x = \log_m(rn) \Rightarrow m^x = rn \;\; or \;\; m = (nr)^{\frac{1}{x}}$ .... (2)

$y = \log_n(mr) \Rightarrow n^y = mr \;\; or \;\; n = (mr)^{\frac{1}{y}}$ .... (3)

$eqn.(1) \div eqn.(2)$, to eliminate n

$\Rightarrow \frac{r^z}{m^x} = \frac{m}{r} \Rightarrow r^{z+1} = m^{x+1}$

$\therefore m = r^{\left(\frac{z+1}{x+1}\right)}$ .... (4)

$eqn.(2) \div eqn.(3)$, to eliminate r

$\Rightarrow \frac{m^x}{n^y} = \frac{n}{m} \Rightarrow m^{x+1} = n^{y+1}$

$\therefore n = m^{\left(\frac{x+1}{y+1}\right)}$ .... (5)

$eqn.(4) in\ to\ eqn.(5)$

$\Rightarrow n = \left[r^{\left(\frac{z+1}{x+1}\right)}\right]^{\left(\frac{x+1}{y+1}\right)} = r^{\left(\frac{z+1}{y+1}\right)}$ .... (6)

Now eqn. (4), (6) in to eqn. (1)

$r^z = mn$, becomes

$\Rightarrow r^z = r^{\left(\frac{z+1}{x+1}\right)} . r^{\left(\frac{z+1}{y+1}\right)}$

$\Rightarrow z = \left(\frac{z+1}{x+1}\right) + \left(\frac{z+1}{y+1}\right)$

$\therefore z = \frac{yz+y+z+xz+x+z+1}{xy+x+y+1}$

$\Rightarrow xyz + xz + yz + z = yz + y + z + 1 + xz + x + z + 1$

$\therefore xyz + z = x + y + 2z + 2$

$\therefore x + y + z = xyz - 2$

g) $\log_a N = x \Rightarrow a^x = N \;\; or \;\; a = N^{\frac{1}{x}}$.................................. (1)

$\log_b N = y \Rightarrow b^y = N \;\; or \;\; b = N^{\frac{1}{y}}$.................................. (2)

Multiplying the equations

$\Rightarrow a.b = N^{\frac{1}{x}}.N^{\frac{1}{y}} = N^{\left(\frac{1}{x}+\frac{1}{y}\right)}$

Introducing logarithm to base ab

$\Rightarrow \log_{ab} ab = \log_{ab} N^{\left(\frac{1}{x}+\frac{1}{y}\right)}$

$\Rightarrow 1 = \left(\frac{1}{x} + \frac{1}{y}\right)\log_{ab} N \quad or \quad 1 = \left(\frac{x+y}{xy}\right)\log_{ab} N$

$\therefore \log_{ab} N = \frac{xy}{x+y}$

Task

1. *Prove that*
   i) $\log_{b^2} a \times \log_{x^2} b = \frac{1}{4}\log_x a$
   ii) If $\log_a\left(1+\frac{1}{8}\right) = l$, $\log_a\left(1+\frac{1}{15}\right) = m$ and $\log_a\left(1+\frac{1}{24}\right) = n$, then $\log_a\left(1+\frac{1}{80}\right) = l - m - n$
   iii) If $\log_a n = x$ and $\log_c n = y$ where $n \neq 1$ then $\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$
2. a) *Given that* $\log_3 x = p$ *and* $\log_{18} x = q$, *show that* $\log_6 3 = \frac{q}{p-q}$ (***Uneb* 2002**).
   b) *Show that* $\log_8 x = \frac{2}{3}\log_4 x$. *Hence without using tables/ calculator, evaluate* $\log_8 6$ *to* 3 *d.p, if* $\log_4 3 = 0.7925$. (***Uneb* 2000**)
   c) *Prove that* $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. *Hence given that* $\log_3 2 = 0.631$, *find without using tables/ calculator* $\log_6 4$ *correct to* 3 *s.f.* (***Uneb* 1998 *March***)
   d) *If* x *and* y *are positive numbers and each is greater than 1,*
   $x^{x+y} = y^{12}$ and $y^{x+y} = x^3$
   Prove that $\log x = 2\log y$. *Hence find* x *and* y

ANSWER:

a) $\log_3 x = p \Rightarrow 3^p = x$ *or* $3 = x^{\frac{1}{p}}$ ………………………………. (1)

$\log_{18} x = q \Rightarrow 18^q = x$ *or* $18 = x^{\frac{1}{q}}$ ……………………… (2)

eqn. (2) $\div$ *eqn.* (1)

$\Rightarrow 6 = x^{\frac{1}{q}} \div x^{\frac{1}{p}} = x^{\frac{1}{q}-\frac{1}{p}} = x^{\left(\frac{p-q}{pq}\right)}$

Now introducing logarithm to base 6

$\Rightarrow \log_6 6 = \log_6 x^{\left(\frac{p-q}{pq}\right)} = \left(\frac{p-q}{pq}\right)\log_6 x$

$\Rightarrow 1 = \left(\frac{p-q}{pq}\right)\log_6 x \quad \therefore \log_6 x = \frac{pq}{p-q}$, dividing through by p

$\Rightarrow \frac{\log_6 x}{p} = \frac{q}{p-q}$

$\Rightarrow \frac{q}{p-q} = \frac{\log_6 x}{\log_3 x} = \frac{\log_6 x}{\frac{\log_6 x}{\log_6 3}}$

$\therefore \log_6 3 = \frac{q}{p-q}$

Alternatively

From $\log_{18} x = q$

$q = \frac{\log_3 x}{\log_3 18} = \frac{\log_3 x}{\log_3 (3\times 6)}$

$q = \frac{\log_3 x}{\log_3 3 + \log_3 6} = \frac{p}{1+\log_3 6}$

$\Rightarrow q(1+\log_3 6) = p$

$\Rightarrow q + q\log_3 6 = p$ *or* $p - q = q\log_3 6$

$\Rightarrow \log_3 6 = \frac{p-q}{q}$, but $\log_3 6 = \frac{\log_6 6}{\log_6 3} = \frac{1}{\log_6 3}$

$\therefore \frac{p-q}{q} = \frac{1}{\log_6 3}$ *or* $\log_6 3 = \frac{q}{p-q}$

b) To show that $\log_8 x = \frac{2}{3}\log_4 x$

LHS: $\log_8 x = \frac{\log_4 x}{\log_4 8} = \log_4 x . \frac{1}{\log_4 8}$

But $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3\log_2 2}{2\log_2 2} = \frac{3}{2}$

$\therefore \log_8 x = \log_4 x \,.\, \frac{1}{\log_4 8} = \frac{2}{3}\log_4 x$

Putting $x = 6$ in to the proof,

$\Rightarrow \log_8 6 = \frac{2}{3}\log_4 6 = \frac{2}{3}\log_4(2 \times 3)$

$= \frac{2}{3}[\log_4 2 + \log_4 3]$

$= \frac{2}{3}\left[\frac{\log_2 2}{\log_2 4} + \log_4 3\right] = \frac{2}{3}\left[\frac{1}{2\log_2 2} + \log_4 3\right] = \frac{2}{3}[0.5 + 0.7925] = 0.862$

c) To prove $\log_6 x = \frac{\log_3 x}{1 + \log_3 2}$

Let $\log_6 x = y \Rightarrow 6^y = x \;\; or \;\; 6 = x^{\frac{1}{y}}$

Introducing logarithm to base 3

$\Rightarrow \log_3 6 = \log_3 x^{\frac{1}{y}} = \frac{1}{y}\log_3 x$

$\Rightarrow \log_3 x = y\log_3 6 \quad or \;\; y = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3(2\times 3)}$

$\Rightarrow y = \frac{\log_3 x}{\log_3 2 + \log_3 3} = \frac{\log_3 x}{1 + \log_3 2}$

$\therefore \log_6 x = \frac{\log_3 x}{1 + \log_3 2}$

Now setting $x = 4$

$\Rightarrow \log_6 4 = \frac{\log_3 4}{1 + \log_3 2} = \frac{2\log_3 2}{1 + \log_3 2}$

$\therefore \log_6 4 = \frac{2\times 0.631}{1 + 0.631} = \frac{1.262}{1.631} = 0.774$

d) $x^{x+y} = y^{12} \Rightarrow \log x^{x+y} = \log y^{12}$

$\Rightarrow (x + y)\log x = 12\log y$ …………………………………………… (1)

$y^{x+y} = x^3 \Rightarrow \log y^{x+y} = \log x^3$

$\Rightarrow (x + y)\log y = 3\log x$ …………………………………………… (2)

$eqn.(1) \div eqn.(2)$

$\Rightarrow \frac{(x+y)\log x}{(x+y)\log y} = \frac{12\log y}{3\log x} \quad or \;\; \frac{\log x}{\log y} = \frac{4\log x}{\log y}$, let $\log x = a$, $\log y = b$

$\Rightarrow \frac{a}{b} = \frac{4b}{a} \quad or \; a^2 = 4b^2$

$\therefore a = 2b$

$\therefore \log x = 2\log y$

Now from $\log x = 2\log y \Rightarrow x = y^2$ …………………………… (3)

$x^{x+y} = y^{12}$, becomes

$\Rightarrow (y^2)^{(y^2+y)} = y^{12} \quad or \; y^{2y^2+2y} = y^{12}$

$\Rightarrow 2y^2 + 2y = 12 \;\; or \; y^2 + y - 6 = 0$

$\Rightarrow (y + 3)(y - 2) = 0 \;\; \therefore y = -3\,, 2$

From eqn. (3), $x = y^2$

For $y = -3\,,\; x = (-3)^2 = 9$

$y = 2\,,\; x = (2)^2 = 4$

$\therefore x = 9\,, y = -3\,; x = 4\,, y = 2$

Task

1. *Prove that* $abc = 1$ *if* $a = \log_b c$, $b = \log_a b$, $\log_a c$
2. *Show that* $\frac{\log\sqrt{27} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}$ *without using tables/calculator.*
3. *Prove that* $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$
4. *Prove that* $\log_4 2\,.\log_2 6\,.\log_6 4 = 1$
5. *Prove that* $\log_2[\log_2\{\log_3(\log 27^3)\}] = 0$

SIMULTANEOUS EQUATIONS INVOLVING LOGARITHMS

1) Solve the following simultaneous equations

a) $2\log_x y = 1$
$xy = 64$

b) $6\log_3 x + 6\log_{27} y = 7$
$4\log_9 x + 4\log_3 y = 9$

c) $y\log_2 8 = x$
$2^x + 8^y = 8192$

d) $\log x - \log 2 = 2\log y$
$x - 5y + 2 = 0$

e) $\log_2 x + 2\log_4 y = 4$
$x + 12y = 52$

f) $\ln 6 + \ln(x-3) = 2\ln y$
$2y - x = 3$

g) $2 + \log_2(2x+1) = 2\log_2 y$
$x = 22 - y$

Solution

a) $2\log_x y = 1 \Rightarrow \log_x y^2 = 1 \;\; \therefore y^2 = x$ .... (1)
$xy = 64$ .... (2)
$\text{eqn. (1)in to eqn. (2)}$
$\Rightarrow y.y^2 = 64 \;\; or \;\; y^3 = 4^3 \;\; \therefore y = 4$ .... (3)
From eqn. (1), $x = 4^2 = 16$
$\therefore x = 16,\; y = 4$

b) $6\log_3 x + 6\log_{27} y = 7$
Changing to the same base
$\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{\log_3 27} = 7$
$\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{\log_3 3^3} = 7$ or $\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{3} = 7$
$\therefore 6\log_3 x + 2\log_3 y = 7$ .... (1)
$4\log_9 x + 4\log_3 y = 9 \Rightarrow 4\frac{\log_3 x}{\log_3 9} + 4\log_3 y = 9$
$\therefore 2\log_3 x + 4\log_3 y = 9$ .... (2)
Eliminating $\log_3 x$ from eqn. (1) and eqn. (2)
$\text{eqn. (1)} - 3\times\text{eqn. (2)}$
$\Rightarrow 6\log_3 x + 2\log_3 y = 7$
$\underline{-\;6\log_3 x + 12\log_3 y = 27}$
$-10\log_3 y = -20 \;\Rightarrow \log_3 y = 2$
$\Rightarrow y = 3^2 = 9$ .... (3)
$\text{eqn. (3) in to eqn. (2)}$
$\Rightarrow 2\log_3 x + 4\log_3 9 = 9$
$\Rightarrow \;\; 2\log_3 x + 8 = 9$
$\Rightarrow 2\log_3 x = 1 \;\; or \;\; \log_3 x^2 = 1$
$\Rightarrow x^2 = 3 \;\; \therefore x = +\sqrt{3}$ .... (4)
$\therefore x = +\sqrt{3}\,; y = 9$

c) $y\log_2 8 = x \Rightarrow y\log_2 2^3 = x \;\; or \;\; x = 3y$………………………. (1)
$2^x + 8^y = 8192$ ……………………………………………… (2)
eqn. (1)in to eqn. (2)
$\Rightarrow 2^{3y} + 8^y = 8192$
$\Rightarrow \;\; 8^y + 8^y = 8192$
$\Rightarrow 2(8^y) = 8192 \;\; or \;\; 8^y = 4096 = 8^4$
$\therefore y = 4$ ……………………………………………………… (3)
eqn. (3)in to eqn. (1)
$\Rightarrow x = 3 \times 4 = 12$
$\therefore x = 12\,,\; y = 4$

d) Left as an exercise **ANSWER:** $\boldsymbol{x = \frac{1}{2}\,,\; y = \frac{1}{4}\,; x = 8\,, y = 2}$

e) Left as an exercise **ANSWER:** $\boldsymbol{x = 4\,,\; y = 4\,; x = 48\,, y = \frac{1}{3}}$

f) Left as an exercise **ANSWER:** $\boldsymbol{x = 9\,,\; y = 6}$

g) Left as an exercise **ANSWER:** $\boldsymbol{x = 12\,,\; y = 10}$

SURDS

Definition:

A surd is an irrational root of a rational number.

A rational number is a number with exact n^{th} root $.i.e. \sqrt[n]{b} = Real\, number$

$e.g.$ $\sqrt{4} = 2$, $\sqrt[4]{16} = 2$, $\sqrt[3]{8} = 2$, these are rational numbers.

If $a = \sqrt[n]{b}$ for some rational number b such that a itself is an irrational number, then $a = \sqrt[n]{b}$ is a surd.

When $a = \sqrt[n]{b}$ is a surd, n is called *order* of a and b is called *a radicand* of a. Surds of 2^{nd} , 3^{rd} , 4^{th} and higher orders are respectively called *quadratic, cubic, quartic* and *quintic*.

Compound, Simple and Similar surds

A compound surd is a surd with coefficient different from ± 1. $e.g. 2\sqrt{2}, 3\sqrt{5}, -3\sqrt{7}\, etc$

A simple surd is a surd with coefficient ± 1. $e.g. \sqrt{2}, \sqrt[4]{\frac{1}{4}}, \sqrt[3]{7}\, etc$

Similar surds are surds if in reduced form (simplified form) contain same surds. $e.g$ $\sqrt{45}$ and $\sqrt{80}$ are similar surds, since $\sqrt{45} = 3\sqrt{5}$ and $\sqrt{80} = 4\sqrt{5}$. Similar surds can be added or subtracted easily after being reduced to their simplest forms.

$e.g. \sqrt{75} + \sqrt{27} = 5\sqrt{3} + 3\sqrt{3} = 8\sqrt{3}$ and $2\sqrt{128} - \sqrt{162} = 16\sqrt{2} - 9\sqrt{2} = 7\sqrt{2}$.

Rules of simplifying surds

i) $\sqrt[a]{a} \times \sqrt[n]{b} = \sqrt[n]{a.b}$; surds of the same order can be multiplied or divided

ii) $\sqrt[a]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$

iii) $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

iv) $(m\sqrt{a} + n\sqrt{b})(m\sqrt{a} - n\sqrt{b}) = (m\sqrt{a})^2 - (n\sqrt{b})^2$

v) $N\sqrt[a]{a} \times M\sqrt[n]{b} = MN\sqrt[n]{a.b}$

vi) If $a + \sqrt{b} = 0$ if and only if $a = b = 0$

vii) If $a + \sqrt{b} = \sqrt{c}$ if and only if $a = 0, b = c$

viii) If $a + \sqrt{b} = c + \sqrt{d}$ if and only if $a = c, b = d$

ix) If $\sqrt{a + \sqrt{b}} = \sqrt{x} - \sqrt{y}$ then $\sqrt{a - \sqrt{b}} = \pm(\sqrt{x} - \sqrt{y})$

Rules /properties from (viii) to (ix) represent quadratic mixed surds properties.

RATIONALIZATION

If the product of two surds is a rational number and if one of them is multiplied by the other, the process is called *Rationalization.*

Thus $\sqrt{3} \times \sqrt{3} = 3$, so multiplication by $\sqrt{3}$ is rationalization of $\sqrt{3}$.

Similarly $\sqrt{5} - \sqrt{3}$ then multiplied by $\sqrt{5} + \sqrt{3}$ is rationalized to 2 as

$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 3 = 2$

NOTE:

Rationalization can be in the numerator/denominator of a given surd.

Rationalization of Denominator

Case I: *Two quadratic surds Denominator.* $\frac{a}{\sqrt{b}+\sqrt{c}}$

$$\frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})} = \frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b})^2-(\sqrt{c})^2}$$

$$= \frac{a(\sqrt{b}-\sqrt{c})}{b-c} = \frac{a}{b-c}\sqrt{b} - \frac{a}{b-c}\sqrt{c}$$

Note:

$\sqrt{b}-\sqrt{c}$ is a surd conjugate of $\sqrt{b}+\sqrt{c}$

Case II: *Three quadratic surds Denominator.* $\frac{a}{\sqrt{b}+\sqrt{c}+\sqrt{d}}$

First multiply both numerator and denominator by $(\sqrt{b}+\sqrt{c})-\sqrt{d}$

Denominator becomes $(\sqrt{b}+\sqrt{c})^2-(\sqrt{d})^2$ *or* $b+c-d+2\sqrt{bc}$

Then multiply both numerator and denominator by $(b+c-d)-2\sqrt{bc}$

Denominator becomes $(b+c-d)^2-(2\sqrt{bc})^2 = (b+c-d)^2-4bc$ which is a rational quantity.

Example

1. *Express as equivalent fraction with a rational denominator*

   a) $\frac{1}{\sqrt{2}+\sqrt{3}}$

   b) $\frac{7+4\sqrt{3}}{7-4\sqrt{3}}$

   c) $\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{2\sqrt{3}-\sqrt{2}}$

Solution

a) $\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2})^2-(\sqrt{3})^2} = \frac{\sqrt{2}-\sqrt{3}}{-1} = \sqrt{3}-\sqrt{2}$

b) $\frac{7+4\sqrt{3}}{7-4\sqrt{3}} = \frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})} = \frac{49+28\sqrt{3}+28\sqrt{3}+16\sqrt{9}}{7^2-(4\sqrt{3})^2}$

$$= \frac{49+48+56\sqrt{3}}{49-48} = 74+56\sqrt{3}$$

c) $\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{2\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}+\frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})}$

$$= \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}+\frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3})^2-(\sqrt{2})^2} = \sqrt{3}-\sqrt{2}+\frac{2\sqrt{3}+\sqrt{2}}{12-2}$$

$$= \frac{10\sqrt{3}-10\sqrt{2}+2\sqrt{3}+\sqrt{2}}{10} = \frac{12\sqrt{3}-9\sqrt{2}}{10}$$

$$= \frac{3}{10}(4\sqrt{3}-3\sqrt{2})$$

2. *Express as equivalent fraction with a rational denominator*

   a) $\frac{1}{1+\sqrt{2}-\sqrt{3}}$

   b) $\frac{1}{2+\sqrt{3}+\sqrt{5}}$

Solution

a) $\frac{1}{1+\sqrt{2}-\sqrt{3}} = \frac{1}{(1+\sqrt{2})-\sqrt{3}} = \frac{(1+\sqrt{2})+\sqrt{3}}{[(1+\sqrt{2})-\sqrt{3}](1+\sqrt{2})+\sqrt{3}}$

$$= \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2} = \frac{1+\sqrt{2}+\sqrt{3}}{1+2\sqrt{2}+2-3}$$

$$= \frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}} = \frac{(1+\sqrt{2}+\sqrt{3})\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}$$

$$= \frac{\sqrt{2}(1+\sqrt{2}+\sqrt{3})}{4} = \frac{\sqrt{2}+2+\sqrt{6}}{4}$$

b) $\frac{1}{2+\sqrt{3}+\sqrt{5}} = \frac{1}{(2+\sqrt{3})+\sqrt{5}} = \frac{(2+\sqrt{3})-\sqrt{5}}{[(2+\sqrt{3})+\sqrt{5}](2+\sqrt{3})-\sqrt{5}}$

$= \frac{2+\sqrt{3}-\sqrt{5}}{(2+\sqrt{3})^2-(\sqrt{5})^2} = \frac{2+\sqrt{3}-\sqrt{5}}{4+4\sqrt{3}+3-5}$

$= \frac{2+\sqrt{3}-\sqrt{5}}{2+4\sqrt{3}} = \frac{(2+\sqrt{3}-\sqrt{5})(2-4\sqrt{3})}{(2+4\sqrt{3})(2-4\sqrt{3})}$

$= \frac{4-8\sqrt{3}+2\sqrt{3}-4\sqrt{9}-2\sqrt{5}+4\sqrt{15}}{2^2-(4\sqrt{3})^2} = \frac{-8-6\sqrt{3}-2\sqrt{5}+4\sqrt{15}}{4-48}$

$= \frac{1}{44}\left(8+6\sqrt{3}+2\sqrt{5}-4\sqrt{15}\right)$

Task

Express as equivalent fractions with rational denominators

a) $\frac{12}{3+\sqrt{5}-2\sqrt{2}}$ **ANSWER**: $1+\sqrt{5}+\sqrt{10}-\sqrt{2}$

b) $\frac{1}{2+\sqrt{3}-\sqrt{5}}$ **ANSWER**: $\frac{1}{44}\left(8+6\sqrt{3}+2\sqrt{5}-4\sqrt{15}\right)$

3. *Express as equivalent fractions with rational denominators*

a) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

b) $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

Solution

a) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} = \frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}} = \frac{\sqrt{2}[(\sqrt{2}+\sqrt{3})+\sqrt{5}]}{[(\sqrt{2}+\sqrt{3})-\sqrt{5}]((\sqrt{2}+\sqrt{3})+\sqrt{5})}$

$= \frac{\sqrt{4}+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2} = \frac{2+\sqrt{6}+\sqrt{10}}{2+2\sqrt{6}+3-5}$

$= \frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}} = \frac{\sqrt{6}(2+\sqrt{6}+\sqrt{10})}{2\sqrt{6}\times\sqrt{6}}$

$= \frac{2\sqrt{6}+6+\sqrt{60}}{12} = \frac{6+2\sqrt{6}+2\sqrt{15}}{12}$

$= \frac{3+\sqrt{6}+\sqrt{15}}{6}$

b) $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{1}{(\sqrt{2}+\sqrt{3})+\sqrt{5}} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{[(\sqrt{2}+\sqrt{3})+\sqrt{5}](\sqrt{2}+\sqrt{3})-\sqrt{5}}$

$= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2+2\sqrt{6}+3-5}$

$= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}} = \frac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2\sqrt{6}\times\sqrt{6}}$

$= \frac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12} = \frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

Task

Express as equivalent fractions with rational denominators

$\frac{\sqrt{10}-\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{10}-\sqrt{5}}$ **ANSWER**: $\frac{3\sqrt{30}+5\sqrt{15}-10\sqrt{2}-12}{7}$

4. *If* $x = \frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$ *, find the value of;*

i) $\frac{x+\sqrt{8}}{x-\sqrt{8}}$

ii) $\frac{x+\sqrt{12}}{x-\sqrt{12}}$

iii) $\frac{x+\sqrt{8}}{x-\sqrt{8}} + \frac{x+\sqrt{12}}{x-\sqrt{12}}$

Solution

$x = \frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}} = \frac{2\sqrt{6\times4}}{\sqrt{2}+\sqrt{3}} = \frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

$= \frac{4\sqrt{6}(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{4\sqrt{12}-4\sqrt{18}}{(\sqrt{2})^2-(\sqrt{3})^2}$

$= \frac{4(2\sqrt{3})-4(3\sqrt{2})}{2-3} = 12\sqrt{2} - 8\sqrt{3}$

i) Now $\frac{x+\sqrt{8}}{x-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{8}}{12\sqrt{2}-8\sqrt{3}-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{2}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{2}}$

$= \frac{14\sqrt{2}-8\sqrt{3}}{10\sqrt{2}-8\sqrt{3}} = \frac{7\sqrt{2}-4\sqrt{3}}{5\sqrt{2}-4\sqrt{3}}$

$= \frac{(7\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}{(5\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}$

$= \frac{70-48+18\sqrt{6}}{(5\sqrt{2})^2-(4\sqrt{3})^2} = \frac{22+8\sqrt{6}}{50-48} = 11 + 4\sqrt{6}$

ii) $\frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{12}}{12\sqrt{2}-8\sqrt{3}-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{3}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{3}}$

$= \frac{12\sqrt{2}-6\sqrt{3}}{12\sqrt{2}-10\sqrt{3}} = \frac{6\sqrt{2}-3\sqrt{3}}{6\sqrt{2}-5\sqrt{3}}$

$= \frac{(6\sqrt{2}-3\sqrt{3})(6\sqrt{2}+5\sqrt{3})}{(6\sqrt{2}-5\sqrt{3})(6\sqrt{2}+5\sqrt{3})}$

$= \frac{72-45+30\sqrt{6}-18\sqrt{6}}{(6\sqrt{2})^2-(5\sqrt{3})^2} = \frac{27+12\sqrt{6}}{72-75}$

$= \frac{27+12\sqrt{6}}{-3} = -9 - 4\sqrt{6}$

iii) $\frac{x+\sqrt{8}}{x-\sqrt{8}} + \frac{x+\sqrt{12}}{x-\sqrt{12}} = 11 + 4\sqrt{6} + \ -9 - 4\sqrt{6} \ = (11-9) + \left(4\sqrt{6} - 4\sqrt{6}\right) = 2$

5. *If* $x = \sqrt{3} + \frac{1}{\sqrt{3}}$ *, calculate;*

i) $x - \frac{\sqrt{126}}{\sqrt{42}}$

ii) $x - \frac{1}{x-\frac{2\sqrt{3}}{3}}$

iii) $\left(x - \frac{\sqrt{126}}{\sqrt{42}}\right).\left(x - \frac{1}{x-\frac{2\sqrt{3}}{3}}\right)$, correct to 2 $d.p$

Solution

$x = \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{\sqrt{9}+1}{\sqrt{3}} = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

i) $x - \frac{\sqrt{126}}{\sqrt{42}} = x - \sqrt{\left(\frac{126}{42}\right)} = x - \sqrt{3}$

$\therefore\ x - \frac{\sqrt{126}}{\sqrt{42}} = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{4-\sqrt{9}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

ii) $x - \frac{1}{x-\frac{2\sqrt{3}}{3}} = \frac{4}{\sqrt{3}} - \frac{1}{\frac{4}{\sqrt{3}}-\frac{2\sqrt{3}}{3}}$

$= \frac{4\sqrt{3}}{3} - \frac{1}{\frac{4\sqrt{3}}{3}-\frac{2\sqrt{3}}{3}} = \frac{4\sqrt{3}}{3} - \frac{3}{2\sqrt{3}}$

$= \frac{8\sqrt{9}-9}{6\sqrt{3}} = \frac{15}{6\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}$

iii) $\left(x - \frac{\sqrt{126}}{\sqrt{42}}\right).\left(x - \frac{1}{x-\frac{2\sqrt{3}}{3}}\right) = \frac{1}{\sqrt{3}} \times \frac{5\sqrt{3}}{6} = \frac{5}{6} = 0.83$

Proofs in surds

1. *If* $x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ *,* $y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ *, show that* $3x^2 - 5xy + 3y^2 = 289$

Solution

$x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{3+2-2\sqrt{6}}{(\sqrt{3})^2-(\sqrt{2})^2}$

$= 5 - 2\sqrt{6}$

$x^2 = (5-2\sqrt{6})^2 = 25 - 20\sqrt{6} + 24$

$= 49 - 20\sqrt{6}$

$x.y = \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = 1$

$y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{1}{x} = \frac{1}{5-2\sqrt{6}}$

$= \frac{5+2\sqrt{6}}{(5-2\sqrt{6})(5+2\sqrt{6})} = \frac{5+2\sqrt{6}}{5^2-(2\sqrt{6})^2}$

$= 5 + 2\sqrt{6}$

$y^2 = (5+2\sqrt{6})^2 = 25 + 20\sqrt{6} + 24$

$= 49 + 20\sqrt{6}$

$\therefore 3x^2 - 5xy + 3y^2 = 3(49-20\sqrt{6}) - 5(1) + 3(49+20\sqrt{6})$

$= 147 - 60\sqrt{6} - 5 + 147 + 60\sqrt{6} = 289$

2. *If* $x = \frac{5-\sqrt{21}}{2}$, *prove that* $x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$

Solution

Let $u = x + \frac{1}{x} = \frac{5-\sqrt{21}}{2} + \frac{2}{5-\sqrt{21}}$

$= \frac{(5-\sqrt{21})^2+2}{2(5-\sqrt{21})} = \frac{25-10\sqrt{21}+21+4}{2(5-\sqrt{21})}$

$= \frac{50-10\sqrt{21}}{2(5-\sqrt{21})} = \frac{10(5-\sqrt{21})}{2(5-\sqrt{21})} = \frac{10}{2} = 5$

Now $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x} = u^2 - 2$

$= 5^2 - 2 = 23$

$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$= 5^3 - 3 \times 5 = 125$

$\therefore x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 110 - 5(23) + 5 = 0$

3. *If* $x = \frac{-3+\sqrt{5}}{2}$, *show that* $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0$

Solution

$x = \frac{-3+\sqrt{5}}{2} \Rightarrow \frac{1}{x} = \frac{2}{-3+\sqrt{5}} = \frac{2(-3-\sqrt{5})}{(-3+\sqrt{5})(-3-\sqrt{5})}$

$\frac{1}{x} = \frac{-6-2\sqrt{5}}{(-3)^2-(\sqrt{5})^2} = \frac{-2(3+\sqrt{5})}{9-5} = -\frac{3+\sqrt{5}}{2}$

Now $x + \frac{1}{x} = \frac{-3+\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2} = -3$

$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

$\therefore x^2 + \frac{1}{x^2} = (-3)^2 - 2 = 7$

$\therefore \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 7 - 3 - 4 = 0$

Task

1. *If* $x = 7 - \sqrt{48}$, *prove that* $x^3 + \frac{1}{x^3} - 15\left(x^2 + \frac{1}{x^2}\right) + 20\left(x + \frac{1}{x}\right) - 72 = 0$
2. *If* $x = 7 - \sqrt{5}$, *prove that* $x^4 - x^3 - 20x^2 - 16x + 39 = 15$
3. *It is given that* $\sqrt{3} = 1.732$ *and* $\sqrt{5} = 2.236$, *evaluate* $\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$

   HINT: $\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}} = \frac{(\sqrt{5}+\sqrt{3})(4-\sqrt{15})}{(4+\sqrt{15})(4-\sqrt{15})} = \frac{4\sqrt{5}+4\sqrt{3}-\sqrt{75}-\sqrt{45}}{16-15}$

   $= 4\sqrt{5} + 4\sqrt{3} - 5\sqrt{3} - 3\sqrt{5} = \sqrt{5} - \sqrt{3}$

   $= 2.236 - 1.732 = \mathbf{0.504}$

4. *If* $2\sqrt{54} + 5\sqrt{294} + \frac{19}{30}\sqrt{6} - \sqrt{\frac{27}{50}} - \sqrt{\frac{2}{3}} = a\sqrt{b}$, *find a.*

   ANSWER: $a = 41$

TO FIND THE SQUARE ROOT OF A MIXED QUADRATIC SURD

A surd of the form $a + \sqrt{b}$ is a mixed quadratic surd. To find $\sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{b}}}$, proceed as follows.

Let $\sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{b}}} = \sqrt{x} + \sqrt{y}$, squaring both sides

$\left(\sqrt{a + \sqrt{b}}\right)^2 = \left(\sqrt{x} + \sqrt{y}\right)^2$

$\Rightarrow a + \sqrt{b} = x + y + 2\sqrt{xy}$

$\Rightarrow a = x + y$ .... (1)

$\Rightarrow \sqrt{b} = 2\sqrt{xy}$ *or* $b = 4xy$

$\Rightarrow b = 4xy$ .... (2)

From $(x - y)^2 = x^2 + y^2 - 2xy$

$\Rightarrow (x - y)^2 = (x + y)^2 - 2xy - 2xy$

$\Rightarrow (x - y)^2 = (x + y)^2 - 4xy$

$\therefore x - y = \pm\sqrt{a^2 - b}$ .... (3)

Now taking the positive sign in the equation

$\Rightarrow x - y = +\sqrt{a^2 - b}$ .... (4)

Adding eqn. (1) and (4)

$\Rightarrow x = \frac{1}{2}\left(a + \sqrt{a^2 - b}\right)$

$\Rightarrow y = \frac{1}{2}\left(a - \sqrt{a^2 - b}\right)$

Hence $\sqrt{a + \sqrt{b}}$ can be determined

NOTE:

If we take a $-ve$ sign from eqn. (3), the value of x and y will be interchanged and we shall get a second square root of $\sqrt{a + \sqrt{b}}$

Example

1. *Find the square root of*

   $31 + 4\sqrt{21}$

   Solution

   Let $\sqrt{31 + 4\sqrt{21}} = \sqrt{x} + \sqrt{y}$, squaring both sides

   $\Rightarrow 31 + 4\sqrt{21} = x + y + 2\sqrt{xy}$

   $\Rightarrow x + y = 31$ .... (1)

$2\sqrt{xy} = 4\sqrt{21}$,squaring
$\Rightarrow 4xy = 336 \ \ or\ xy = 84$ .... (2)
But $(x-y)^2 = (x+y)^2 - 4xy$
$\Rightarrow \mathrm{x} - \mathrm{y} = \sqrt{31^2 - 336} = \pm 25$
Taking the $+ve$ sign
$\Rightarrow x - y = 25$ .... (3)
Adding eqn. (1) and (3)
$2x = 56 \ \Rightarrow x = 28$
From eqn. (1) $y = 31 - x = 31 - 28 = 3$
$\therefore \sqrt{31 + 4\sqrt{21}} = \pm\left(\sqrt{28} + \sqrt{3}\right) = \pm\left(2\sqrt{7} + \sqrt{3}\right)$

2. *Find the square root of* $\sqrt{32} - \sqrt{24}$
Solution
$\sqrt{32} - \sqrt{24} = 4\sqrt{2} - 2\sqrt{6} = \sqrt{2}\left(4 - 2\sqrt{3}\right)$
Then $\sqrt{\sqrt{32} - \sqrt{24}} = \sqrt{\sqrt{2}\left(4 - 2\sqrt{3}\right)} = 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}}$
Evaluating $\sqrt{4 - 2\sqrt{3}}$
Let $\sqrt{4 - 2\sqrt{3}} = \sqrt{x} - \sqrt{y}$, squaring both sides
$\Rightarrow 4 - 2\sqrt{3} = x + y - 2\sqrt{xy}$
$\Rightarrow x + y = 4$ .... (1)
$2\sqrt{xy} = 2\sqrt{3} \ \ or\ \sqrt{xy} = \sqrt{3}$, squaring
$\Rightarrow xy = 3$... (2)
But $(x-y)^2 = (x+y)^2 - 4xy$
$\Rightarrow \mathrm{x} - \mathrm{y} = \sqrt{4^2 - 3} = \pm 2$
Taking the $+ve$ sign
$\Rightarrow x - y = 2$ .... (3)
Adding eqn. (1) and (3)
$2x = 6 \ \Rightarrow x = 3$
From eqn. (1) $y = 4 - x = 4 - 3 = 1$
$\therefore \sqrt{\sqrt{32} - \sqrt{24}} = 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}} = \pm 2^{\frac{1}{4}}\left(\sqrt{3} - 1\right)$

3. *If* $x = \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)}$, *show that* $x^2(x-14)^2 = 1$
Solution
$x = \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)} \Rightarrow x = \sqrt{\frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})}}$
$\Rightarrow x = \frac{7+4\sqrt{3}}{49-48} = 7 + 4\sqrt{3}$
Now $(x-14)^2 = \left(7 + 4\sqrt{3} - 14\right)^2 = \left(-7 + 4\sqrt{3}\right)^2$
$\Rightarrow \ (x-14)^2 = 49 - 56\sqrt{3} + 48$
$= 97 - 56\sqrt{3}$
Also $x^2 = \left(7 + 4\sqrt{3}\right)^2$
$= 49 + 56\sqrt{3} + 48 = 97 + 56\sqrt{3}$
$\therefore x^2(x-14)^2 = \left(97 + 56\sqrt{3}\right)\left(97 - 56\sqrt{3}\right) = 97^2 - \left(56\sqrt{3}\right)^2 = 1$

4. *If* $2x = \sqrt{a} + \frac{1}{\sqrt{a}}$, *show that* $\frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{a-1}{2}$

Solution

$2x = \sqrt{a} + \frac{1}{\sqrt{a}} \Rightarrow x = \frac{a+1}{2\sqrt{a}}$

Now $x^2 = \frac{(a+1)^2}{4a} = \frac{a^2+2a+1}{4a}$

Also $x^2 - 1 = \frac{a^2+2a+1}{4a} - 1 = \frac{a^2-2a+1}{4a} = \frac{(a-1)^2}{4a}$

$$\Rightarrow \frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{\sqrt{\left(\frac{(a-1)^2}{4a}\right)}}{\frac{a+1}{2\sqrt{a}} - \sqrt{\left(\frac{(a-1)^2}{4a}\right)}}$$

$$= \frac{\frac{a-1}{2\sqrt{a}}}{\frac{a+1}{2\sqrt{a}} - \frac{a-1}{2\sqrt{a}}} = \frac{a-1}{2\sqrt{a}} \div \frac{2a}{2\sqrt{a}} = \frac{a-1}{2}$$

Task

1. Simplify $\frac{\sqrt{3}-2}{2\sqrt{3}+3}$ $(\boldsymbol{Uneb}\ \mathbf{2007})$
2. *Express as an equivalent fraction with a rational denominator*
   $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ $(\boldsymbol{Uneb}\ \mathbf{1996})$ **ANSWER**: $\frac{6+2\sqrt{6}+2\sqrt{15}}{12}$
3. *Express* $\frac{1+\sqrt{3}}{(\sqrt{3}-1)^3}$ *in the form* $a + b\sqrt{c}$ **ANSWER**: $\frac{7}{2} + 2\sqrt{3}$
4. *Find the square root of* $14 + 6\sqrt{5}$ **ANSWER**: $\pm\left(3 + \sqrt{5}\right)$
5. *Simplify* $\frac{2\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$
6. *Find the rational numbers a and b such that*
   $3 + \sqrt{2} = \left(a + b\sqrt{2}\right)\left(6 - 2\sqrt{3}\right)$
   ANSWER: $a + b\sqrt{2} = \frac{13}{14} + \frac{3}{7}\sqrt{2} \Rightarrow a = \frac{13}{14}, b = \frac{3}{7}, c = 2$

THE THEORY OF QUADRATIC EQUATIONS

An equations of degree two is a quadratic equation and generally takes the form $\boxed{\boldsymbol{ax^2+bx+c=0}}$, where $a\,,b,c$ are rational numbers.

Types of Quadratic Equations

Are of two types $.i.e.$

a) **Pure Quadratic Equation**

Is an equation of the form $\boxed{\boldsymbol{ax^2+c=0}}$, where $b=0$ in $ax^2+bx+c=0$

b) **Affected Quadratic Equation**

Is an equation which is not a pure quadratic equation, and takes on the form, with $ab\neq 0 \Leftrightarrow a\neq 0\,,b\neq 0$ $\boxed{\boldsymbol{ax^2+bx+c=0}}$

Methods of solving a Quadratic Equation $\boldsymbol{ax^2+bx+c=0}$

I) **Method of Factorization**

If the expression $\boldsymbol{ax^2+bx+c}$ can be factored in to linear factors, then each of the factors put to zero, provides us with a root of the quadratic equation. Thus if

$$\boxed{\boldsymbol{ax^2+bx+c=a(x-\alpha)(x-\beta)}}$$

, then the roots of the equation $\boldsymbol{ax^2+bx+c=0}$ are $\boxed{\boldsymbol{x=\alpha} \text{ and } \boldsymbol{x=\beta}}$

Example

Solve:

a) $x^2-5x+6=0$
b) $2x^2-x-3=0$
c) $4x^2-x-3=0$

Solution

a) $x^2-5x+6=0$

Sum of roots$=-5$, (coefficient of middle term)
Product$=6$, (product of the constant term and coefficient of x^2)
Numbers $=(-2\,,-3)$
$\therefore x^2-2x-3x+6=0$
$\Rightarrow x(x-2)-3(x-2)=0$
$\Rightarrow (x-3)(x-2)=0$
$\therefore x-3=0\; or\; x=3$
$\therefore x-2=0\; or\; x=2$
Hence roots are 3 and 2

b) $2x^2-x-3=0$

Sum of roots$=-5$
Product$=2\times -3=-6$
Numbers $=(-3\,,2)$
$\therefore 2x^2-3x+2x-3=0$
$\Rightarrow x(2x-3)+1(2x-3)=0$
$\Rightarrow (x+1)(2x-3)=0$
$\therefore x+1=0\; or\; x=-1$

$\therefore 2x-3=0$ or $x=\frac{3}{2}$

Hence roots are -1 and $\frac{3}{2}$

c) Left as an exercise: **ANSWER**: $x=1\,;\,-\frac{3}{4}$

II) **Method of Perfect Square**

Let $\boldsymbol{ax^2+bx+c=0}$ **be given equation.**

Step 1: Divide both sides by a

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Step 2: Transpose the constant term (i.e. term independent of x) on the **R.H.S**

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Step 3: Add $\frac{b^2}{4a^2}$ to both sides

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

or $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

NOTE: $\frac{b^2}{4a^2}=\left(\frac{b}{a}\div 2\right)^2=\left(\frac{coeff\ .of\ x}{2}\right)^2$

Step 4: Find the square root on both sides

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Now solving for x

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. This is a quadratic formula for solving quadratic equations.

$$\boxed{\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}\textbf{, for } \boldsymbol{ax^2+bx+c=0}}$$

Example

Solve:

a) $x^2-5x+6=0$
b) $2x^2-x-3=0$

Solution

By quadratic formula

a) $x^2-5x+6=0 \Rightarrow a=1, b=-5\,, c=6$

$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)}=\frac{5\pm1}{2}$

$\therefore x=\frac{5+1}{2}=3$

$\therefore x=\frac{5-1}{2}=2$

$\therefore$ Roots are 3 and 2

b) $2x^2-x-3=0 \Rightarrow a=2\,, b=-1\,, c=-3$

$\therefore x=\frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-3)}}{2(2)}=\frac{1\pm5}{2}$

$\therefore x=\frac{1+5}{4}=\frac{3}{2}$

$\therefore x=\frac{1-5}{2}=-2$

$\therefore$ Roots are $\frac{3}{2}$ and -2

Nature of Roots of a Quadratic Equation

The equation $\boldsymbol{ax^2+bx+c=0}$ has roots $\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

Nature of roots depend on $\boldsymbol{(b^2-4ac)}$ inside the radical, called **DISCRIMINANT**

Let the roots be α and β

Let $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

Deduction

Case I:

If b^2-4ac is *positive*. *i.e.* $b^2-4ac>0$, α and β are *real and unequal or real and distinct*

Case II:

If b^2-4ac is zero. *i.e.* $b^2-4ac=0$, α and β are *real and equal* each reducing in this case to $-\frac{b}{2a}$ or *roots are coincident/ double/ repeated.*

Case III:

If b^2-4ac is *negative* . *i.e.* $b^2-4ac<0$, α and β are *imaginary and unequal*

Case IV:

If b^2-4ac is a *perfect square,* α and β are *rational and unequal*

By applying these tests the nature of roots of any quadratic equation may be determined without solving.

Graphical Interpretation of $\boldsymbol{y=ax^2+bx+c}$

The graph $y=ax^2+bx+c$ is a parabola with vertical axis of symmetry.

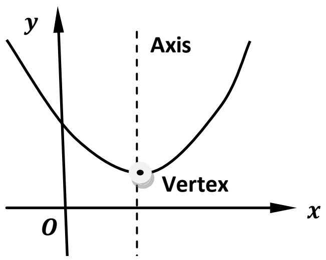


Fig. $y=ax^2+bx+c, a>0$
(Opens upwards and has a minimum values)

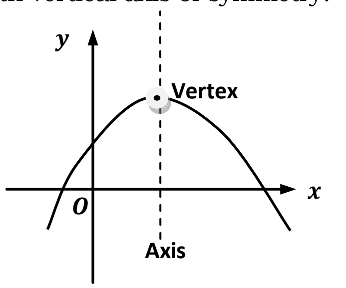
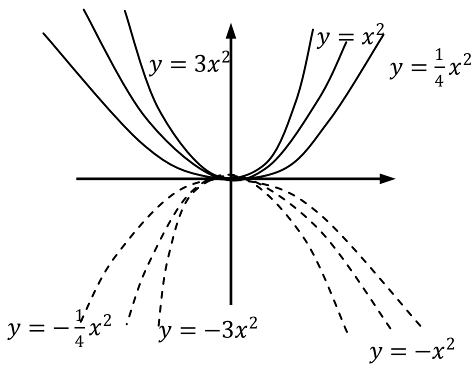


Fig. $y=ax^2+bx+c, a<0$
(Opens down wards and has a minimum values)



Note: The bigger $|a|$ the narrower the parabola.

For an equation $ax^2 + bx + c = 0$

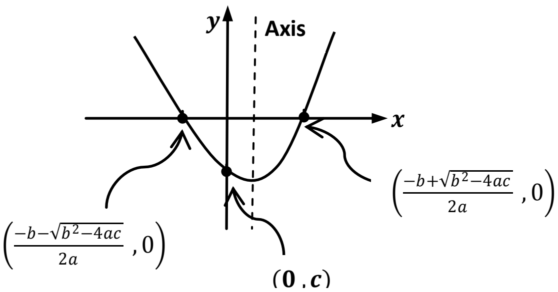


Fig I. $\boldsymbol{b^2 - 4ac} > 0$ There are two intercepts

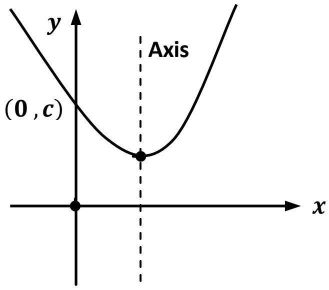


Fig II. $\boldsymbol{b^2 - 4ac} < 0$. No $x-$ intercepts

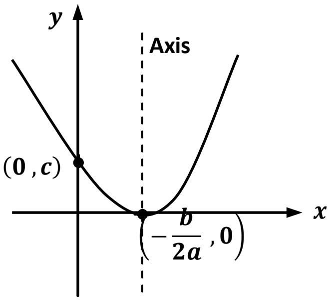


Fig III. $\boldsymbol{b^2 - 4ac = 0}$. There is one intercept. (at the point where the parabola meets the x –axis and $x - axis$ in a

Illustrative examples on deduced cases on roots of a quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$

1. *Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real value of x.*

   Solution

   From discriminate, $b^2 - 4ac, a = 2, b = -6, c = 7$

   $\Rightarrow (-6)^2 - 4(2)(7) = -20$. Thus roots are imaginary.

2. *If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.*

   Solution

   For equal roots $b^2 - 4ac = 0, a = 1, b = 2(k+2), c = 9k$

   $\Rightarrow [2(k+2)]^2 - 4(1)(9k) = 0$ ....÷ (4)

   $\Rightarrow (k+2)^2 - 9k = 0$

   $\Rightarrow k^2 + 4k + 4 - 9k = 0$ *or* $k^2 - 5k + 4 = 0$

   $\Rightarrow (k-1)(k-4) = 0 \therefore k = 1, 4$

3. *Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.*

   Solution

   For rational roots, $b^2 - 4ac$ is a perfect square

   $a = 1, b = -2p, c = p^2 - q^2 + 2qr - r^2$

   $\Rightarrow b^2 - 4ac = (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$

   $= 4p^2 - 4p^2 + 4(q^2 + 2qr + r^2) = 4(q^2 + 2qr + r^2)$

   $= 4(q-r)^2 = [2(q-r)]^2$, hence roots are rational.

4. *Prove that the roots of the equation* $x^2 - 2ax + a^2 - b^2 - c^2 = 0$ *are real.*
   Solution
   For real roots, $b^2 - 4ac \geq 0$
   Note: for real and distinct roots, $b^2 - 4ac > 0$
   $\Rightarrow (-2a)^2 - 4(1)(a^2 - b^2 - c^2) \geq 0$
   $\Rightarrow 4a^2 - 4a^2 + 4(b^2 + c^2) \geq 0$
   $\Rightarrow 4(b^2 + c^2) \geq 0$
   R.H.S will always be true for any value of b and c.
5. *Prove that the roots of the equation*$(a - b - c)x^2 + ax + b + c = 0$ *are real if* a, b, c *are real*
   Solution
   For real roots $B^2 - 4AC \geq 0$
   $\Rightarrow a^2 - 4(a - b - c)(b + c) \geq 0$
   Let $b + c = n$
   $\Rightarrow a^2 - 4(a - n)n \geq 0$
   $\Rightarrow a^2 - 4an + 4n^2 \geq 0$
   $\Rightarrow (a - 2n)^2 \geq 0$
   $\therefore [a - 2(b + c)]^2 \geq 0$
   This inequality will be satisfied if a , b and c are real, hence a , b and c are real.

Task

1. *Prove that the roots of the equations are real rational*
   a) $(a + c - b)x^2 + 2cx + (b + c - a = 0)$
   b) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$
2. *Prove that the roots of the equation* $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0$ *are real*
3. *For what values of m will the equations*
   a) $x^2 - 15 - m(2x - 8) = 0$
   b) $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$, have equal roots?

ANSWER: $a)$ **3 , 5** $b)$ $\mathbf{2 , -\frac{10}{9}}$

Relationship between roots of a quadratic equation $.i.e. \boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$ **and its coefficients in** $\boldsymbol{ax^2 + bx + c = 0}$

From $ax^2 + bx + c = 0$, with roots α and β,

$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$ $\left[\because x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$

Then $\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}$

$$\boxed{\boldsymbol{\alpha + \beta = -\frac{b}{a}}}$$

Also $\alpha.\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$

$= \frac{(-b+\sqrt{b^2-4ac}).(-b-\sqrt{b^2-4ac})}{4a^2}$

$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$

$= \frac{4ac}{4a^2} = \frac{c}{a}$

$$\boxed{\boldsymbol{\alpha.\beta = \frac{c}{a}}}$$

From $ax^2 + bx + c = 0$, re writing in the form

$$\boxed{\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}}$$

From

$\alpha + \beta = -\frac{b}{a}$ *or* $\frac{b}{a} = -(\alpha + \beta)$ and $\alpha.\beta = \frac{c}{a}$

∴ the quadratic equation becomes

$$\boxed{\boldsymbol{x^2 - x(\alpha + \beta) + \alpha.\beta = 0}}$$

In general, if a quadratic equation $ax^2 + bx + c = 0$ can be written with coefficient of the first term as unity $\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}$, then

a) Sum of roots = coefficient of x with its sign changed
b) Product of roots = the third term

Note:

Re writing $\boldsymbol{x^2 - x(\alpha + \beta) + \alpha.\beta = 0}$ by factorization,

$\boxed{\boldsymbol{(x - \alpha).(x - \beta) = 0}}$, is a quadratic equation with coefficient of $\boldsymbol{x^2}$ being unity.

In general, the quadratic equation with given roots is

$$\boxed{\boldsymbol{x^2} - (sum\ of\ roots)\boldsymbol{x} + (product\ of\ roots) = \boldsymbol{0}}$$

Illustrative examples

1. *Form the equation whose roots are;*
   a) 3 and -2
   b) $2 + \sqrt{3}$ and $2 - \sqrt{3}$

Solution

Required equation is $\boldsymbol{x^2} - (sum\ of\ roots)\boldsymbol{x} + (product\ of\ roots) = \boldsymbol{0}$

a) Sum of roots $= 3 + \ -2 = 1$

Product of roots $= 2(-3) = -6$

∴ Required equations is $x^2 - x - 6 = 0$

b) Sum of roots $= \left(2 + \sqrt{3}\right) + \ \left(2 - \sqrt{3}\right) = 4$

Product of roots $= \left(2 + \sqrt{3}\right).\left(2 - \sqrt{3}\right) = 2^2 - \left(\sqrt{3}\right)^2 = 1$

∴ Required equations is $x^2 - 4x + 1 = 0$

2. *If α and β are the roots of the equation $x^2 - px + q = 0$, find;*
   i) Values of $\alpha^2 + \beta^2$
   ii) Values of $\alpha^3 + \beta^3$
   iii) Values of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
   iv) Value of $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$

Solution

$x^2 - px + q = 0\ , \alpha\ , \beta$ are roots

$\Rightarrow \alpha + \beta = -(-p) = p\ [i.e.\ sign\ of\ coeff.of\ x\ changed]$

$\Rightarrow \alpha.\beta = q$

i) From $(\alpha+\beta)^2 = \alpha^2+\beta^2+2\,\alpha.\beta$

$\therefore \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\,\alpha.\beta$

$= p^2 - 2q$

ii) From $(\alpha+\beta)^3 = \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\,\alpha.\beta(\alpha+\beta)$

$= p^3 - 3pq$

iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2.\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{p^2-2q}{q^2}$

iv) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^2}{\beta^2} - 2.\frac{\alpha}{\beta}.\frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2}$

$= \frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2} - 2 = \frac{\alpha^4+\beta^4}{(\alpha\beta)^2} - 2$

From $(\alpha+\beta)^4 = \alpha^4+4\alpha^3\beta+6\alpha^2\beta^2+4\alpha\beta^3+\beta^4$

$\therefore \alpha^4+\beta^4 = (\alpha+\beta)^4 - (4\alpha^3\beta+6\alpha^2\beta^2+4\alpha\beta^3)$

$= (\alpha+\beta)^4 - [4\alpha.\beta(\alpha^2+\beta^2)+6(\alpha\beta)^2]$

$= p^4 - [4q(p^2-2q)+6q^2]$

$= p^4 - 4q(p^2-2q) - 6q^2$

$= p^4 - 4qp^2 + 8q^2 - 6q^2$

$= p^4 - 4qp^2 + 2q^2$

3. *If α and β are the roots of the equation $x^2-7x+2=0$, evaluate;*

   i) $\alpha^2+\beta^2$

   ii) $\frac{2-\alpha}{3+\beta}+\frac{2-\beta}{3+\alpha}$

Solution

$x^2-7x+2=0$, with roots α and β, then $\alpha+\beta=7$,$\alpha.\beta=2$

i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\,\alpha.\beta$

$= 7^2 - 2(2) = 45$

ii) $\frac{2-\alpha}{3+\beta}+\frac{2-\beta}{3+\alpha} = \frac{(2-\alpha)(3+\beta)+(2-\beta)(3+\beta)}{(3+\beta)(3+\alpha)}$

$= \frac{6+2\alpha-3\alpha-\alpha^2+6+2\beta-3\beta-\beta^2}{9+3\alpha+3\beta+\alpha\beta}$

$= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-(\alpha^2+\beta^2)}{9+3(\alpha+\beta)+\alpha\beta}$

$= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-[(\alpha+\beta)^2-2\,\alpha.\beta\,]}{9+3(\alpha+\beta)+\alpha\beta}$

$= \frac{12+14-21-(7^2-2(2))}{9+21+2} = -\frac{28}{32} = -\frac{7}{8}$

4. *If α and β are the roots of the equation $x^2-px+q=0$, form an equation whose roots are $\alpha\beta+\alpha+\beta$ and $\alpha\beta-\alpha-\beta$*

Solution

Required equation is $\boldsymbol{x^2} - (sum\ of\ roots)\boldsymbol{x} + (product\ of\ roots) = \boldsymbol{0}$

From $x^2-px+q=0$ with roots α *and* β,

$\alpha+\beta = -(-p) = p$, $\alpha.\beta = q$

Sum of roots = $\alpha\beta+\alpha+\beta+\ \alpha\beta-\alpha-\beta = 2\alpha\beta = 2q$

Product of roots = $(\,\alpha\beta+\alpha+\beta).(\,\alpha\beta-\alpha-\beta)$

$= (q+p)+(q-p) = q^2-p^2$

$\therefore$ Equation is $x^2-2qx+q^2-p^2=0$

5. *If α and β are the roots of the equation $2x^2-4x+1=0$, form an equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$*

Solution
From $2x^2-4x+1=0$ with roots α and β,
$\Rightarrow \alpha+\beta=-\left(-\frac{4}{2}\right)=2,\ \alpha.\beta=\frac{1}{2}$
Required equation is $\boldsymbol{x^2}-(sum\ of\ roots)\boldsymbol{x}+(product\ of\ roots)=\mathbf{0}$
Sum of roots $=\ \alpha^2+\beta+\ \beta^2+\alpha=\alpha^2+\beta^2+\alpha+\beta$
$=(\alpha+\beta)^2-2\,\alpha.\beta+\alpha+\beta$
$=2^2-2.\frac{1}{2}+2=4-1+2=5$
Product of roots $=(\ \alpha^2+\beta).(\beta^2+\alpha)$
$=\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta$
$=(\alpha\beta)^2+\alpha\beta+(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
$=\frac{1}{4}+2+8-3.\frac{1}{2}.2=\frac{23}{4}$
$\therefore x^2-5x+\frac{23}{4}=0\ \ or\ \ 4x^2-20x+23=0$

6. *If α and β are the roots of the equation $ax^2+bx+c=0$, find an equation whose roots are $a\alpha+b\beta$ and $b\alpha+a\beta$*
Solution
Sum of roots $=\ a\alpha+b\beta+\ b\alpha+a\beta$
$=\alpha(a+b)+\beta(a+b)=(\alpha+\beta)(a+b)$
$=-\frac{b}{a}\times(a+b)=\frac{-b(a+b)}{a}$
Product of roots $=(\ a\alpha+b\beta).(\ b\alpha+a\beta)$
$=ab\alpha^2+a^2\beta\alpha+b^2\alpha\beta+ab\beta^2$
$=ab(\alpha^2+\beta^2)+\alpha\beta(a^2+b^2)$
$=ab[(\alpha+\beta)^2-2\,\alpha.\beta]+\alpha\beta(a^2+b^2)$
$=ab\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)+\frac{c(a^2+b^2)}{a}$
$=\frac{b(b^2-2ac)+c(a^2+b^2)}{a}$
Hence the required equation is
$x^2-\frac{b(a+b)}{a}x+\frac{b(b^2-2ac)+c(a^2+b^2)}{a}=0$
$or\ \ ax^2-b(a+b)x+b(b^2-2ac)+c(a^2+b^2)=0$
$or\ \ ax^2-bax-b^2x+b^3-2abc+a^2c+b^2c=0$
$or\ \ ax(x-b)-b^2(x-b)+c(a^2-2ab+b^2)=0$
$or\ \ (x-b)(ax-b^2)+c(a-b)^2=0$

7. *The roots of a quadratic equation $x^2-px+q=0$ are α and β. Determine the equation whose roots are $\alpha^2+\beta^{-2}$ and $\beta^2+\alpha^{-2}$ expressing the coefficients in terms of p and q. prove further that if p and q are both real, then this equation can have roots if $p=0$ or $p^2=4q$.*
Solution
Required equation: $x^2-(A+B)x+A.B=0$
Let $A=\alpha^2+\beta^{-2}=\alpha^2+\frac{1}{\beta^2}$ and $B=\beta^2+\alpha^{-2}=\beta^2+\frac{1}{\alpha^2}$
$A+B=\alpha^2+\frac{1}{\beta^2}+\beta^2+\frac{1}{\alpha^2}$
$=(\alpha^2+\beta^2)+\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$$= (\alpha+\beta)^2 - 2\alpha\beta + \frac{\alpha^2+\beta^2}{\alpha^2.\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta + \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha.\beta)^2}}$$

But from the equation $x^2 - px + q = 0$, $\alpha\ and\ \beta$ are roots then $\alpha + \beta = p\,, q = \alpha\beta$

$\therefore A + B = p^2 - 2q + \frac{p^2-2q}{q^2}$

$= (p^2 - 2q)\left(1 + \frac{1}{q^2}\right) = \frac{(p^2-2q)(q^2+1)}{q^2}$

$A.B = (\alpha^2 + \beta^2).\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$

$= \alpha^2.\beta^2 + 1 + 1 + \frac{1}{\alpha^2.\beta^2}$

$= (\alpha.\beta)^2 + 2 + \frac{1}{(\alpha.\beta)^2}$

$= q^2 + 2 + \frac{1}{q^2} = \frac{q^4+2q^2+1}{q^2}$

$= \frac{(q^2+1)^2}{q^2}$

Required equation is; $x^2 - \frac{(p^2-2q)(q^2+1)}{q^2}x + \frac{(q^2+1)^2}{q^2} = 0$

If p and q are real then x must also be real $.i.e.$

$\left[-\frac{(p^2-2q)(q^2+1)}{q^2}\right]^2 \geq 4\frac{(q^2+1)^2}{q^2} \Rightarrow \frac{(p^2-2q)^2(q^2+1)^2}{q^4} \geq 4\frac{(q^2+1)^2}{q^2}$

Dividing both sides by $\frac{(q^2+1)^2}{q^4}$

$\Rightarrow (p^2 - 2q)^2 \geq 4q^2\ \ or\ \ (p^2 - 2q)^2 - 4q^2 \geq 0$

By factoring using a difference of two squares identity,

$\Rightarrow (p^2 - 2q + 2q)(p^2 - 2q - 2q) \geq 0\ \ or\ p^2(p^2 - 4q) \geq 0$

$\Rightarrow p^2 \geq 0\ \ or\ p^2 - 4q \geq 0$

$\therefore p = 0\ \ or\ \ p = 4q$

8. *If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $p{:}\,q$, prove that $ac(p+q)^2 = b^2pq$*

Solution

Let the roots of the equation be $\alpha\ and\ \beta$

$\Rightarrow \alpha + \beta = -\frac{b}{a}\ \ , \alpha\beta = \frac{c}{a}$ .... (1)

But $\alpha{:}\beta = \frac{p}{q}\ \ or\ \frac{\alpha}{\beta} = \frac{p}{q}$

$\therefore \alpha = \frac{p}{q}\beta$ .... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow \frac{p}{q}\beta + \beta = -\frac{b}{a}$, also $\beta.\frac{p}{q}\beta\ = \frac{c}{a}$

$\therefore \beta\left(\frac{p}{q} + 1\right) = -\frac{b}{a}\ \ \ or\ \beta = -\frac{bq}{a(p+q)}\ \ , \beta^2 = \frac{qc}{ap}$

Solving these equations

$\Rightarrow \left[-\frac{bq}{a(p+q)}\right]^2 = \frac{qc}{ap}\ \ \ or\ \ \frac{b^2q^2}{a^2(p+q)^2} = \frac{qc}{ap}$

$\therefore b^2q^2.ap = a^2(p+q)^2.qc$, simplifying

$\Rightarrow ac(p+q)^2 = b^2pq$

Task

a) *If $\alpha\ and\ \beta$ are the roots of the equation $ax^2 + bx + c = 0$, and r is the ratio of the roots, show that $ac(r+q)^2 = b^2r$*

b) *If the roots of the equation* $lx^2 + nx + n = 0$ *are in the ratio* $p:q$ *prove that* $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$

9. *If the roots of the equation* $ax^2 + 2bx + c = 0$ *are* α *and* β *and those of the equation* $Ax^2 + 2Bx + C = 0$ *are* $\alpha + \delta$ *and* $\beta + \delta$, *prove that* $\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$

Solution

Given: $ax^2 + 2bx + c = 0$, α , β are roots, then

$$\alpha + \beta = -\frac{2b}{a} \;\; , \alpha\beta = \frac{c}{a} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1)$$

Given: $Ax^2 + 2Bx + C = 0$, α + δ , β + δ are roots, then

$$\alpha + \delta + \beta + \delta = -\frac{2B}{A}$$

$$\Rightarrow \alpha + \beta + 2\delta = -\frac{2B}{A}$$

$Or\ 2\delta = -\frac{2B}{A} - (\alpha + \beta)$, from eqn. (1)

$\Rightarrow 2\delta = -\frac{2B}{A} - \left(-\frac{2b}{a}\right)$

$\therefore \delta = \frac{b}{a} - \frac{B}{A} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2)$

Also $(\alpha + \delta)(\beta + \delta) = \frac{C}{A}$

$\Rightarrow \alpha\beta + \alpha\delta + \beta\delta + \delta^2 = \frac{C}{A} \quad or \;\; \alpha\beta + \delta(\alpha + \beta) + \delta^2 = \frac{C}{A}$

$\Rightarrow \frac{c}{a} + \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A}$

$\therefore \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A} - \frac{c}{a} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3)$

Eqn. (2) in to eqn. (3)

$\Rightarrow \left(\frac{b}{a} - \frac{B}{A}\right)\left(-\frac{2b}{a}\right) + \left(\frac{b}{a} - \frac{B}{A}\right)^2 = \frac{C}{A} - \frac{c}{a}$

Let $\frac{b}{a} = p \;\; \frac{B}{A} = q$

$\Rightarrow (p - q)(-2p) + (p - q)^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow -2p^2 + 2pq + p^2 - 2pq + q^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow q^2 - p^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow \frac{B^2}{A^2} - \frac{b^2}{a^2} = \frac{C}{A} - \frac{c}{a} \quad or \quad \frac{B^2}{A^2} - \frac{C}{A} = \frac{b^2}{a^2} - \frac{c}{a}$

$\therefore \frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$ as required.

To find the condition that the roots of the equation $ax^2 + bx + c = 0$ **should be;**

i) Equal in magnitude and opposite in sign
ii) Reciprocals
iii) With one of the roots as zero
iv) Both positive

i) The roots will be *equal in magnitude* and *opposite in sign* if their *sum is zero.* Hence the required condition is:
$\alpha + \beta = -\frac{b}{a} = 0 \;\; or \;\; \boxed{\boldsymbol{b = 0}}$, where α *and* β are roots of the equation.

ii) The roots will be *reciprocals* if their *product is unity.* Hence the required condition is:
$\alpha\beta = \frac{c}{a} = 1 \;\; or \;\; \boxed{\boldsymbol{c = a}}$

iii) When one of the roots is zero then the product must be zero. Hence the required condition is:
$\alpha\beta = \frac{c}{a} = 0$ *or* $\boxed{\boldsymbol{c = 0}}$

iv) The roots will be both positive if;
   a) The *product* of the roots is *positive* $.i.e. \alpha\beta = \frac{c}{a} = +ve$
   This is only true if c and a are of *like signs*
   b) The *sum* of the roots is also *positive* $.i.e. \alpha + \beta = -\frac{b}{a} = +ve$
   This is possible if b and a are of *unlike signs*. Thus the required condition is that the signs of a and c are alike and opposite to that of b.

10. *The roots of the equation* $x^2 + ax + b = 0$ *are* α *and* β. *Find the equation whose roots are* $p\alpha + q\beta$ *and* $p\beta + q\alpha$.
If the original equation is $x^2 - 4x - 5 = 0$, *find the values of* $\frac{p}{q}$ *in order that the new equation shall have one zero root.*

Solution

Required equation: $x^2 - (A + B)x + A.B = 0$
Let $A = p\alpha + q\beta$ and $B = p\beta + q\alpha$
$A + B = p\alpha + q\beta + p\beta + q\alpha$
$\quad = \alpha(p + q) + \beta(p + q) = (p + q)(\alpha + \beta)$
But $\alpha + \beta = -a$, $\alpha\beta = b$
$\therefore A + B = -a(p + q)$
$A.B = (p\alpha + q\beta)(p\beta + q\alpha)$
$\quad = p^2\alpha\beta + pq\alpha^2 + pq\beta^2 + q^2\alpha\beta$
$\quad = (p^2 + q^2)\alpha\beta + pq(\alpha^2 + \beta^2)$
$\quad = (p^2 + q^2).b + pq[(\alpha + \beta)^2 - 2\alpha\beta]$
$\quad = (p^2 + q^2).b + pq(a^2 - 2b)$
$\therefore$ Required equation is: $x^2 + a(p + q)x + (p^2 + q^2).b + pq(a^2 - 2b) = 0$
Now to find the values of $\frac{p}{q}$ in original equation $x^2 - 4x - 5 = 0$, so that the new equation shall have one zero root,
Comparing the equations $x^2 - 4x - 5 = 0$ and $x^2 + ax + b = 0$,
$\Rightarrow a = -4, b = -5$
Substituting these values in the obtained equation gives;
$x^2 - 4(p + q)x - 5(p^2 + q^2) + pq((-4)^2 - 2 \times -5) = 0$
$\Rightarrow x^2 - 4(p + q)x - 5(p^2 + q^2) + 26pq = 0$
This equation will have one zero root if the product is zero
$\Rightarrow -5(p^2 + q^2) + 26pq = 0$ *or* $5p^2 - 26pq + 5q^2 = 0$
Factoring
$\Rightarrow 5p^2 - 25pq - pq + 5q^2 = 0$
$\Rightarrow 5p(p - 5q) - q(p - 5q) = 0$
$\Rightarrow (p - 5q)(5p - q) = 0$
$\therefore p - 5q = 0$ *or* $\frac{p}{q} = 5$
$\therefore 5p - q = 0$ *or* $\frac{p}{q} = \frac{1}{5}$
$\therefore \frac{p}{q} = 5, \frac{1}{5}$

Other examples on quadratic equation

11. *If α and β are roots of the equation $ax^2 + 2bx + c = 0$ while δ and γ are the roots of the equation $a_1x^2 + 2b_1x + c_1 = 0$, show that if $(\alpha - \gamma)(\beta - \delta) + (\alpha - \delta)(\beta - \gamma) = 0$, then $ac_1 + a_1c = 2bb_1$.*

Solution

From $ax^2 + 2bx + c = 0$, $\alpha + \beta = -\frac{2b}{a}$, $\alpha\beta = \frac{c}{a}$

From $a_1x^2 + 2b_1x + c_1 = 0$, $\gamma + \delta = -\frac{2b_1}{a_1}$, $\gamma\delta = \frac{c_1}{a_1}$

$(\alpha - \gamma)(\beta - \delta) + (\alpha - \delta)(\beta - \gamma) = 0$

$\Rightarrow \alpha\beta - \alpha\delta - \gamma\beta + \gamma\delta + \alpha\beta - \alpha\gamma - \delta\beta + \gamma\delta = 0$

$\Rightarrow 2\alpha\beta + 2\gamma\delta - \delta(\alpha + \beta) - \gamma(\delta + \gamma) = 0$

$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} - (\alpha + \beta)(\delta + \gamma) = 0$

$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} + \frac{2b}{a}.-\frac{2b_1}{a_1} = 0$

$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} - \frac{4bb_1}{aa_1} = 0$ ....$\left(\times \frac{aa_1}{2}\right)$

$\Rightarrow a_1c + ac_1 - 2bb_1 = 0$

$\therefore ac_1 + a_1c = 2bb_1$

12. *If x is a real number, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ doe not lie strictly between 2 and 6.*

Solution

Let $y = \frac{x^2+2x-11}{2(x-3)}$

$\Rightarrow 2yx - 6y = x^2 + 2x - 11$ *or* $x^2 + (2 - 2y)x + 6y - 11 = 0$

$\Rightarrow x = \frac{-(2-2y)\pm\sqrt{[-(2-2y)]^2-4(6y-11)}}{2}$

$= \frac{-(2-2y)\pm\sqrt{4-8y+4y^2-24+44}}{2}$

$= \frac{-(2-2y)\pm\sqrt{4y^2-32y+48}}{2}$

x will be real if $4y^2 - 32y + 48 \geq 0$ *or* $y^2 - 8y + 12 \geq 0$

$\Rightarrow (y - 2)(y - 6) \geq 0$

Between 2 and 6 $.i.e. 2 < y < 6$, one of the factors on L.H.S will be $+ve$ and the other $-ve$, so the product will be negative. Hence $\frac{x^2+2x-11}{2(x-3)}$ will not lie between 2 and 6

13. *If x is a real number, prove that the expression $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$and 1.*

Solution

Let $y = \frac{x}{x^2-5x+9}$ *or* $yx^2 + (-5y - 1)x + 9y = 0$

$\Rightarrow x = \frac{(5y+1)\pm\sqrt{(5y+1)^2-36y^2}}{2y}$

For real values of x, $(5y + 1)^2 - 36y^2 \geq 0$

$\Rightarrow 25y^2 + 10y + 1 - 36y^2 \geq 0$

$\Rightarrow -11y^2 + 10y + 1 \geq 0$ *or* $11y^2 - 10y - 1 \leq 0$

$\Rightarrow 11y^2 - 11y + y - 1 \leq 0$

$\Rightarrow 11y(y - 1) + (y - 1) \leq 0$

$\Rightarrow (11y + 1)(y - 1 \leq 0)$

Between $-\frac{1}{11} < y < 1$, the inequality is true for any value of y. Hence $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ *and* 1.

14. *For real x, $f(x) = \frac{x^2-k}{x-2}$ can take any real value. Find the range of values of k can take.*

Solution

Let $\frac{x^2-k}{x-2} = y$ *or* $yx - 2y = x^2 - k$

$\therefore x^2 - yx + 2y - k = 0$

$\Rightarrow x = \frac{y \pm \sqrt{y^2-4(2y-k)}}{2}$

For real value of x, $y^2 - 4(2y-k) \geq 0$

$\Rightarrow y^2 - 8y + 4k \geq 0$

$\Rightarrow y^2 - 8y \geq -4k$, completing squares,

$\Rightarrow y^2 - 8y + 16 \geq -4k + 16$

$\therefore (y-4)^2 \geq 16 - 4k$

Since L.H.S is always $+ve$, taking its smallest value say 0 . $i.e.$ $y = 4$, then

$0 \geq 16 - 4k \ \Rightarrow k \geq 4$

$\therefore k \geq 4$

15. *Find the range of values of k can take given that for real x, $f(x) = \frac{x^2+3x}{x+k}$ can take any real value.*

Solution

Let $\frac{x^2+3x}{x+k} = y$ *or* $yx + ky = x^2 + 3x$

$\therefore x^2 + (3-y)x - ky = 0$

$\Rightarrow x = \frac{-(3-y) \pm \sqrt{(3-y)^2+4ky}}{2}$

$= \frac{(y-3) \pm \sqrt{(3-y)^2+4ky}}{2}$

For real values of x, $(3-y)^2 + 4ky \geq 0$

$\Rightarrow 9 - 6y + y^2 + 4ky \geq 0$

$\Rightarrow y^2 + (4k-6)y + 9 \geq 0$

$\Rightarrow y^2 + (4k-6)y \geq -9$, completing squares

$\Rightarrow y^2 + (4k-6)y + (2k-3)^2 \geq -9 + (2k-3)^2$

$\therefore \left(y + (2k-3)\right)^2 \geq -9 + (2k-3)^2$

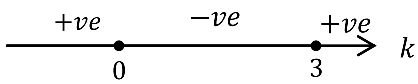
Since y can take on any real value, let its smallest value be $-(2k-3)^2$

$\Rightarrow 0 \geq -9 + (2k-3)^2$

$\Rightarrow 4k^2 - 12k + 9 - 9 \leq 0$

$\therefore 4k^2 - 12k \leq 0$ *or* $k^2 - 3k \leq 0$

For $0 \leq k \leq 3$, the inequality is true.



16. *Given that $y = \frac{3x+k}{x^2-1}$, where x is real and k is a constant, show that y can take all real values if $|k| < 3$.*

Solution

$y = \frac{3x+k}{x^2-1}$ *or* $yx^2 - y = 3x + k$

$\therefore yx^2 - 3x - y - k = 0$

For real values of x, $(-3)^2 \geq 4y(-y-k)$

$\Rightarrow 9 + 4y^2 + 4ky \geq 0$

$\Rightarrow 4y^2 + 4ky \geq -9 \quad or \quad y^2 + ky \geq -\frac{9}{4}$, completing squares,

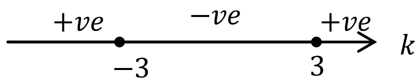
$\Rightarrow y^2 + ky + \frac{k^2}{4} \geq -\frac{9}{4} + \frac{k^2}{4}$

$\therefore \left(y + \frac{k}{2}\right)^2 \geq \frac{k^2-9}{4}$

Since y can take on any real value, let its smallest value be $-\frac{k}{2}$

$\Rightarrow 0 \geq \frac{k^2-9}{4} \quad or \quad k^2 - 9 \leq 0$

$\therefore (k+3)(k-3) \leq 0$



For interval $-3 \leq k \leq 3$, the inequality holds. Thus $|k| < 3$ is the required interval.

Task

1. *If α and β are the roots of the equation $ax^2 + bx + c = 0$, find the value of;*
   a) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **ANSWER**: $\frac{b^2-2ac}{ac}$
   b) $\alpha^4 + \beta^4$ **ANSWER**: $\frac{b^4-4ab^2c+2a^2c^2}{a^4}$
   c) $(\alpha^2 - \beta)^2 + (\beta^2 - \alpha)^2$ **ANSWER**: $\frac{b^4-4ab^2c+2a^2c^2+a^2b^2-2a^3c+2a^2bc}{a^4}$
   d) $\alpha^4\beta^7 + \alpha^7\beta^4$ **ANSWER**: $\frac{\boldsymbol{b}c^4(3ac-b^2)}{\boldsymbol{a^7}}$
2. *The roots of the equation are α and β*
   i) Show that if $\alpha - \beta = 1$, then $a^2 = 4(b^2 - ac)$
   ii) Find the quadratic equation whose roots are $\alpha + \alpha\beta$ and β
3. *If α and β are the roots of the equation $x^2 - px + q = 0$, show that* $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2$
4. *If p, q are the roots of the equation $3x^2 + 6x + 2 = 0$, form an equation whose roots are $-\frac{p^2}{q}$ and $-\frac{q^2}{p}$.* **ANSWER**: $3x^2 - 18x + 2 = 0$
5. *If α and β are the roots of the equation $ax^2 + bx + c = 0$, form an equation whose roots are;*
   i) $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$ **ANSWER**: $(a - b + c)x^2 - 2x(a - c) + a + b + c = 0$
   ii) $\alpha^2 + \beta$, $\alpha^{-2} + \beta^{-2}$
   ANSWER: $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$
6. *If the roots of the equation $ax^2 + 8(b - a)x + 4(4a - 8b + c) = 0$ are $(4 - 2\alpha)$ and $(4 - 2\beta)$, find the equation whose roots are α and β.*
   ANSWER: $x^2 - \frac{4b}{a}x + \frac{c}{a} = 0$
7. *If $(a + b + c)x^2 + x(b + 2c) + c = 0$ has roots α and β, find the equation whose roots are $\frac{\alpha}{\alpha+1}$, $\frac{\beta}{\beta+1}$ expressing the coefficients in terms of a, b and c.*
   ANSWER: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
8. *Given that the roots of the equation $x + \frac{1}{x} = 4$ are α and β, evaluate*
   i) $(\alpha - \beta)^2$
   ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

iii) $\alpha^3 + \beta^3$
iv) $2\alpha^2 + 2\beta^2 + 5\alpha\beta$

9. *Given that the roots of the equation* $ax^2 + bx + c = 0$ *are* β *and* $n\beta$*, show that* $(n+1)^2ac = nb^2$

10. *Given that* α *and* β *are the roots of the equation* $x^2 - px + q = 0$*, obtain the quadratic equation whose roots are* α^{-2}*,* β^{-2}*,*
ANSWER: $q^2x^2 - (p^2 - 2p)x + 1 = 0$

11. *If the roots of a quadratic equation* $x^2 + 2bx + c = 0$ *differ by 4, show that* $8a = -c \pm \sqrt{(c^2 + 16b^2)}$ *. Hence find two values of a for which the equation* $ax^2 + 2x + 3 = 0$ *has roots differing by 4.* **ANSWER**: $a = -1, \frac{1}{4}$

12. *The roots of the equation* $9x^2 + 6x + 1 = 4kx$ *where k is a real constant, are denoted as* α *and* β*.*
    a) *Show that the equation whose roots are* α^{-1} *and* β^{-1} *is* $x^2 + 6x + 9 = 4kx$
    b) *Find the set of values of k for which* α *and* β *are real.*
    ANSWER: $k \le 0, k \ge 3$
    c) *Find also the set of values of k for which* α *and* β *are real and positive.* **ANSWER**: $k \ge 3$

13. *If* α *and* β *are the roots of the equation* $x^2 + bx + c = 0$*, express* $(\alpha - \beta^2)(\alpha^2 - \beta)$ *in terms of b and c. hence or otherwise show that if one root of the equation is the square of the other, then* $b^3 + c^3 + c = 3bc$

14. *Given that the roots of the equation* $x^2 + px + q = 0$ *has roots* α *and* β*, express p and q in terms of* α *and* β*. Find the equation, the coefficients expressed in terms of p and q, whose roots are* $3\alpha - \beta$ *and* $3\beta - \alpha$*.*
ANSWER: $p = -\alpha - \beta$, $q = \alpha\beta$, $x^2 + 2px + 16q - 3q^2 = 0$

15. *Given that the roots of the equation* $x^2 + px + q = 0$ *are* α *and* β*, express* $(\alpha - 2\beta)(\beta - 2\alpha)$ *in terms of p and q. hence/otherwise show that the condition for one root of the equation to be double the other is* $2p^2 = 9q$*.*
ANSWER: $9q - 2p^2$

16. *Given that* α *and* β *are the roots of the equation* $2x^2 - 7x - 17 = 0$*, show that the equation with roots* $(\alpha - 4), (\beta - 4)$ *is* $2x^2 + 9x - 13 = 0$*.*

17. *Given that* α *and* β *are the roots of the equation* $x^2 - px + q = 0$*, prove that* $\alpha + \beta = p$ *,* $\alpha\beta = q$*.Prove also that;*
    a) $\alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2q^n$ $[\boldsymbol{Hint}: Let\ \alpha^n = A\ , \beta^n = B]$
    b) $\alpha^4 + \beta^4 = p^4 - 4p^2q + 2q^2$
Hence/otherwise form a quadratic equation whose roots are the fourth powers of those of the equation $x^2 - 3x + 1 = 0$

18. *The roots of the equation* $x^2 + 6x + c = 0$ *differ by* $2n$*, where n is real and non zero, show that* $n^2 = 9 - c$*. Given that the roots also have opposite signs, find set of possible values of n.*
ANSWER: $n > 3$, $n < -3$

19. *If the roots of the equation* $x^2 - px + q = 0$ *are* α *and* β*, find the value of;*
    i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$ **ANSWER**: $\frac{p(p^2-4q)(p^2-q)}{q}$
    ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$ **ANSWER**: $\frac{p^4-4p^2q+2q^2}{q^4}$

20. a) *For what values of m will the equation* $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ *have roots equal in magnitude but opposite in sign?* **ANSWER**: $m = \frac{a-b}{a+b}$

b) *If x is real, prove that* $\frac{x^2+34x-71}{x^2+2x-7}$ *cannot lie strictly between 5 and 9*

c) *Show that for real x,* $-\frac{1}{4} \le \frac{x}{x^2+4} \le \frac{1}{4}$

d) *The roots of a quadratic equation* $x^2+px+q=0$ *are* α *and* β. *Show that the equation whose roots are* $\alpha^2-q\alpha$ *and* $\beta^2-q\beta$ *is given by* $x^2-(p^2+pq-2q)x+q^2(p+q+1)=0$ ***Ubeb* 2011**

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Let $ax^2+bx+c=0$ and
$lx^2+mx+n=0$
, be the two equations with a common root $x=\alpha \neq 0$
Substituting $x=\alpha$ in to the equations
$\Rightarrow a\alpha^2+b\alpha+c=0$ .... (1)
$\Rightarrow l\alpha^2+m\alpha+n=0$ .... (2)
Eliminating α^2 from eqn. (1) and eqn. (2)
$l \times eqn.(1) - m \times eqn.(2)$, gives
$\alpha(lb-am)+lc-an=0$
$\therefore \alpha = \frac{an-lc}{bl-am}$ .... (3)
Now eliminating constants c and n from eqn. (1) and eqn. (2)
$n \times eqn.(1) - c \times eqn.(2)$, gives
$\alpha^2(an-lc)+\alpha(bn-mc)=0$
$\Rightarrow \alpha[\alpha(an-lc)+(bn-mc)]=0$
Either $\alpha=0$, but $\alpha \neq 0$
, or $\alpha(an-lc)+(bn-mc)$
$\Rightarrow \alpha = \frac{mc-bn}{an-lc}$ .... (4)
Equating equations (3) and (4)
$\Rightarrow \frac{an-lc}{bl-am} = \frac{mc-bn}{an-lc}$
$\therefore (an-lc)^2 = (mc-bn)(bl-am)$

Illustrative examples

1) *Given that* $x^2+px+q=0$ *and* $x^2+rx+s=0$ *share a common root* $\alpha \neq 0$, *prove that* $\frac{ps-qr}{q-s} = \frac{q-s}{r-p}$

Solution

$x^2+px+q=0\,, x=\alpha\,, \alpha^2+p\alpha+q=0$ .... (1)
$x^2+rx+s=0\,, x=\alpha\,, \alpha^2+r\alpha+s=0$ .... (2)
Eliminating α^2 from eqn. (1) and eqn. (2)
$eqn.(1) - eqn.(2)$, gives
$\Rightarrow (p-r)\alpha+q-s=0$
$\therefore \alpha = \frac{s-q}{p-r} = \frac{q-s}{r-p}$ .... (3)
Now eliminating constants q and s from eqn. (1) and eqn. (2)
$q \times eqn.(2) - s \times eqn.(1)$, gives
$\Rightarrow \alpha^2(s-q)+\alpha(ps-qr)=0$

$\Rightarrow \alpha[\alpha(s-q)+(ps-qr)]=0$
Either $\alpha=0$, but $\alpha \neq 0$
, or $\alpha(s-q)+(ps-qr)$
$\Rightarrow \alpha=\frac{qr-ps}{s-q}=\frac{ps-qr}{q-s}$ ………………………………………………………………… (4)
Equating equations (3) and (4)
$\Rightarrow \frac{ps-qr}{q-s}=\frac{q-s}{r-p}$, as required

2) *Given that* $ax^2+bx+c=0$ *and* $bx^2+cx+a=0$ *have a common root, then* $a^3+b^3+c^3=3abc$

Solution
Let $x=\alpha \neq 0$ be the common root.
$\Rightarrow a\alpha^2+b\alpha+c=0$ ……………………………………………………… (1)
$\Rightarrow b\alpha^2+c\alpha+a=0$ ……………………………………………………… (2)
Eliminating α^2 from eqn. (1) and eqn. (2)
$b \times eqn.(1)-a \times eqn.(2)$, gives
$\Rightarrow (b^2-ac)\alpha+bc-a^2=0$
$\therefore \alpha=\frac{a^2-bc}{b^2-ac}$ ……………………………………………………………………… (3)
Now eliminating constants a and c from eqn. (1) and eqn. (2)
$a \times eqn.(1)-c \times eqn.(2)$, gives
$\Rightarrow \alpha^2(a^2-bc)+\alpha(ab-c^2)=0$
$\Rightarrow \alpha[\alpha(a^2-bc)+(ab-c^2)]=0$
Either $\alpha=0$, but $\alpha \neq 0$
, or $\alpha(a^2-bc)+(ab-c^2)=0$
$\Rightarrow \alpha=\frac{c^2-ab}{a^2-bc}$ ……………………………………………………………………… (4)
Equating equations (3) and (4)
$\Rightarrow \frac{a^2-bc}{b^2-ac}=\frac{c^2-ab}{a^2-bc}$
$\Rightarrow (a^2-bc)^2=(b^2-ac)(c^2-ab)$
$\Rightarrow a^4-2a^2bc+b^2c^2=b^2c^2-ac^3-ab^3+a^2bc$
$\Rightarrow a^4-2a^2bc=-ac^3-ab^3+a^2bc$ …………………………….$(\div a)$
$\Rightarrow a^3-2abc=-c^3-b^3+abc$
$\therefore a^3+b^3+c^3=3abc$

3) *If* $x^2-px+q=0$ *and* $x^2-qx+p=0$ *have a common root, prove that* $p+q+1=0$

Solution
Let $x=\alpha \neq 0$ be the common root.
$\Rightarrow \alpha^2-p\alpha+q=0$ ……………………………………………………… (1)
$\Rightarrow \alpha^2-q\alpha+p=0$ ……………………………………………………… (2)
Eliminating α^2 from eqn. (1) and eqn. (2)
$eqn.(1)-eqn.(2)$, gives
$(-p+q)\alpha+q-p=0$
$\Rightarrow \alpha=-\frac{q-p}{q-p}=-1$
$\therefore \alpha=-1$ ……………………………………………………………………… (3)
$eqn.(3) in\ to\ eqn.(1)\ \ or\ (2)$
$\Rightarrow (-1)^2-p(-1)+q=0$
$\therefore 1+p+q=0$. As required

4) *Find* p *such that* $x^2-5x+6=0$ *and* $x^2+px-10=0$ *have a common root.*

Solution

Let $x = \alpha \neq 0$ be the common root.

$\Rightarrow \alpha^2 - 5\alpha + 6 = 0$ ………………………………………………………… (1)

$\Rightarrow \alpha^2 + p\alpha - 10 = 0$ ……………………………………………………… (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$eqn.(1) - eqn.(2)$, gives

$(-5 - p)\alpha + 16 = 0$

$\Rightarrow \alpha = \frac{-16}{-5-p} = \frac{16}{p+5}$ ……………………………………………………… (3)

$eqn.(3)\ \ in\ to\ (1)$

$\Rightarrow \left(\frac{16}{p+5}\right)^2 - 5\left(\frac{16}{p+5}\right) + 6 = 0$

$\Rightarrow 256 - 80(p + 5) + 6(p + 5)^2 = 0$

$\Rightarrow 256 - 80p - 400 + 6p^2 + 60p + 150 = 0$

$\Rightarrow 6 - 20p + 6p^2 = 0$ …………………………………………… $(\div 2)$

$\therefore 3p^2 - 10p + 3 = 0$

$\Rightarrow 3p^2 - 9p - p + 3 = 0$

$\Rightarrow 3p(p - 3) - (p - 3) = 0$

$\Rightarrow (3p - 1)(p - 3) = 0$

$\therefore p = 3, \frac{1}{3}$

Alternatively

Eliminating constants 6 and -10 from eqn. (1) and eqn. (2)

$10 \times eqn.(1) + 6 \times eqn.(2)$, gives

$16\alpha^2 + (6p - 50)\alpha = 0$

$\Rightarrow \alpha = \frac{50-6p}{16} = \frac{25-3p}{8}$ ………………………………………………… (4)

Equating equations (3) and (4)

$\Rightarrow \frac{16}{p+5} = \frac{25-3p}{8}$, simplifying the equation gives

$3p^2 - 10p + 3 = 0$, solving the equation

$\therefore p = 3, \frac{1}{3}$

5) *Prove that if the equations* $x^2 + ax + b = 0$ *and* $cx^2 + 2ax - 3b = 0$ *have a common root and nether a nor b is zero, then* $b = \frac{5a^2(c-2)}{(c+3)^2}$

Solution

Let $x = \alpha \neq 0$ be the common root.

$\Rightarrow \alpha^2 + a\alpha + b = 0$ ……………………………………………………………… (1)

$\Rightarrow c\alpha^2 + 2a\alpha - 3b = 0$ ………………………………………………………… (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$c \times eqn.(1) - eqn.(2)$, gives

$\alpha(ac - 2a) + bc + 3b = 0$

$\therefore \alpha = -\frac{(bc+3b)}{(ac-2a)} = \frac{-b(c+3)}{a(c-2)}$ …………………………………………………… (3)

Eliminating constants $b, -3b$ from eqn. (1) and eqn. (2)

$3 \times eqn.(1) + eqn.(2)$, gives

$\alpha^2(3 + c) + 5a\alpha = 0$

$\Rightarrow \alpha[\alpha(3 + c) + 5a] = 0$

$\therefore \alpha = -\frac{5a}{c+3}$ …………………………………………………………………… (4)

Equating equations (3) and (4)

$\Rightarrow \frac{-b(c+3)}{a(c-2)} = -\frac{5a}{c+3}$
$\Rightarrow b(c+3)^2 = 5a^2(c-2)$
$\therefore b = \frac{5a^2(c-2)}{(c+3)^2}$, as required.

Task

1. *Show that if the equations* $x^2 + px + q = 0$ *and* $x^2 + 2Px + Q = 0$ *have a common root then* $(q-Q)^2 + 4(P-p)(Pp-pQ) = 0$
2. *If the equations* $a_1x^2 + b_1x + c_1 = 0$ *and* $a_2x^2 + b_2x + c_2 = 0$ *have a common root, then* $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$
3. *Show that if the equations* $x^2 + bx + c = 0$, $x^2 + px + q = 0$ *have a common root, then* $(c-q)^2 = (b-p)(cp-bq)$
4. *Show that if the equations* $x^2 + ax + 1 = 0$ *and* $x^2 + x + b = 0$ *have a common root, then* $(b-1)^2 = (a-1)(1-ab)$

COMMON FACTORS

If two functions $f(x)$ and $g(x)$ have a common factor $(x-a)$, then
$f(x) = (x-a).F(x)$ ………………………………….. (1)
$g(x) = (x-a).G(x)$ ………………………………….. (2)
For any constant k,
$f(x) + kg(x) = (x-a).F(x) + k(x-a).G(x)$

$$\boxed{f(x) + kg(x) = (x-a)[F(x) + kG(x)]}$$

Thus for any constant k, $(x-a)$ is a common factor of $f(x) + kg(x)$

NOTE:

Equations (1) and (2) are identities. Thus , $\boxed{f(a) = 0\,, \; g(a) = 0}$, which gives a pair of simultaneous equations which provides another approach to problems involving a common factor.

Examples

1. *Find constants p and q such that* $(x-2)$ *is a common factor of* $x^3 - x^2 - 2px + 3q$ *and* $qx^3 - px^2 + x + 2$

Solution

Let $f(x) = x^3 - x^2 - 2px + 3q$
$g(x) = qx^3 - px^2 + x + 2$
Since $(x-2)$ is a factor to both $f(x)$ and $g(x)$, then $f(2) = 0$ $g(2) = 0$
$\Rightarrow 8 - 4 - 4p + 3q = 0$ *or* $-4p + 3q = -4$ ……………………………………… (1)
$\Rightarrow 8q - 4p + 2 + 2 = 0$ *or* $-4p + 8q = -4$ ……………………………………… (2)
Solving eqn. (1) and eqn. (2) simultaneously,
$eqn.(1) - eqn.(2)$
$\Rightarrow -5q = 0 \; \therefore q = 0$
From eqn. (1), for $q = 0$, $p = 1$
$\therefore p = 1\,, q = 0$

2. *Find constants p and q such that* $(x-1)$ *is a common factor of* $x^4 + x^2 - q$ *and* $x^4 - 2px^2 + 2$

Solution

Let $f(x) = x^4 + x^2 - q$, $f(1) = 1 + 1 - q = 0$ $\therefore q = 2$

$g(x) = qx^3 - px^2 + x + 2$, $g(1) = 1 - 2p + 2 = 0$ $\therefore p = \frac{3}{2}$

$\therefore p = \frac{3}{2}$, $q = 2$

3. *Show that if* $(x+1)$ *is a common factor of* $x^3 - ax^2 + b$ *and* $x^4 - ax^3 + bx^2 + c$, *then* $2a = 2b - 2 = -2 - c$.

Solution

Let $f(x) = x^3 - ax^2 + b$

$g(x) = x^4 - ax^3 + bx^2 + c$

For any constant k, $(x+1)$ is a factor of $f(x) + kg(x)$

$\Rightarrow f(-1) + kg(-1) = 0$, for any value of k.

$\therefore -1 - a - b + k(1 + a + b + c) = 0$

For $k = 0$, $a - b = -1$ .... (1)

For $k = 1$, $2b + c = 0$ .... (2)

From eqn. (1), multiplying through by 2

$\Rightarrow 2a - 2b = -2$

From eqn. (2), $2b + c = 0$ *or* $b = -\frac{c}{2}$.................... (3)

Eqn. (3) in to eqn. (1)

$a + \frac{c}{2} = -1$ *or* $2a + c = -2$ *or* $2a = -2 - c$

$\therefore 2a = 2b - 2 = -2 - c$

4. *Find constant a for which the function* $f(x) = ax^2 + 2x - 1$ *and* $g(x) = x^2 + 4x + a$ *have a common factor.*

Solution

Let $x - \alpha$ be a common factor

$\Rightarrow f(x) + kg(x)$ has a factor of $x - \alpha$

$\Rightarrow f(\alpha) + kg(\alpha) = 0$

$\Rightarrow a\alpha^2 + 2\alpha - 1 + k(\alpha^2 + 4\alpha + a) = 0$

For $k = -a$, to eliminate α^2

$\Rightarrow a\alpha^2 + 2\alpha - 1 - a\alpha^2 - 4a\alpha - a^2 = 0$

$\Rightarrow \alpha(2 - 4a) - 1 - a^2 = 0$

$\therefore \alpha = \frac{a^2+1}{2-4a}$ .... (1)

For $k = \frac{1}{a}$, to eliminate constants

$\Rightarrow a\alpha^2 + 2\alpha - 1 + \frac{\alpha^2}{a} + \frac{4\alpha}{a} + 1 = 0$

$\Rightarrow \alpha^2\left(a + \frac{1}{a}\right) + \alpha\left(2 + \frac{4}{a}\right) = 0$

$\Rightarrow \alpha\left[\alpha\left(a + \frac{1}{a}\right) + \left(2 + \frac{4}{a}\right)\right] = 0$

$\Rightarrow \alpha\left(a + \frac{1}{a}\right) + \left(2 + \frac{4}{a}\right) = 0$

$\therefore \alpha = -\frac{\left(2+\frac{4}{a}\right)}{\left(a+\frac{1}{a}\right)} = \frac{-(2a+4)}{a^2+1}$ .... (2)

Equating equations (1) and (2)

$\Rightarrow \frac{a^2+1}{2-4a} = \frac{-(2a+4)}{a^2+1}$

$\Rightarrow a^4 + 2a^2 + 1 = -(2a+4)(2-4a)$

$\Rightarrow a^4 + 2a^2 + 1 = -(4a - 8a^2 + 8 - 16a) = 12a + 8a^2 - 8$

$\therefore a^4 - 6a^2 - 12a + 9 = 0$

Let $h(a) = a^4 - 6a^2 - 12a + 9$
For $a = 3$, $h(3) = 0$ $\therefore a - 3$ is a factor of $h(a)$

$$\begin{array}{r|l} & a^3 + 3a^2 + 3a - 3 \\ \hline a - 3 & a^4 + 0a^3 - 6a^2 - 12a + 9 \\ & a^4 - 3a^3 \\ \hline & \quad 3a^3 - 6a^2 - 12a + 9 \\ & \quad 3a^3 - 9a^2 \\ \hline & \qquad 3a^2 - 12a + 9 \\ & \qquad 3a^2 - 9a \\ \hline & \qquad\quad 3a + 9 \\ & \qquad\quad 3a + 9 \\ \hline \end{array}$$

$\therefore$ $h(a) = (a - 3)(a^3 + 3a^2 + 3a - 3) = 0$
$\Rightarrow a = 3,\ 0.26$
$\therefore a = 3$

5. *Determine the value of m such that the equations $x^3 + mx - 1 = 0$ and $x^3 - 3x + m = 0$ have a common root.*

Solution
Let common root be $x = \alpha \neq 0 \Rightarrow (x - \alpha)$ is a common factor.
Let $f(x) = x^3 + mx - 1$, $g(x) = x^3 - 3x + m$
$\Rightarrow f(x) + kg(x)$, has a common factor $(x - \alpha)$
$\Rightarrow f(\alpha) + kg(\alpha) = 0$
$\Rightarrow \alpha^3 + m\alpha - 1 + k(\alpha^3 - 3\alpha + m) = 0$
For $k = -1$ to eliminate α^3
$\Rightarrow \alpha^3 + m\alpha - 1 - \alpha^3 + 3\alpha - m = 0$
$\Rightarrow \alpha(m + 3) = m + 1 \therefore \alpha = \frac{m+1}{m+3}$
Now from the equation $x^3 + mx - 1 = 0$ with $x = \alpha$ as a root, then $\alpha^3 + m\alpha - 1 = 0$ must have the root $\alpha = \frac{m+1}{m+3}$
$\Rightarrow \left(\frac{m+1}{m+3}\right)^3 + m\left(\frac{m+1}{m+3}\right) - 1 = 0$ ....$\times (m + 3)^3$
$\Rightarrow (m + 1)^3 + (m^2 + m)(m + 3)^2 - (m + 3)^3 = 0$
$\Rightarrow m^3 + 3m^2 + 3m + 1 + (m^2 + m)(m^2 + 6m + 9) - (m^3 + 9m^2 + 27m + 27) = 0$
$\Rightarrow m^3 + 3m^2 + 3m + 1 + m^4 + 6m^3 + 9m^2 + m^3 + 6m^2 + 9m - m^3 - 9m^2 - 27m - 27 = 0$
$\Rightarrow m^4 + 7m^3 + 9m^2 - 15m - 26 = 0$
Solving by factor theorem,
Let $f(m) = m^4 + 7m^3 + 9m^2 - 15m - 26$
For $m = -2$, $f(-2) = (-2)^4 + 7(-2)^3 + 9(-2)^2 - 15(-2) - 26 = 0$
$\Rightarrow m + 2$, is a factor of $f(m)$

$$\begin{array}{r|l} & m^3 + 5m^2 - m - 13 \\ \hline m + 2 & m^4 + 7m^3 + 9m^2 - 15m - 26 \\ & m^4 + 2m^3 \\ \hline & \quad 5m^3 + 9m^2 - 15m - 26 \\ & \quad 5m^3 + 10m^2 \\ \hline & \qquad -m^2 - 15m - 26 \\ & \qquad -m^2 - 2m \\ \hline & \qquad\quad -13m - 26 \\ & \qquad\quad -13m - 26 \\ \hline \end{array}$$

$\therefore (m+2)(m^3+5m^2-m-13)=0$
$\therefore m=-2\,,1.49\,,-4.60\,,-1.89$

6. *Show that If the cubic functions* x^3+ax^2+b *and* ax^3+bx^2+x-a *have a common factor, it is also a common factor of the quadratic function* $(b-a^2)x^2+x-a(1+b)=0$

Solution
Let the common factor be $(x-\alpha)$
Let $f(x)=x^3+ax^2+b\ ,g(x)=ax^3+bx^2+x-a$
$\Rightarrow f(x)+kg(x)$, has a common factor $(x-\alpha)$
$\Rightarrow f(\alpha)+kg(\alpha)=0$
$\Rightarrow \alpha^3+a\alpha^2+b+k(a\alpha^3+b\alpha^2+\alpha-a)=0$
For $k=-\frac{1}{a}$ to eliminate α^3
$\Rightarrow \alpha^3+a\alpha^2+b-\frac{1}{a}(a\alpha^3+b\alpha^2+\alpha-a)=0$
$\Rightarrow a\alpha^3+a^2\alpha^2+ab-a\alpha^3-b\alpha^2-\alpha+a=0$
$\Rightarrow \alpha^2(a^2-b)-\alpha+a(1+b)=0$
$\therefore (b-a^2)\alpha^2+\alpha-a(1+b)=0$. this equation is as a result of $x=\alpha$ being a true root of the quadratic equation $(b-a^2)x^2+x-a(1+b)=0$. Hence the common factor of the above two equations is also a common factor of the quadratic equation.

Task
Show that if $(x+\alpha)$ *is a common factor of* x^3+px^2+q *and* ax^3+bx+c*, then it is also a factor of* $apx^2-bx+aq-c$.

REPEATED FACTORS
Let $(x-a)$ be a repeated factor of $f(x)$
$\Rightarrow f(x)=(x-a)^2.g(x)$ $.i.e.$ repeated twice
Differentiating with respect to x
$f'(x)=g(x).2(x-a)+(x-a)^2.g'(x)$
Thus if $(x-a)$ is a repeated factor of $f(x)$, then it is also a factor of $f'(x)$
NOTE:
$(x-a)$ is a repeated factor of $f(x)$ if $f'(a)=0$

Examples

1. *Determine whether* $f(x)\equiv 3x^4-8x^3-6x^2+24x-13$ *has any repeated factors, and if so find them.*

Solution
$f'(x)\equiv 12x^3-24x^2-12x+24$
$\equiv 12(x^3-2x^2-x+2)$
$\equiv 12[x^2(x-2)-(x-2)]$
$\equiv 12(x-2)(x^2-1)\equiv 12(x-1)(x+1)(x-2)$
Now $f'(x)=0$ if $x=\pm1\,,2$
Checking the value of $f(x)$ for these values of x
$\Rightarrow f(1)=0\,,f(-1)\neq 0\ \,,f(2)\neq 0$
So $(x+1)$ and $(x-2)$ are not factors of $f(x)$
Hence $(x-1)$ is the only repeated factor of $f(x)$

2. *If the equation* $3x^4+2x^3-6x^2-6x+p=0$ *has equal roots, find the possible values of p.*

Solution

Let $f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$

The equation $f(x) = 0$ has equal roots if $f(x)$ has repeated factors.

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$\Rightarrow f'(x) \equiv 12x^3 + 6x^2 - 12x - 6$

$\equiv 6(2x^3 + x^2 - 2x - 1)$

$\equiv 6[x^2(2x+1) - (2x+1)]$

$\equiv 6(2x+1)(x^2-1)$

$\equiv 6(x-1)(x+1)(2x+1)$

If $(x-1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 3 + 2 - 6 - 6 + p = 0 \ \therefore p = 7$

Similarly $(x+1)$ is a repeated factor of $f(x)$, then $f(-1) = 0$

$\Rightarrow 3 - 2 - 6 + 6 + p = 0 \ \therefore p = -1$

Also $(2x+1)$ is a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$

$\Rightarrow 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0 \ \therefore p = -\frac{23}{16}$

3. *If the equation* $2x^3 - 9x^2 + 12x + p = 0$ *has two equal roots, find the possible values of p.*

Solution

Let $f(x) = 2x^3 - 9x^2 + 12x + p = 0$

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$\Rightarrow f'(x) \equiv 6x^2 - 18x + 12$

$\equiv 6(x^2 - 3x + 2)$

$\equiv 6(x-2)(x-1)$

If $(x-2)$ is a repeated factor of $f(x)$, then $f(2) = 0$

$\Rightarrow 16 - 36 + 24 + p = 0 \ \therefore p = -4$

Similarly $(x-1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 2 - 9 + 12 + p = 0 \ \therefore p = -5$

$\therefore p = -4, -5$

Task

1. *Find the possible values of a for which the equation* $2x^3 - 3x^2 - 12x + a = 0$ *has repeated roots.* **ANSWER**: $-7, 20$
2. *Find the value of a for which the function* $2x^3 - ax^2 - 12x - 7$ *has a repeated factor.* **ANSWER**: 3

MISCELLANIOUS EQUATIONS

Special quadratic equations

Some equations do not appear quadratic but they can be transformed in to quadratic equations using a *suitable substitution.*

TYPE I: *Equation of the form* $\frac{\boldsymbol{u}}{\boldsymbol{c}}+\frac{\boldsymbol{a}}{\boldsymbol{u}}=\boldsymbol{b}$ where a and b are constants and u is a suitable substitution.

Example

Solve the equations:

a) $x^2+\frac{9}{x^2}=10$

b) $x^{4/3}+16x^{-4/3}=17$

c) $x^2+2x=34+\frac{35}{x^2+2x}$

d) $y(y+1)+\frac{12}{y(y+1)}=8$

e) $9x^{2/3}+16x^{-2/3}=37$

f) $x^2+2x+\frac{12}{x^2+2x}=7$

Solution

a) $x^2+\frac{9}{x^2}=10$, let $x^2=u$

$\Rightarrow u+\frac{9}{u}=10$ *(Multiplying both sides by u and re-arranging)*

$\Rightarrow \mathrm{u}^2-10\mathrm{u}+9=0$ *or* $(u-9)(u-1)=0$

$\therefore u=9\,,1$

But $x^2=u \Rightarrow x^2=9\,,x^2=1$

$\therefore x=\pm 1\,,\pm 3$

Note:

The values of x are *four* because the equation is of degree $4.\,i.e.\,4^0$

b) $x^{4/3}+16x^{-4/3}=17$ *or* $x^{4/3}+\frac{16}{x^{4/3}}=17$, let $x^{4/3}=u$

$\Rightarrow u+\frac{16}{u}=17$ *or* $\mathrm{u}^2-17\mathrm{u}+16=0$

$\Rightarrow (u-16)(u-1)=0\ \therefore u=16\,,1$

But $x^{4/3}=u \Rightarrow x^{4/3}=16\,,x^{4/3}=1$

Multiplying the powers by $\frac{3}{4}$

$x=16^{3/4}=8\,;x=(1)^{3/4}=1$

$\therefore x=1\,,8$

c) $x^2+2x=34+\frac{35}{x^2+2x}$, let $x^2+2x=u$

$\Rightarrow u=34+\frac{35}{u}$ *or* $\mathrm{u}^2-34\mathrm{u}-35=0$

$\Rightarrow (u-35)(u+1)=0\ \therefore u=35\,,-1$

But $x^2+2x=u$,

$\Rightarrow x^2+2x=35$ *or* $x^2+2x-35=0$

$\Rightarrow (x+7)(x-5)=0\ \therefore x=5\,,-7$

Also $x^2+2x=-1$ *or* $x^2+2x+1=0$

$\Rightarrow (x+1)^2=0\ \therefore x=-1\,,-1$

$\therefore x=5\,,-1\,,-1\,,-7$

d) $y(y+1)+\frac{12}{y(y+1)}=8$, let $y(y+1)=u$

$\Rightarrow u+\frac{12}{u}=8$ *or* $\mathrm{u}^2-8\mathrm{u}+12=0$

$\Rightarrow (u-6)(u-2)=0\ \therefore u=6\,,2$

$\Rightarrow y(y+1)=2 \ \ or\ y^2+y-2=0$
$\Rightarrow (y+2)(y-1)=0 \ \therefore y=1\,,-2$
Also $y(y+1)=6 \ \ or\ y^2+y-6=0$
$\Rightarrow (y+3)(y-2)=0 \ \therefore y=2\,,-3$
$\therefore y=1\,,\pm2\,,-3$

e) $9x^{2/3}+16x^{-2/3}=37 \quad or\ 9x^{2/3}+\frac{16}{x^{2/3}}=37$, let $x^{2/3}=u$
$\Rightarrow 9u+\frac{16}{4}=37 \quad or\ 9\mathrm{u}^2-37\mathrm{u}+4=0$
$\Rightarrow 9\mathrm{u}^2-36\mathrm{u}-\mathrm{u}+4=0$
$\Rightarrow 9u(u-4)-(u-4)=0$
$\Rightarrow (9u-1)(u-4)=0 \ \therefore u=\frac{1}{9}\,,4$
Now $x^{2/3}=u$
$\Rightarrow x^{2/3}=\frac{1}{9} \quad or\ x=\left(\frac{1}{9}\right)^{3/2}=\frac{1}{27}$
$\Rightarrow x^{2/3}=4 \quad or\ x=(4)^{3/2}=8$
$\therefore x=8\,,\frac{1}{27}$

f) Left as an exercise. **ANSWER**: $\pm1\ -3,\pm\sqrt{5}$

Task
Solve the equations:

a) $\sqrt{3^x}+\frac{3}{\sqrt{3^x}}=4$ ***Answer***: $0\,,2$
b) $(x^2-2x)^2+24=11(x^2-2x)$ **ANSWER**: $-2\,,-1\,,3\,,4$
Note: the equation can be expressed in the form $x^2-2x+\frac{24}{x^2-2x}=11$
c) $\frac{x^2+4x}{3}+\frac{84}{x^2+4x}=11$ **ANSWER**: $-7\,,-6\,,2\,,3$
d) $(3x^2+2x)^2+8=9(3x^2+2x)$ **ANSWER**: $-2\,,-1\,,\frac{1}{3}\,,\frac{4}{3}$

TYPE II: *Indicial equations reducible to quadratic equation .i.e.* $\boldsymbol{ax^2+bx+c=0}$

Example

1. *Solve the equations:*

i) $3^{2x}+9=10.3^x$
ii) $4^x-3.2^{x+3}=-128$
iii) $5(5^x+5^{-x})=26$
iv) $5^x+5^{2-x}=26$
v) $2-5e^{-x}+2e^{-2x}=0$
vi) $3^{2(x+1)}-10\times3^x+1=0$

Solution

i) $3^{2x}+9=10.3^x$
$\Rightarrow (3^x)^2+9=10\times3^x$, let $3^x=u$
$\Rightarrow u^2+9=10u \ \ or\ u^2-10u+9=0$
$\Rightarrow (u-9)(u-1)=0 \ \therefore u=9\,,1$
But $3^x=u$
$\Rightarrow 3^x=9=3^2 \ \therefore x=2$
$\Rightarrow 3^x=1=3^0 \ \therefore x=0$
$\therefore x=2\,,0$

ii) $4^x - 3.2^{x+3} = -128$

$\Rightarrow (2^x)^2 - 3 \times 2^x \times 2^3 = -128$ *or* $(2^x)^2 - 24 \times 2^x = -128$, let $2^x = u$

$\Rightarrow u^2 - 24u = -128$ *or* $u^2 - 24u + 128 = 0$

$\Rightarrow u = \frac{24 \pm \sqrt{24^2 - 4 \times 128}}{2} = \frac{24 \pm 8}{2}$

$\therefore u = 16 , 8$

But $2^x = u$

$\Rightarrow 2^x = 16 = 2^4 \ \therefore x = 4$

$\Rightarrow 2^x = 8 = 2^3 \ \therefore x = 3$

$\therefore x = 3 , 4$

iii) $5(5^x + 5^{-x}) = 26$ *or* $5\left(5^x + \frac{1}{5^x}\right) = 26$, let $5^x = u$

$\Rightarrow 5\left(u + \frac{1}{u}\right) = 26$ *or* $5u^2 - 26u + 5 = 0$

$\Rightarrow 5u^2 - 25u - u + 5 = 0$

$\Rightarrow (5u - 1)(u - 5) = 0 \ \therefore u = 5 , \frac{1}{5}$

Now $5^x = u$

$\Rightarrow 5^x = 5 \ \therefore x = 1$

$\Rightarrow 5^x = \frac{1}{5} = 5^{-1} \ \therefore x = -1$

$\therefore x = \pm 1$

v) $2 - 5e^{-x} + 2e^{-2x} = 0$ *or* $2 - 5e^{-x} + 2(e^{-x})^2 = 0$, let $e^{-x} = u$

$\Rightarrow 2 - 5u + 2u^2 = 0$

$\Rightarrow (2u - 1)(u - 2) = 0 \ \therefore u = 2 , \frac{1}{2}$

Now $e^{-x} = u$

$\Rightarrow e^{-x} = 2$, introducing logarithm to base e

$\Rightarrow \ln e^{-x} = \ln 2 \ \therefore x = -\ln 2$

Also $e^{-x} = \frac{1}{2}$

$\Rightarrow \ln e^{-x} = \ln\left(\frac{1}{2}\right) \ \therefore x = +\ln 2$

$\therefore x = \pm \ln 2$

vi) $3^{2(x+1)} - 10 \times 3^x + 1 = 0$ *or* $3^{2x}.3^2 - 10 \times 3^x + 1 = 0$, let $3^x = u$

$\Rightarrow 9u^2 - 10u + 1 = 0$

$\Rightarrow (9u - 1)(u - 1) = 0 \ \therefore u = 1 , \frac{1}{9}$

But $3^x = u$

$\Rightarrow 3^x = 1 = 3^0 \ \therefore x = 0$

$\Rightarrow 3^x = \frac{1}{9} = 3^{-2} \ \therefore x = -2$

$\therefore x = 0 , -2$

Task

Solve: i) $4^x - 5.2^x + 4 = 0$ **ANSWER**: $0 , 2$

ii) $3^{2x+1} - 3^{x+1} - 3^x + 1 = 0$ **ANSWER**: $-1 , 0$

iii) $5^{2x} - 5^{x+1} + 4 = 0 \quad 0$ **ANSWER**: $0 , 0.861$

iv) $2^{2x+8} - 32(2^x) + 1 = 0$ **ANSWER**: -4

v) $2^{2x+3} - 57 = 65(2^x - 1) = 0$ **ANSWER**: ± 3

2. *Solve the equation*

a) $t - 1.324\sqrt{t} - 2.896 = 0$

b) $x + \sqrt{x} = \frac{6}{25}$

Solution

a) $t - 1.324\sqrt{t} - 2.896 = 0$, let $\sqrt{t} = u \Rightarrow u^2 = t$

$\therefore u^2 - 1.324u - 2.896 = 0$

$\therefore u = \frac{1.324 \pm \sqrt{(-1.324)^2 + 4 \times 2.896}}{2} = \frac{1.324 \pm 3.652}{2}$

$\therefore u = 2.488 , -1.164$

$\Rightarrow t = (2.488\ )^2 = 6.19$; $(-1.164\ )^2 = 1.35$

$\therefore t = 6.19 , 1.35$

b) $x + \sqrt{x} = \frac{6}{25}$, let $\sqrt{x} = u \Rightarrow x = u^2$

$\Rightarrow u^2 + u = \frac{6}{25}$ *or* $25u^2 + 25u - 6 = 0$

$\Rightarrow u = \frac{-25 \pm \sqrt{25^2 + 4 \times 6 \times 25}}{2 \times 25} = \frac{-25 \pm 5}{50}$

$\therefore u = -\frac{2}{5} , -\frac{3}{5}$

$\therefore x = \frac{4}{25} , \frac{9}{25}$

Task

Solve the equations:

a) $6\sqrt{x} = 5x^{-\frac{1}{2}} - 13$; **ANSWER**: $\frac{1}{9} , \frac{25}{4}$

b) $\sqrt{2^x} + \frac{1}{\sqrt{2^x}} = 2$; **ANSWER**: 0

TYPE III: *Equation of the form* $\boldsymbol{Ax^4 + Bx^2 + C = 0}$. *This equation contains only* $3 - terms$

Solve the equation:

i) $x^4 - 13x^2 + 36 = 0$

ii) $2x^4 - x^2 = 6$

Solution

i) $x^4 - 13x^2 + 36 = 0$, let $x^2 = u$

$\Rightarrow u^2 - 13u + 36 = 0$ *or* $(u - 9)(u - 4) = 0$

$\therefore u = 9 , 4$

But $x^2 = u$

$\Rightarrow x^2 = 9 \ \therefore x = \pm 3$

$\Rightarrow x^2 = 4 \ \therefore x = \pm 2$

$\therefore x = \pm 2 , \pm 3$

ii) $2x^4 - x^2 = 6$, let $x^2 = u$

$\Rightarrow 2u^2 - u - 6 = 0$ *or* $(2u + 3)(u - 2) = 0$

$\therefore u = -\frac{3}{2} , 2$

But $x^2 = u$

$\Rightarrow x^2 = 2 \ \therefore x = \sqrt{2}$

$\Rightarrow x^2 = -\frac{3}{2} \ \therefore x = \sqrt{\left(-\frac{3}{2}\right)}$, no real values of x

$\therefore x = \pm\sqrt{2}$

TYPE IV: *Equation of the form* $(\boldsymbol{x}+\boldsymbol{a})(\boldsymbol{x}+\boldsymbol{b})(\boldsymbol{x}+\boldsymbol{c})(\boldsymbol{x}+\boldsymbol{d})=\boldsymbol{n}$*, where* $a+b=c+d$ *and* a,b,c,d *and* n *are constants*

Example:

Solve the equations:

i) $(x+1)(x+3)(x+4)(x+6)=72$
ii) $(x-7)(x-3)(x+5)(x+1)=1680$
iii) $(x-5)(x-7)(x+6)(x+4)=504$

Solution

i) $(x+1)(x+3)(x+4)(x+6)=72$

Since $1+6=3+4=7$, then re-grouping

$\Rightarrow (x+1)(x+6).(x+3)(x+4)=72$

$\Rightarrow (x^2+7x+6).(x^2+7x+12)=72$

Now let $x^2+7x=u$

$\Rightarrow (u+6)(u+12)=72$

$\Rightarrow u^2+18u+72=72 \quad or\ u(u+18)=0$

$\therefore u=0\,,-18$

But $x^2+7x=u$

$\Rightarrow x^2+7x=0 \ \therefore x=0\,,-7$

$\Rightarrow x^2+7x=-18 \ or \ x^2+7x+18=0$

$\therefore x=\frac{-7\pm\sqrt{7^2-4\times18}}{2}=\frac{-7\pm\sqrt{-23}}{2}=\frac{-7\pm i\sqrt{23}}{2}$

$\therefore x=0\,,-7\,,\frac{-7\pm i\sqrt{23}}{2}$

ii) $(x-7)(x-3)(x+5)(x+1)=1680$

Since $-7+5=-3+1=-2$, then re-grouping

$\Rightarrow (x-7)(x+5).(x-3)(x+1)=1680$

$\Rightarrow (x^2-2x-35).(x^2-2x-3)=1680$

Now let $x^2-2x=u$

$\Rightarrow (u-35)(u-3)=1680$

$\Rightarrow u^2-38u+105=1680 \quad or\ u^2-38u-1575=0$

$\Rightarrow u=\frac{38\pm\sqrt{38^2+4\times1575}}{2}=\frac{38\pm88}{2}$

$\therefore u=63\,,-25$

But $x^2-2x=u$

$\Rightarrow x^2-2x=63 \quad or \ \ x^2-2x-63=0$

$\Rightarrow (x-9)(x+7)=0$

$\therefore x=9\,,-7$

$\Rightarrow x^2-2x=-25 \ or \ x^2-2x+25=0$

$\therefore x=\frac{2\pm\sqrt{(-2)^2-4\times25}}{2}=\frac{2\pm\sqrt{-96}}{2}=\frac{2\pm i\sqrt{96}}{2}=\frac{2\pm i4\sqrt{6}}{2}$

$\therefore x=9\,,-7\,,1\pm i2\sqrt{6}$

iii) Left as an exercise. **ANSWER**: $3\,,-2\,,8\,,-7$

Task

Solve the equations:

i) $x(2x+1)(x-2)(2x-3)=63$ **ANSWER**: $3,-\frac{3}{2}\,,\frac{3\pm\sqrt{-47}}{4}$

ii) $16x(x+1)(x+2)(x+3)=9$ **ANSWER**: $-\frac{3}{2}\,,-\frac{3}{2}\,,\ \frac{3\pm\sqrt{10}}{2}$

TYPE IV: **e***quation of the form* $\boldsymbol{ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0}$, *where coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of the quadratic.*
Equations of this form are known as *Reciprocal Equations*, and so are named because they are not altered when x is changed in to its $\frac{1}{x}$.

Example:
Solve the equations:

a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$
b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$
c) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$
d) $x^4 + x^3 - 4x^2 + x + 1 = 0$
e) $x^4 + 1 - 3(x^3 + x) = 2x^2$

Solution

a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$
Dividing through by x^2
$\Rightarrow 12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$
$\Rightarrow 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0$
Now let $\left(x + \frac{1}{x}\right) = u$
$\Rightarrow u^2 = x^2 + \frac{1}{x^2} + 2 \;\; or \;\; x^2 + \frac{1}{x^2} = u^2 - 2$
The substitutions that solve reciprocal equations are;

$$\boxed{\boldsymbol{x + \frac{1}{x} = u \;\; , \;\; x^2 + \frac{1}{x^2} = u^2 - 2}}$$

$\Rightarrow 12(u^2 - 2) - 56u + 89 = 0$
$\Rightarrow 12u^2 - 56u + 65 = 0$
$\Rightarrow u = \frac{56 \pm \sqrt{(56)^2 - 4\times12\times65}}{2\times12} = \frac{56\pm4}{48}$
$\therefore u = \frac{5}{2}, \frac{13}{6}$ (**Note:** left as fractions)
But $\left(x + \frac{1}{x}\right) = u$
$\Rightarrow x + \frac{1}{x} = \frac{13}{6} \;\; or \;\; 6x^2 - 13x + 6 = 0$
$\Rightarrow (3x - 2)(2x - 3) = 0 \;\; \therefore x = \frac{3}{2}, \frac{2}{3}$
Also $x + \frac{1}{x} = \frac{5}{2} \;\; or \;\; 2x^2 - 5x + 2 = 0$
$\Rightarrow (2x - 1)(x - 2) = 0 \;\; \therefore x = 2, \frac{1}{2}$
$\therefore x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$

b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$
Dividing through by x^2
$\Rightarrow x^2 + 2x - 1 + \frac{2}{x} + \frac{1}{x^2} = 0$
$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 1 = 0$
Now let $x + \frac{1}{x} = u, x^2 + \frac{1}{x^2} = u^2 - 2$ (as derived as before)
$\Rightarrow u^2 - 2 + 2u - 1 = 0$
$\Rightarrow u^2 + 2u - 3 = 0$

$\Rightarrow (u+3)(u-1) = 0 \quad \therefore u = -1, -3$

But $x + \frac{1}{x} = u$

$\Rightarrow x + \frac{1}{x} = -1 \;\; or \;\; x^2 + x + 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

Also $x + \frac{1}{x} = -3 \;\; or \;\; x^2 + 3x + 1 = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$\therefore x = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm \sqrt{5}}{2}$

c) Left as an exercise. **ANSWER**: $\frac{1}{3}, \frac{1}{2}, 2, 3$

d) Left as an exercise. **ANSWER**: $-\frac{1}{4}, -4$

e) Left as an exercise. **ANSWER**: $1, \frac{-3 \pm \sqrt{5}}{2}$

f) $x^4 + 1 - 3(x^3 + x) = 2x^2$

Re arranging the equation,

$\Rightarrow x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$

Dividing through by x^2

$\Rightarrow x^2 - 3x - 2 - \frac{3}{x} + \frac{1}{x^2} = 0$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 2 = 0$

Now let $x + \frac{1}{x} = u, x^2 + \frac{1}{x^2} = u^2 - 2$ (as derived as before)

$\Rightarrow u^2 - 2 - 3u - 2 = 0$

$\Rightarrow u^2 - 3u - 4 = 0$

$\Rightarrow (u+1)(u-4) = 0 \quad \therefore u = -1, 4$

But $x + \frac{1}{x} = u$

$\Rightarrow x + \frac{1}{x} = -1 \;\; or \;\; x^2 + x + 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

Also $x + \frac{1}{x} = 4 \;\; or \;\; x^2 - 4x + 1 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\therefore x = \frac{-1 \pm i\sqrt{3}}{2}, 2 \pm \sqrt{3}$

Solving equations which are not reciprocal but solved in a similar manner as reciprocal

Example

Solve the equation:

a) $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$

b) $x^2 + \frac{9}{x^2} - 4\left(x - \frac{3}{x}\right) - 2 = 0$

Solution

a) $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$

Dividing through by x^2

$\Rightarrow 6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0$

$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0$

Now let $x-\frac{1}{x}=u$

$u^2=x^2+\frac{1}{x^2}-2 \Rightarrow u^2+2=x^2+\frac{1}{x^2} \;\Rightarrow u^2-2-3u-2=0$

$\Rightarrow 6(u^2+2)-25u+12=0 \;\; or \;\; 6u^2-25u+24=0$

$\Rightarrow u=\frac{25\pm\sqrt{25^2-4\times6\times24}}{2\times6}=\frac{25\pm7}{12} \;\;\therefore u=\frac{32}{12}=\frac{8}{3}\;,\frac{18}{12}=\frac{3}{2}$

Now $x-\frac{1}{x}=\frac{8}{3} \;\; or \;\; 3x^2-8x-3=0$

$\Rightarrow 3x^2-9x+x-3=0 \;\, or\; 3x(x-3)+1(x-3)=0$

$\Rightarrow (3x+1)(x-3)=0 \;\;\therefore x=-\frac{1}{3}\,,3$

Also $x-\frac{1}{x}=\frac{3}{2} \;\; or \;\; 2x^2-3x-2=0$

$\Rightarrow 2x^2-4x+x-2=0 \;\, or\; 2x(x-2)+1(x-2)=0$

$\Rightarrow (2x+1)(x-2)=0 \;\;\therefore x=-\frac{1}{2}\,,2$

$\therefore x=-\frac{1}{3}\,,3\,,2\,,-\frac{1}{2}$

b) $x^2+\frac{9}{x^2}-4\left(x-\frac{3}{x}\right)-2=0$

Let $x-\frac{3}{x}=u\,,u^2=x^2+\frac{9}{x^2}-6 \;\Rightarrow x^2+\frac{9}{x^2}=u^2+6$

$\Rightarrow u^2+6-4u-2=0 \;\, or\; u^2-4u+4=0$

$\Rightarrow (u-2)^2=0 \;\therefore u=2$

$\Rightarrow x-\frac{3}{x}=2 \;\; or \;\; x^2-2x-3=0$

$\Rightarrow (x+1)(x-3)=0 \;\therefore x=-1\,,3$

Task

Solve the equation: $x^2+\frac{9}{x^2}-4\left(x-\frac{3}{x}\right)-6=0$

EQUATIONS INVOLVING SQUARE ROOTS

Steps followed:

1. Isolate one square root on either L.H.S or R.H.S of the equation and square both sides
2. If there is another remaining square root that persists, isolate it again and square both sides
3. Continue to square both sides until there is no more square root, then solve the resulting equation.

NOTE:

These equations yield *extraneous* roots which do not satisfy the equation being solved. Thus for correct roots, test the roots in the original equation, that which satisfy is the correct root.

TYPE A: $\boldsymbol{P\sqrt{bx+c}+Q\sqrt{mx+n}+R\sqrt{\alpha x+\beta}=k,}$ where $P\,,Q\,,R$ are constants.

Case I: For $R=0\,.i.e.$ $\boldsymbol{P\sqrt{bx+c}+Q\sqrt{mx+n}=k}$

Example:

Solve the equations:

a) $\sqrt{x+1}+\sqrt{x-2}=3$

b) $\sqrt{3x-3}-\sqrt{x}=1$

c) $\sqrt{x-5}+\sqrt{x}=5$

d) $\sqrt{x-1}+2\sqrt{x-4}=4$

Solution

a) $\sqrt{x+1}+\sqrt{x-2}=3$; let $A=x+1\,,B=x-2$

$\Rightarrow \sqrt{A}+\sqrt{B}=3$

Isolating $\sqrt{A}$ and squaring both sides

$\Rightarrow \left(\sqrt{A}\right)^2=\left(3-\sqrt{B}\right)^2$

$\Rightarrow A=9-6\sqrt{B}+B$

Isolating $\sqrt{B}$ and squaring both sides

$\Rightarrow 6\sqrt{B}=9+B-A$

$\Rightarrow 6\sqrt{x-2}=9+x-2-x-1$

$\Rightarrow 6\sqrt{x-2}=6 \;\; or \; \sqrt{x-2}=1$

$\therefore \left(\sqrt{x-2}\right)^2=1 \Rightarrow x-2=1$

$\therefore x=3$

Testing $x=3$ in to the original equation,

$\sqrt{3+1}+\sqrt{3-2}=3=R.H.S$

$\therefore x=3$ is a root.

b) $\sqrt{3x-3}-\sqrt{x}=1$; let $A=3x+3$

$\Rightarrow \sqrt{A}+\sqrt{x}=1$

Isolating $\sqrt{A}$ and squaring both sides

$\Rightarrow \left(\sqrt{A}\right)^2=\left(1+\sqrt{x}\right)^2$

$\Rightarrow A=1+2\sqrt{x}+x$

$\Rightarrow 3x-3=1+2\sqrt{x}+x$

Now $2\sqrt{x}=2x-4 \;\; or \; \sqrt{x}=x-2$

Squaring both sides again,

$\Rightarrow x=(x-2)^2=x^2-4x+4$

$\Rightarrow x^2-5x+4=0 \;\; or \;\; (x-1)(x-4)=0$

$\Rightarrow x=1\,,4$

Testing the values,

For $x=1\,,\sqrt{3-3}-\sqrt{1}=0\neq 1$

Hence $x=1$ is not a correct root. $i.e.$ it is an extraneous root.

For $x=4\,,\sqrt{12-3}-\sqrt{4}=1=\mathrm{R.H.S.}$

Only $x=4$ is the correct root.

c) $\sqrt{x-5}+\sqrt{x}=5$, let $B=x-5$

$\Rightarrow \sqrt{B}+\sqrt{x}=5$

Isolating $\sqrt{B}$ and squaring both sides

$\Rightarrow \left(\sqrt{B}\right)^2=\left(5-\sqrt{x}\right)^2$

$\Rightarrow B=25-10\sqrt{x}+x$

$\Rightarrow x-5=25-10\sqrt{x}+x \;\; or \;\; 10\sqrt{x}=30$

$\Rightarrow \sqrt{x}=3$, squaring both sides,

$\Rightarrow x=9$

Testing $x=9,\sqrt{9-4}+\sqrt{9}=5=\mathrm{R.H.S}$

$\therefore x=9$ is the correct root.

d) $\sqrt{x-1}+2\sqrt{x-4}=4$, let $x-1=A\,,x-4=B$

$\Rightarrow \sqrt{A}+2\sqrt{B}=4$

Isolating $2\sqrt{B}$ and squaring both sides,

$\Rightarrow \left(2\sqrt{B}\right)^2 = \left(4-\sqrt{A}\right)^2$

$\Rightarrow 4B = 16 - 8\sqrt{A} + A$

Isolating $8\sqrt{A}$ and squaring both sides,

$\Rightarrow \left(8\sqrt{A}\right)^2 = (A - 4B + 16)^2$

$\Rightarrow 64A = (x - 1 - 4(x-4) + 16)^2$

$\Rightarrow 64(x-1) = (x - 1 - 4x + 16 + 16)^2 = (-3x+31)^2$

$\Rightarrow 64x - 64 = 9x^2 - 186x + 961 \quad or \quad 9x^2 - 250x + 1025 = 0$

$\Rightarrow x = \frac{250 \pm \sqrt{(-250)^2 - 4\times 9\times 1025}}{18} = \frac{250\pm 160}{18}$

$\Rightarrow x = \frac{205}{9}, 5$

Testing the values,

For $x = 5, \sqrt{5-1} + 2\sqrt{5-4} = 4 = \text{R.H.S}$

Hence $x = 5$ is root.

For $x = \frac{205}{9}, \sqrt{\left(\frac{205}{9} - 1\right)} + 2\sqrt{\left(\frac{205}{9} - 4\right)} = \frac{40}{3} \neq \text{R.H.S.}$

Only $x = 5$ is the correct root.

Task

Solve the equations:

a) $2\sqrt{2x-12} - \sqrt{2x-3} = 3$ **ANSWER**: $14 \; not \; 6$

b) $2\sqrt{x+4} - \sqrt{x-1} = 4$ **ANSWER**: $5 \; not \; 1\frac{4}{9}$

c) $\sqrt{x-5} + 2 = \sqrt{x+7}$ **ANSWER**: 9

d) $\sqrt{3x+4} - \sqrt{x-3} = 4$ **ANSWER**: $4 \;, 7$

Case II: For $R \neq 0 \;. i.e. \; \boldsymbol{P\sqrt{bx+c} + Q\sqrt{mx+n} + R\sqrt{\alpha x + \beta} = k}$

Example:

Solve the equations:

a) $2\sqrt{x+1} - 3\sqrt{2x-5} = \sqrt{x-2}$

b) $\sqrt{3-x} - \sqrt{7+x} = \sqrt{16+2x}$

c) $\sqrt{x+6} - \sqrt{x+3} = \sqrt{2x+5}$

Solution

a) $2\sqrt{x+1} - 3\sqrt{2x-5} = \sqrt{x-2}$

Let $x+1 = A, 2x-5 = B, x-2 = C$

$\Rightarrow 2\sqrt{A} - 3\sqrt{B} = \sqrt{C}$, squaring both sides,

$\Rightarrow \left(2\sqrt{A} - 3\sqrt{B}\right)^2 = C$

$\Rightarrow 4A + 9B - 12\sqrt{AB} = C$

Isolating $12\sqrt{AB}$ and squaring both sides,

$\Rightarrow \left(12\sqrt{AB}\right)^2 = (4A + 9B - C)^2$

$\Rightarrow 144AB = (4A + 9B - C)^2$, substituting for A, B and C

$\Rightarrow 144(x+1)(2x-5) = \left(4(x+1) + 9(2x-5) - (x-2)\right)^2$

$\Rightarrow 144(2x^2 - 3x - 5) = \left(4x + 4 + 18x - 45 - x + 2\right)^2$

$\Rightarrow 288x^2 - 432x - 720 = (21x - 39)^2 = 441x^2 - 1638x + 1521$

$\Rightarrow 153x^2 - 1206x + 2241 = 0 \quad or \quad 17x^2 - 134x + 249 = 0$

$\Rightarrow x = \frac{134 \pm \sqrt{(134)^2 - 4\times17\times249}}{34} = \frac{134 \pm 32}{34}$

$\Rightarrow x = 3\,, \frac{83}{17}$

Testing the values,

For $x = 3$, $2\sqrt{3+1} - 3\sqrt{6-5} = \sqrt{3-2} = 1$, hence $x = 3$ is a root

For$x = \frac{83}{17}$, $2\sqrt{\frac{83}{17}+1} - 3\sqrt{\frac{166}{17}-5} \neq \sqrt{\frac{83}{17}-2} = 1$, hence $x = \frac{83}{17}$ is not a root

$\therefore x = 3$ is the only correct root.

b) $\sqrt{3-x} - \sqrt{7+x} = \sqrt{16+2x}$

Let $3 - x = A\,, 7 + x = B\,, 16 + 2x = C$

$\Rightarrow \sqrt{A} - \sqrt{B} = \sqrt{C}$, squaring both sides,

$\Rightarrow \left(\sqrt{A} - \sqrt{B}\right)^2 = C$

$\Rightarrow A + B - 2\sqrt{AB} = C$

Isolating $2\sqrt{AB}$ and squaring both sides,

$= (A + B - C)^2 = 4AB$, substituting for A, B and C

$\Rightarrow 4(3-x)(7+x) = \left(3 - x + 7 + x - (16+2x)\right)^2$

$\Rightarrow 4(21 - 4x - x^2) = (-6 - 2x)^2$

$\Rightarrow 84 - 16x - 4x^2 = 36 + 24x + 4x^2$

$\Rightarrow 8x^2 + 40x - 48 = 0 \;\; or\; x^2 + 5x - 6 = 0$

$\Rightarrow (x+6)(x-1) = 0 \Rightarrow x = 1\,, -6$

Testing the values,

For $x = 1$, $\sqrt{3-1} - \sqrt{7+1} \neq \sqrt{16+2}$, hence $x = 1$ is not a root.

For $x = -6$, $\sqrt{3+6} - \sqrt{7-6} = \sqrt{16-12}$, hence $x = 2$ is a root.

$\therefore x = 2$ is the correct root.

c) Left as an exercise. **ANSWER**: $x = -2$

TYPE B: Any equation which can be thrown in to the form

$\boldsymbol{ax^2 + bx + c + p\sqrt{ax^2 + bx + c} = q}$

This is solved by putting $y = \sqrt{ax^2 + bx + c}$

$\Rightarrow y^2 = ax^2 + bx + c$

$\therefore$ the equation being solved is $y^2 + py - q = 0$

Let α and β be roots of this equation.

$\Rightarrow \alpha = \sqrt{ax^2 + bx + c}\,,\; \beta = \sqrt{ax^2 + bx + c}$

From these equations we obtain *four* values of x.

NOTE:

If *no sign* is prefixed to *a radical* it is usually understood that it is to be taken as a *positive*. Hence if both α and β are both *positive*, all the four values of x satisfy the original equation.

If however, α or β is *negative*, the roots found from the resulting quadratic equation will satisfy the equation $\boldsymbol{ax^2 + bx + c - p\sqrt{ax^2 + bx + c} = q}$ but not the original equation.

Example

Solve the equations:

a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$

c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$

d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$

e) $3x^2 - 18 + \sqrt{3x^2 - 4x + 6} = 4x$

Solution

a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

Adding 3 on both sides

$\Rightarrow x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} = 15$

Now let $y = \sqrt{x^2 - 5x + 3} \Rightarrow y^2 = x^2 - 5x + 3$

$\Rightarrow y^2 + 2y = 15 \;\; or \; y^2 + 2y - 15 = 0$

$\Rightarrow (y + 5)(y - 3) = 0 \therefore y = -5, 3$

But $y = \sqrt{x^2 - 5x + 3}$

$\Rightarrow \sqrt{x^2 - 5x + 3} = -5$ .... (1)

$\Rightarrow \sqrt{x^2 - 5x + 3} = 3$ .... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 - 5x + 3 = 25 \;\; or \; x^2 - 5x - 22 = 0$

$\therefore x = \frac{5 \pm \sqrt{25+88}}{2} = \frac{5 \pm \sqrt{113}}{2}$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 - 5x + 3 = 9 \;\; or \; x^2 - 5x - 6 = 0$

$\Rightarrow (x - 6)(x + 1) = 0 \;\; \therefore x = -1, 6$

From the discussion above,

$x = -1, 6$ satisfy the equation $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $2\sqrt{x^2 - 5x + 3}$ is *positive* and $y = +3$

But for $x = \frac{5 \pm \sqrt{113}}{2}$ satisfy the equation $x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $-2\sqrt{x^2 - 5x + 3}$ is *negative* and $y - 5$

b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$, re arranging

$\Rightarrow x^2 + 6x + 2\sqrt{x^2 + 6x} = 24$

Let $y = \sqrt{x^2 + 6x} \Rightarrow y^2 = x^2 + 6x$

$\Rightarrow y^2 + 2y = 24 \;\; or \; y^2 + 2y - 24 = 0$

$\Rightarrow (y + 6)(y - 4) = 0 \therefore y = -6, 4$

But $y = \sqrt{x^2 - 5x + 3}$

$\Rightarrow \sqrt{x^2 - 5x + 3} = 4$ .... (1)

$\Rightarrow \sqrt{x^2 - 5x + 3} = -6$ .... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 + 6x = 16 \;\; or \; x^2 + 6x - 16 = 0$

$\Rightarrow (x + 8)(x - 2) = 0 \;\; \therefore x = -8, 2$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 + 6x = 36 \;\; or \; x^2 + 6x - 36 = 0$

$\therefore x = \frac{-6 \pm \sqrt{36+4\times36}}{2} = -3 \pm 3\sqrt{5}$

$\therefore$ Roots to $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$ are $\therefore x = -8, 2$ for $y = +4$ and roots of

$x^2 - 2\sqrt{x^2+6x} = 24 - 6x$ are $-3 \pm 3\sqrt{5}$ for $y = -4$. The roots of the latter equation are extraneous.

c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$
Subtracting 3 from both sides,
$\Rightarrow x^2 - 6x + 6 = 4\sqrt{x^2 - 6x + 6} - 3$
Now let $y = \sqrt{x^2 - 6x + 6} \Rightarrow y^2 = x^2 - 6x + 6$
$\Rightarrow y^2 = 4y - 3 \;\; or\; y^2 - 4y + 3 = 0$
$\Rightarrow (y -)(y - 3) = 0 \therefore y = 1\,,3$
But $y = \sqrt{x^2 - 6x + 6}$
$\Rightarrow \sqrt{x^2 - 6x + 6} = 1$ .... (1)
$\Rightarrow \sqrt{x^2 - 6x + 6} = 3$ .... (2)
From eqn. (1), squaring both sides,
$\Rightarrow x^2 - 6x + 6 = 1 \;\; or\; x^2 - 6x + 5 = 0$
$\Rightarrow (x - 5)(x - 1) = 0 \;\therefore x = 1\,,5$
From eqn. (2), squaring both sides,
$\Rightarrow x^2 - 6x + 6 = 9 \;\; or\; x^2 - 6x - 3 = 0$
$\therefore x = \frac{6 \pm \sqrt{36+4\times 3}}{2} = 3 \pm 2\sqrt{3}$
$\therefore x = 1\,,5\,,3 \pm 2\sqrt{3}$ satisfy the original equation.

d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$, re arranging the equation,
$x^2 + 3x - 40 + 10\sqrt{x^2 + 3x + 16} = 0$
Adding 56 on both sides,
$\Rightarrow x^2 + 3x + 16 + 10\sqrt{x^2 + 3x + 16} = 56$
Let $y = \sqrt{x^2 + 3x + 16} \Rightarrow y^2 = x^2 + 3x + 16$
$\Rightarrow y^2 + 10y = 56 \;\; or\; y^2 + 10y - 56 = 0$
$\Rightarrow (y + 14)(y - 4) = 0 \therefore y = 4\,,-14$
But $y = \sqrt{x^2 + 3x + 16}$
$\Rightarrow \sqrt{x^2 + 3x + 16} = 4$ .... (1)
$\Rightarrow \sqrt{x^2 + 3x + 16} = -14$ .... (2)
From eqn. (1), squaring both sides,
$\Rightarrow x^2 + 3x + 16 = 16 \;\; or\; x^2 + 3x = 0$
$\Rightarrow x(x + 3) = 0 \;\therefore x = 0\,,-3$
From eqn. (2), squaring both sides,
$\Rightarrow x^2 + 3x + 16 = 196 \;\; or\; x^2 + 3x - 180 = 0$
$\therefore x = \frac{-3 \pm \sqrt{9+4\times 180}}{2} = \frac{3 \pm 27}{2} \Rightarrow x = -15\,,12$
Since $y = +4$ and $10\sqrt{x^2 + 3x + 16}$ is also $+ve$, then $x = 0\,,-3$ satisfy the equation $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$.
Since $y = -14$ and $10\sqrt{x^2 + 3x + 16}$ is $+ve$, then $x = -15\,,12$ satisfy the equation $x^2 + x - 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$. These are extraneous roots.

e) Left as an exercise.
ANSWER: $x = 3\,,-\frac{5}{3}\; and\; \frac{2 \pm \sqrt{70}}{3}$

NOTE:

- In general some equations of degree 4^0 can be converted in to a quadratic
- Before clearing an equation of a radical, it is advisable whether any common factor can be removed by division.

Example:

Solve: $2\sqrt{x^2-x-2}-3\sqrt{2x^2-9x+10}=x-2$

$\Rightarrow 2\sqrt{(x+1)(x-2)}-3\sqrt{(2x-5)(x-2)}=x-2$

Dividing through by $\sqrt{x-2}$

$\Rightarrow 2\sqrt{x+1}-3\sqrt{2x-5}=\sqrt{x-2}$, which is as case II for $R\neq 0$

Solving the equation, $x=3$.

NOTE:

When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of a quadratic equation as illustrated below.

Example:

Solve the equation:

i) $(1-a^2)(x+a)-2a(1-x^2)=0$

ii) $\frac{(1-x)^2}{2-x^2}=\frac{(1-a)^2}{2-a^2}$

Solution

i) $(1-a^2)(x+a)-2a(1-x^2)=0$

By inspection, clearly $x=a$ is a root to the equation.

Re writing the equation,

$\Rightarrow x+a-a^2x-a^3-2a+2ax^2=0$

$\therefore 2ax^2+(1-a^2)x-a(1+a^2)=0$

Let α and β be roots of this equation, then

$\Rightarrow \alpha\beta=-\frac{a(1+a^2)}{2a}=-\frac{(1+a^2)}{2}$

Since one root of the equation is a $.i.e.$ say $\alpha=a$, then the other root β is;

$\beta==-\frac{(1+a^2)}{2\alpha}=-\frac{(1+a^2)}{2a}$

ii) Left as an exercise.

Task

Solve the equation:

a) $\sqrt{4x^2-7x-15}-\sqrt{x^2-3x}=\sqrt{x^2-9}$ **ANSWER**: $1\,,3$

b) $\sqrt{2x^2-9x+4}+3\sqrt{2x-1}=\sqrt{2x^2+21x-11}$ **ANSWER**: $5\,,\frac{1}{2}$

c) $\sqrt{x^2-7ax+10a^2}-\sqrt{x^2+ax-6a^2}=x-2a$ **ANSWER**: $2a\,,-\frac{10a}{3}$

SERRIES
Arithmetic and Geometric Progressions
$(\mathbf{A.ps}$ and $\mathbf{G.ps})$

Basic concepts:

a) A *Sequence* is a set of quantities/numbers $a_1, a_2, a_3 \dots\dots\dots\dots$, stated in a definite order and each term formed according to a fixed pattern.
$i.e. a_r = f(r)$.
Example
- $1, 3, 5, 7 \dots\dots\dots\dots$, is a sequence (the next term would be 9)
- $2, 6, 18, 54 \dots\dots\dots$, is a sequence (the next term would be 162)
- $1^2, -2^2, 3^2, -4^2 \dots\dots\dots$, is a sequence (the next term would be $+5^2$)

i) A *finite sequence* contains only a finite number of terms.
ii) An *infinite sequence* is unending.

NOTE:
The numbers in a sequence are called *terms.*

b) A *series* is formed by the *sum* of the terms of a sequence.
Example
For $1, 3, 5, 7 \dots\dots\dots\dots$ is a sequence
$1 + 3 + 5 + 7 \dots\dots\dots\dots$ is a series.

c) **The *Arithmetic Progression (A.P)***
A series in which each term is obtained from the proceeding one by adding or subtracting a constant quantity is called the *Arithmetic Progression.*
Example
- $1 + 3 + 5 + 7 \dots\dots\dots\dots\dots\dots\dots\dots\dots$ *is an A.P*
- $3 - 2 - 7 - \cdots \dots\dots\dots\dots\dots\dots\dots\dots$ *is an A.P*
- $a + (a + d) + (a + 2d) + \cdots \dots\dots\dots$ *is an A.P*

The constant quantity is called *a common difference, d.*
Consider the A.P below;

$$\boxed{a_1 + a_2 + a_3 + \cdots \dots\dots\dots + a_{n-1} + a_n}$$

This is an A.P if the common difference

$$\boxed{d = a_n - a_{n-1}, \text{ for all } n > 1}$$

From the above examples of series,
- $1 + 3 + 5 + 7 \dots\dots\dots\dots, d = 3 - 1 = 5 - 3 = 7 - 5 = 2$
- $3 - 2 - 7 - \cdots \dots\dots\dots, d = -2 - 3 = -7 - -2 = -5$
- $a + (a + d) + (a + 2d) + \cdots \dots\dots\dots$,
$common\ difference = (a + d) - a = (a + 2d) - (a + d) = d$

To find General term$(n^{th} - term)$ of an A.P.
Let $a_1 + a_2 + a_3 + \cdots \dots\dots\dots$, be an A.P.
Let $d = common\ difference$, then
$d = a_n - a_{n-1}$, for all $n > 1$
$\Rightarrow a_n = d + a_{n-1}$
Now for $n = 2, a_2 = d + a_1$
$n = 3,\ a_3 = d + a_2 = d + (d + a_1) = a_1 + 2d$

$$n = 4\,, a_4 = d + a_3 = d + (2d + a_1) = a_1 + 3d$$

Observation of the terms;

$a_2 = a_1 + d = a_1 + (2-1)d$

$a_3 = a_2 + d = a_1 + (3-1)d$

$a_4 = a_3 + d = a_1 + (4-1)d$

$\vdots$

$a_n = a_{n-1} + d = a_1 + (n-1)d$

Thus the general term of an A.P is

$$\boxed{\boldsymbol{a_n = a + (n-1)d}}$$

From the formula,

$a = 1^{st}\ term$

$n = no.of\ terms$

$a_n = last\ term$

$d = common\ difference$

NOTE:

1. The general term of an A.P is $\boldsymbol{a_n = a_1 + (n-1)d}$**,** has four terms. If the *three* are given then the 4^{th} can be evaluated.
2. From the general term of an A.P, the standard form of an A.P is stated as;

$$\boxed{\boldsymbol{a + (a+d) + (a+2d) + \cdots \ldots.\ldots\ldots + [a + (n-1)d]}}$$

3. *The most convenient terms in an A.P with given sum and product:*
   a) If **three terms** are in an A.P, then use

   $$\boxed{\boldsymbol{(a-d)\,, a\,, (a+d)}}$$

   b) If **four terms** are in an A.P, then use

   $$\boxed{\boldsymbol{(a-3d)\,,\ (a-d)\,, (a+d)\,,\ (a-3d)}}$$

NOTICE:

a) If you sum up the terms, the common difference**, d,** vanishes, and the sum is in terms of only the first term**, a.**
b) In case of **odd number** of terms, the common difference is taken as **d** and the middle term as **a.**
c) In case of **even number** of terms, the common difference is taken as **2d** and the two middle terms are($\boldsymbol{a-d}$) *and* ($\boldsymbol{a+d}$) **.**

Properties of an A.P

a) If the constant is *added* to or *subtracted* from every term in an A.P, the resulting series is an A.P.
b) If every term is *multiplied* or *divided* by a constant, the resulting series is also in an A.P.

Examples

1. *Find the* 17^{th} *− term of the series*
   a) $4+6+8+\cdots\ldots\ldots\ldots.$
   b) $-8-7-6-5-\cdots\ldots..$

Solution

a) $4+6+8+\cdots\ldots\ldots\ldots.$
Since $d=6-4=8-6=2$, the series is an A.P
Using $a_n=a+(n-1)d$
Given: $a=4\,, n=17\,, d=2$
Required: a_n
$\Rightarrow a_n=4+(17-1)\times 2=36$
$\therefore$ *the* 17^{th} *− term is* 36

b) $-8-7-6-5-\cdots\ldots..$
$d=-7-\ -8=-6-\,-7=1$
Given: $a=-8\,, n=17\,, d=1$
Required: a_{17}
$\Rightarrow a_{17}=-8+(17-1)\times 1=8$
$\therefore$ *the* 17^{th} *− term is* 8

2. *State the* 20^{th} *− term and the number of terms of the following series;*
   a) $2+4+6+\cdots\ldots\ldots\ldots..+100$
   b) $-8-6-4-\cdots\ldots.+200$

Solution

a) $2+4+6+\cdots\ldots\ldots\ldots..+100$
Given: $a=2\,, n=?\,, d=4-2=2\,, a_n=100$
Required: $a_{20}=?\,, n=?$
Using $a_n=a+(n-1)d$
$100=2+(n-1)\times 2\ \ or\ 50=1+n-1$
$\therefore n=50$
Now $a_{20}=a+(20-1)d=a+19d$
$a_{20}=2+19\times 2=40$
$\therefore$ *No. of terms is* 50 , 20^{th} *− term is* 40

b) $-8-6-4-\cdots\ldots.+200$
Given: $a=-8\,, n=?\,, d=-6-\,-8=2\,, a_n=200$
Required: $a_{20}=?\,, n=?$
Using $a_n=a+(n-1)d$
$200=-8+(n-1)\times 2\ \ or\ 100=-4+n-1$
$\therefore n=105$
Now $a_{20}=a+(20-1)d=a+19d$
$a_{20}=-8+19\times 2=30$
$\therefore$ *No. of terms is* 105 , 20^{th} *− term is* 30

3. *The* 3^{rd} *− term of an A.P is 18 and the* 7^{th} *− term is 30. Find the* 17^{th} *− term.*

Solution

Using $a_n=a+(n-1)d$
Given: $a_3=18\,,\ a_7=30$
Required: $a_{17}=?$

$\Rightarrow a_3 = a + (3-1)d = a + 2d = 18$........................ (1)
$\Rightarrow a_7 = a + (7-1)d = a + 6d = 30$ .... (2)
Solving the equations simultaneously
$Eqn. (1) - eqn. (2)$
$\Rightarrow -4d = -12\,, d = 3$
$Subst.\ d = 3$ in to eqn. (1) or (2)
For $a + 2d = 18 \Rightarrow a = 18 - 2 \times 3 = 12$
Now $a_{17} = a + (17-1)d = a + 16d = 12 + 16 \times 3 = 60$
$\therefore the\ 17^{th} - term\ is\ 60$

4. *The* 3^{rd} *and* 13^{th} *term of an A.P are respectively equal to* -40 *and* 0. *Find the A.P and its* 20^{th} *– term.*
Solution
Using $a_n = a + (n-1)d$
Given: $a_3 = -40\,,\ a_{13} = 0$
Required: $a_{20} = ?$
$\Rightarrow a_3 = a + (3-1)d = a + 2d = -40$........................ (1)
$\Rightarrow a_{13} = a + (13-1)d = a + 12d = 0$ .... (2)
Solving the equations simultaneously
$Eqn. (1) - eqn. (2)$
$\Rightarrow -10d = -40\,,\ d = 4$
$Subst.\ d = 4$ in to eqn. (1) or (2)
For $a + 2d = -40 \Rightarrow a = -40 - 2 \times 4 = -48$
Now $a_{20} = a + (20-1)d = a + 19d = -48 + 19 \times 4 = -12$
$\therefore the\ 20^{th} - term\ is - 12$
The A.P is $a\,, (a+d)\,, (a+2d)\,, \ldots \ldots . \ldots$
$\therefore the\ A.P\ is\ -48\,, -44\,, -40\,, \ldots \ldots \ldots \ldots, 28$

5. *In an A.P* $u_1 + u_2 + \cdots \ldots \ldots \ldots + u_4 = 15$ *and* $u_{11} = -3$. *Find the greatest integer* N *such that* $u_N \geq 0$
Solution
Using $a_n = a + (n-1)d$
$u_1 = a\,, u_2 = a + d\,, u_3 = a + 2d\,, u_4 = a + 3d$
$\Rightarrow a + a + d + a + 2d + a + 3d = 15$
$\therefore 4a + 6d = 15$ .... (1)
Given: $u_{16} = -3 \ \Rightarrow a + 15d = -3$ .... (2)
Solving (1) and (2) simultaneously,
From (1) $a = -3 - 15d$ .... (3)
Eqn. (3) in to (1)
$\Rightarrow 4[-3 - 15d] + 6d = 15$
$\Rightarrow -12 - 60d + 6d = 15\ \ or\ d = -\frac{27}{54} = -\frac{1}{2}$
$Subst.\, d = -\frac{1}{2}\ in\ to\ (3)$
$\Rightarrow a = -3 - 15\left(-\frac{1}{2}\right) = \frac{-6+15}{2} = \frac{9}{2}$
Now $u_N = a + (N-1)d \geq 0$
$\Rightarrow \frac{9}{2} + (N-1) \times \left(-\frac{1}{2}\right) \geq 0$.......................$(\times 2)$
$\Rightarrow 9 + (1 - N) \geq 0\ or\ 10 \geq N$
$\therefore the\ greatest\ integer\ is\ 10$

6. *The* n^{th} *term of a series is* $U_n = a\,3^n + b\,n + c$. *Given that* $U_1 = 4\,, U_2 = 13$ *and* $U_3 = 46$. *Find the values of a, b and c.*

Solution

Given: $U_n = a\,3^n + b\,n + c$

For $U_1 = 4\,, n = 1 \Rightarrow 4 = 3a + b + c$ .... (1)

For $U_2 = 13\,, n = 2 \Rightarrow 13 = 9a + 2b + c$ .... (2)

For $U_3 = 46\,, n = 3 \Rightarrow 46 = 27a + 3b + c$ .... (3)

Solving (1), (2) and (3) simultaneously

$eqn.(1) - eqn.(2)$, to eliminate c

$\Rightarrow -9 = -6a - b$ *or* $6a + b = 9$ .... (4)

$eqn.(1) - eqn.(3)$, to eliminate c

$\Rightarrow -42 = -24a - 2b$ *or* $12a + b = 21$ .... (5)

Now eliminating b from (4) and (5)

$eqn.(4) - eqn.(5)$, to eliminate b

$\Rightarrow -6a = -12 \;\therefore a = 2$

$Subst.\, a = 2\; in\; to\; eqn.(5)\; or\; (4)$

$6a + b = 9\,, a = 2 \;\Rightarrow b = 2 - 6 \times 2 = -3$

$Subst.\, a = 2\,, b = -3\; in\; to\; (1) or\; (2) or\; (3)$

From $\Rightarrow c = 4 - (3a + b) = 4 - (6 - 3) = 1$

$\therefore a = 2\,, b = -3\,, c = 1$

Examples on sum and product of a certain number of terms in an A.P

1. *Find three terms in an A.P such that;*
   a) Their sum is 21 and the product is 315
   b) Their sum is 33 and the sum of their squares is 563

Solution

Let the numbers be $a - d\,, a\; a + d$

a) *Given:* $sum =\; 21,\; a - d + a + a + d = 21$

$\Rightarrow 3a = 21 \;\therefore a = 7$

$Product = 315\,, a\,(a - d)(a + d) = 315$

$\Rightarrow a(a^2 - d^2) = 315$

$\Rightarrow 7(7^2 - d^2) = 315\;$ *or* $49 - d^2 = 45$

$\therefore d = \sqrt{4} = \pm 2$

The numbers are: $[7 - (\pm 2)]\,, 2\,, [7 + (\pm 2)]$

$\therefore 5\,, 7\,, 9\,,\; for\; d = +2$

$\therefore 9\,, 7\,, 5\,,\; for\; d = -2$

b. $sum =\; 33,\; a - d + a + a + d = 33$

$\Rightarrow 3a = 33 \;\therefore a = 11$

$sum\; of\; squares = 563, (a - d)^2 + a^2 + (a + d)^2 = 563$

$\Rightarrow (11 - d)^2 + 11^2 + (11 + d)^2 = 563$

$\Rightarrow 121 - 22d + d^2 + 121 + 121 + 22d + d^2 = 563$

$\Rightarrow 2d^2 + 363 = 563\,, d^2 = 100 \;\therefore d = \pm 10$

Now numbers are:

For $d = 10\,, a = 11 \Rightarrow (11 - 10)\,, 11\,, (11 + 10)\; or\; 1\,, 11\,,\; 21$

For $d = -10\,, a = 11 \Rightarrow (11 - -10)\,, 11\,, (11 - 10)\; or\; 21\,, 11\,, 1$

2. *Find four numbers in an A.P such that their sum is 20 and sum of the squares is 120.*

Solution
Let the numbers be: $(a-3d)\,,(a-d)\,,(a+d)\,,(a+3d)$
$sum = 20 \Rightarrow a-3d+a-d+a+d+a+3d = 20$
$\Rightarrow 4a = 20\ \therefore a = 5$
$sum\ of\ squares = 120\,,(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$
$\Rightarrow (5-3d)^2+(5-d)^2+(5+d)^2+(5+3d)^2 = 120$
$\Rightarrow 25-30d+9d^2+25+30d+9d^2+25-10d+d^2+25+10d+d^2 = 120$
$\Rightarrow 100+20d^2 = 120\ \therefore d = \pm 1$
The numbers are:
For $d=1\,,a=5\,,(5-3)\,,(5-1)\,,(5+1)\,,(5+3)$
$\therefore 2\,,4\,,6\,,8$
For $d=-1\,,a=5\,,(5+3)\,,(5+1)\,,(5-1)\,,(5-3)$
$\therefore 8\,,6\,,4\,,2$

3. *Find five numbers in an A.P such that;*
   a) Their sum is 20 and product of the first and last term is 15
   b) Their sum is 25 and sum of their squares is 135

Solution

a) Let the numbers be: $a-2d\,,a-d\,,a\,,a+d\,,a+2d$
Now sum= $20 = a-2d+\ a-d+\ a+a+d+a+2d$
$\Rightarrow 20 = 5a\ \therefore a = 4$
Product of the first and last= $(a-2d)\times(a+2d\,) = 15$
$\Rightarrow (4-2d)\times(4+2d\,) = 15\ \ or\ 16-4d^2 = 15$
$\Rightarrow d = \pm\frac{1}{2}$
$\therefore numbers\ are{:}\,3\,,\ 3\frac{1}{2}\,,4\,,\ 4\frac{1}{2}\ and\ 5$

b) $Sum = 5a = 25\ \therefore a = 5$
$Sum\ of\ squares = 135 = (a-2d)^2+\ (a-d)^2+\ a^2+(a+d)^2+(a+2d)^2$
$135 = a^2-4ad+4d^2+a^2-2ad+d^2+\ a^2+a^2+2ad+d^2+a^2+4ad+4d^2$
$\Rightarrow 135 = 5a^2+10d^2$, but $a = 5$
$\Rightarrow 135 = 5\times 5^2+10d^2\ or\ 135 = 125+10d^2$
$\therefore d = \pm 1$
$\therefore numbers\ are{:}\,3\,,4\,,5\,,6\ and\ 7$

Task

1. *Find three numbers in an A.P such that*
   a) Sum is 27 and product is 648
   b) Sum is 27 and sum of their squares is 275
   c) Sum is 12 and sum of their cubes is 408
   d) Sum is 15 and sum of squares of its first and third term is 58
   e) Sum is 9 and the sum of their squares is 77
   f) Sum is 6 and their product is -90

ANSWER: $(\boldsymbol{a})\ \mathbf{6\,,9\,,12\,,}\boldsymbol{d}=\pm\mathbf{3}\,,\boldsymbol{a}=\mathbf{9}\ (\boldsymbol{b})\ \mathbf{5\,,9\,,13\,,}\boldsymbol{d}=\pm\mathbf{4}\,,\boldsymbol{a}=\mathbf{9}\ (\boldsymbol{c})\ \mathbf{1\,.4\,,7\,,}\boldsymbol{d}=\pm\mathbf{3}\,,\boldsymbol{a}=\mathbf{4}$
$(\boldsymbol{d})\ \mathbf{3\,,5\,,7}\ (\boldsymbol{e})-\mathbf{2\,,3\,,8}\ \ (\boldsymbol{f})-\mathbf{5\,,2\,,9}$

2. *The sum of four integers of an A.P is 24 and their product is 945. Find the integers.*
   ANSWER: $\mathbf{3\,,5\,,7\,,9}$ **Hint**: $Use\ (a-3d)\,,(a-d)\,,(a+d)\,,(a+3d)$

PROOFS IN A.PS

1) *If a , b and c are respectively the p^{th} , q^{th} and r^{th} terms of an A.P, prove that* $a(q-r)+b(r-p)+c(p-q)=0$

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $U_p = a\,, U_q = b\,, U_r = c$

$\Rightarrow a = A + (p-1)D$ ………………………. (1)

$\Rightarrow b = A + (q-1)D$ ………………………. (2)

$\Rightarrow c = A + (r-1)D$ ………………………. (3)

Note:

A and D are the unknowns, hence solving for them,

$eqn.(1) - eqn.(2)$

$\Rightarrow a - b = [p-1-(q-1)]D = (p-q)D$

$\Rightarrow D = \frac{a-b}{p-q}$ …………………………………………………………. (4)

$eqn.(4)\, in\ to\ (1)\, to\ find\ A$

From (1), $A = a-(p-1)D$

$\Rightarrow A = a-(p-1)\left(\frac{a-b}{p-q}\right) = \frac{ap-aq-(ap-pb-a+b)}{p-q}$

$\therefore A = \frac{-aq+pb+a-b}{p-q}$ ……………………………………………… (5)

$Subst.(4)\, and\ (5)\, in\ to\ (3)$

From $c = A + (r-1)D$

$\Rightarrow c = \frac{-aq+pb+a-b}{p-q} + (r-1)\left(\frac{a-b}{p-q}\right)$

$\Rightarrow c(p-q) = -aq+pb+a-b+ar-br-a+b$

$= -aq+pb+ar-br$

$\therefore c(p-q) = a(r-q)+b(p-r)\ or\ a(q-r)+b(r-p)+c(p-q)=0$

2) *Prove that the n^{th} term is $p+q-n$ when the p^{th} term of an A.P is q and the q^{th} term is p.*

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $U_p = q\,, U_q = p$

$\Rightarrow p = A + (p-1)D$ ………………………………. (1)

$\Rightarrow q = A + (q-1)D$ ………………………………. (2)

Required: $U_n = p+q-n$ ………………………. (3)

Now solving for A and D

$eqn.(1) - eqn.(2)$, to eliminate A

$p-q = (q-p)D = -D(p-q)$

$\therefore D = -1$

From (1) and $D=-1$

$A = p-(q-1)D = p+q-1$

Required: $U_n = A+(n-1)D$

$U_n = p+q-1+(n-1)\times -1 = p+q-1-n+1 = p+q-n$

$\therefore n^{th}\ term = p+q-n$

3. *If p times the p^{th} term of an A.P is equal to q times the q^{th}, prove that the $(p+q)^{th}$ term is zero.*

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $p \times U_p = q \times U_q$

Required: to prove $U_{p+q} = 0$

$U_p = A + (p-1)D$, $U_q = A + (q-1)D$

$\Rightarrow p \times U_p = q \times U_q$ *becomes*

$p \times [A + (p-1)D] = q \times [A + (q-1)D]$ .... (1)

Now $U_{p+q} = A + (p+q-1)D$.. (2)

Expressing eqn. (2) in terms of only D

From (1), $pA + (p^2 - p)D = Aq + (q^2 - q)D$

$\Rightarrow A(p-q) = (q^2 - q)D - (p^2 - p)D$

$A(p-q) = (q^2 - p^2 - q + p)D$

$= -[p^2 - q^2 - (p-q)]D$

$= -[(p-q)(p+q) - (p-q)]D$ .... $[\div (p-q)]$

$A = -[p+q-1]D$

$\therefore A = [1 - (p+q)]D$ .... (3)

$Subst.$ (3) $in\ to$ (2) , must give zero for the proof to have been proved as required.

$U_{p+q} = A + (p+q-1)D$

$= [1 - (p+q)]D + (p+q-1)D$

$= D[1 - p - q + p + q - 1] = 0$, as required.

Task

1. *The* m^{th} *term of an A.P is* $\frac{1}{n}$ *and the* n^{th} *term is* $\frac{1}{m}$ *, prove that the* $(mn)^{th}$ *term is unity.i.e.* $U_{mn} = 1$
2. *If* p, q, r, s *are any four consecutive terms of an A.P, show that* $p^2 - 3q^2 + 3r^2 - s^2 = 0$.
   [**Hint**: *Let* $p = \alpha - 3\beta, q = \alpha - \beta, r = \alpha + \beta, s = \alpha + 3\beta$]
3. *The* 8^{th} *term of an A.P is double the* 13^{th} *term. Prove that the* 2^{nd} *term is double the* 10^{th} *term.*

Other forms of proofs in A.Ps

1. *If* a, b *and* c *are in an A.P, prove that;*
   a) $a^3 + c^3 + 6abc = 8b^3$
   b) $a^2 + 4ac + c^2 = 2(ab + bc + ca)$

Solution

For a, b and c to be in A.P, then $b - a = c - b = d$ $(common\ difference)$

$\Rightarrow b = a + d, c = b + d = a + d + d = a + 2d, 2b = a + c$

a) $\mathbf{L.H.S} = a^3 + c^3 + 6abc = a^3 + (a+2d)^3 + 6a(a+d)(a+2d)$

$= a^3 + a^3 + 6a^2d + 12ad^2 + 8d^3 + 6a(a^2 + 2ad + ad + 2d^2)$

$= 8a^3 + 6a^2d + 12ad^2 + 8d^3 + 18a^2d + 12ad^2$

$= 8a^3 + 24a^2d + 24ad^2 + 8d^3 = 8(a^3 + 3a^2d + 3ad^2 + d^3) = 8(a+d)^3 = 8b^3$

b) $\mathbf{L.H.S} = a^2 + 4ac + c^2 = a^2 + c^2 + 4ac$

$= (a+c)^2 - 2ac + 4ac = (a+c)^2 + 2ac$

$= (a+c)(a+c) + 2ac$

$= (2b)(a+c) + 2ac = 2(ab + bc + ac)$

2. *If* a, b *and* c *are in an A.P, show that*

i) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P

ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P

Solution

i) if a, b, c are in an A.P, then

$b - a = c - b$ .... (1)

Now $\frac{1}{ab}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P if $\frac{1}{ac} - \frac{1}{bc} = \frac{1}{ab} - \frac{1}{ac}$

$\Rightarrow \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{a}\left(\frac{1}{b} - \frac{1}{c}\right)$

$\Rightarrow \frac{b-a}{abc} = \frac{c-b}{abc}$

$\therefore b - a = c - b \Leftrightarrow a, b, c\ are\ in\ an\ A.P$ .hence the result follows.

ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \frac{\sqrt{b}+\sqrt{c}-\sqrt{c}-\sqrt{a}}{(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})} = \frac{\sqrt{c}+\sqrt{a}-\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}}$ $\therefore b - a = c - b \Leftrightarrow a, b, c\ are\ in\ an\ A.P$. Hence the result.

3. *If a^2, b^2 and c^2 are in an A.P, show that* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$

Solution

If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in an A.P, then $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$

$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b}$ $\therefore b^2 - a^2 = c^2 - b^2$, hence a^2, b^2, c^2 are in an A.P

Task:

a. If $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in an A.P, show that a^2, b^2, c^2 are also in an A.P

b. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in an A.P, show that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in an A.P

TO FIND THE SUM OF A FINITE NUMBER OF QUANTITIES IN AN A.P

Let $a_1, a_2, a_3, \dots\dots\dots\dots\dots, a_n$ be $n-$ quantities in an A.P, and let the last term a_n be denoted as l. If their common difference, then

$a_n = a_1 + (n-1)d = l$

Now summing up the n- terms of an A.P

$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots \dots\dots\dots + a_1 + (n-1)d$

$\quad = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots \dots\dots\dots + (l - d) + l$ .... (1)

Re-writing this sum in a reverse order

$\Rightarrow S_n = l + (l - d) + (l - 2d) + \cdots \dots\dots + (a_1 + d) + a_1$ .... (2)

Adding the equations (1) and (2)

$\Rightarrow 2S_n = (a_1 + l) + (a_1 + l) + (a_1 + l) + \cdots \dots\dots + (a_1 + l), (n - times)$

$\Rightarrow 2S_n = n(a_1 + l)$

$\therefore S_n = \frac{n}{2}(a_1 + l) = \frac{n}{2}[a_1 + a_1 + (n-1)d]$

$\quad = \frac{n}{2}[2a_1 + (n-1)d]$

Consequently, $\boxed{\boldsymbol{S_n = \frac{n}{2}[2a_1 + (n-1)d]}}$

In general, $\boxed{\boldsymbol{S_n = \frac{n}{2}[1^{st}\ \textbf{term} + \textbf{last term}]}}$

NOTE:
The above formula contains four quantities. $i.e. S_n , a , n , d$. If three of them are known the fourth can be evaluated

Examples

1. *Find the sum of the series*
   a) $3 + 8 + 13 + \cdots \ldots \ldots \ldots + 12^{th}\ term$
   b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \cdots \ldots \ldots \ldots . . +19^{th}\ term$

Solution

a) $3 + 8 + 13 + \cdots \ldots \ldots \ldots + 12^{th}\ term$
Given: $a_1 = 3 , d = 8 - 3 = 13 - 8 = 5 , n = 12 , a_{12} = ?$
Using $a_n = a_1 + (n-1)d$
$a_{12} = 3 + (12-1) \times 5 = 58$
Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow S_{12} = \frac{12}{2}[2 \times 3 + (12-1) \times 5] = 366$
Alternatively
Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term]$
$\Rightarrow S_{12} = \frac{12}{2}[3 + 58] = 366$

b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \cdots \ldots \ldots \ldots . . +19^{th}\ term$
Given: $a_1 = \frac{3}{4} , d = \frac{2}{3} - \frac{3}{4} = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12} , n = 19 , a_{19} = ?$
$a_{19} = \frac{3}{4} + (19-1) \times \left(-\frac{1}{12}\right) = -\frac{3}{4}$
Now $S_{19} = \frac{19}{2}\left[2 \times \frac{3}{4} + (19-1) \times \left(-\frac{1}{12}\right)\right] = \frac{19}{2}\left[\frac{3}{2} - \frac{18}{12}\right] = 0$
Alternatively
Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term]$
$\Rightarrow S_{19} = \frac{19}{2}\left[\frac{3}{4} + -\frac{3}{4}\right] = 0$

2. *How many terms of the following series may be taken to;*
   i) *Their sum is 66 if* $-9 , -6 , 3 \ldots \ldots \ldots \ldots \ldots . \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$
   ii) *Their sum is* -80 *if* $15 + 13 + 11 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$
   iii) *Their sum is* $n(2n+4)$ *if* $6 + 10 + 14 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$

Solution

i) $-9 , -6 , 3 \ldots \ldots \ldots \ldots \ldots . \ldots \ldots$
Let $S_n = 66 , a_1 = -9 , d = -6 — 9 = 3$
Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow 66 = \frac{n}{2}[2 \times -9 + (n-1) \times 3]$
$132 = -18n + 3(n^2 - n)$
$\therefore 3n^2 - 21n - 132 = 0\ or\ n^2 - 7n - 44 = 0$
$\Rightarrow (n-11)(n+4) = 0\ \therefore n = 11 , -4$
Since n is a positive integer, $n = -4$ is rejected.
$\therefore number\ of\ terms\ is\ 11$

ii) $15 + 13 + 11 + \cdots \ldots \ldots \ldots$
Let $S_n = -80 , a_1 = 15 , d = 13 - 15 = -2$
Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow -80 = \frac{n}{2}[2 \times 15 + (n-1) \times -2]$

$-160 = 30n - 2(n^2 - n)$

$\therefore 2n^2 - 32n - 160 = 0 \; or \; n^2 - 16n - 80 = 0$

$\Rightarrow (n-20)(n+4) = 0 \;\; \therefore n = 20, -4$

Since n is a positive integer, $n = -4$ is rejected.

$\therefore number\ of\ terms\ is\ 20$

iii) $6 + 10 + 14 + \cdots \ldots \ldots \ldots \ldots$

Let $S_N = n(2n+4)$, $a_1 = 6$, $d = 10 - 6 = 4$

Using $S_N = \frac{N}{2}[2a_1 + (N-1)d]$

$\Rightarrow n(2n+4) = \frac{N}{2}[2 \times 6 + \; (N-1) \times 4]$

$\Rightarrow 4n^2 + 8n = 12N + 4N^2 - 4N \; or \; n^2 + 2n = 3N + N^2 - N$

$\therefore N^2 + 2N - (n^2 + 2n) = 0$, solving for N

$\Rightarrow N = \frac{-2 \pm \sqrt{2^2 + 4(n^2 + 2n)}}{2}$

$= \frac{-2 \pm 2\sqrt{1 + n^2 + 2n}}{2}$

$= -1 \pm \sqrt{(n+1)^2} = -1 \pm n + 1$

$\therefore N = -1 + n + 1 = n \;\; or \;\; N = -1 - n - 1 = -2 - n \; (Reject)$

$\therefore number\ of\ terms\ is\ n$

3. *The sum of the series is 525. Its last term is 39 and the first term is 3. If the series is an A.P, find its common difference.*

Solution

Given: $S_n = 525$, $a_n = 39$, $a_1 = 3$

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 525 = \frac{n}{2}[3 + 39] = 21n \;\; \therefore n = 25$

From $a_n = a_1 + (n-1)d \; or \; from \; S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$From \; S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 525 = \frac{25}{2}[2 \times 3 + (25-1)d]$

$\Rightarrow 1050 = 150 + 600d \;\; \therefore d = \frac{3}{2}$

$\therefore Common\ difference\ is\ \frac{3}{2}$

4. *The first and last terms of an A.P are -4 and 146 respectively. The sum of the A.P is 7171. Find the number of terms and sum of the first 20 terms.*

Solution

Given: $S_n = 7171$, $a_n = 146$, $a_1 = -4$

Required: $n = ?, d = ?$

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 7171 = \frac{n}{2}[-4 + 146] = 71n$

$\Rightarrow 7171 = 71n \;\; \therefore n = 101$

$\therefore No. of\ terms\ is\ 101$

Now finding d using either $a_n = a_1 + (n-1)d \;\; or \;\; S_n = \frac{n}{2}[2a_1 + (n-1)d]$

This time let us use $a_n = a_1 + (n-1)d$

$\Rightarrow 146 = -4 + (101-1)d$

$\Rightarrow 150 = 100d \;\; \therefore d = \frac{3}{2}$

$S_{20} = \frac{20}{2}\left[2 \times -4 + (20-1) \times \frac{3}{2}\right] = 205$
$\therefore S_{20} = 205$

5. *The sum of a certain number of terms in an A.P is 5500. The firs and the last terms are 100 and 1000 respectively. Find the number of terms and the sum of the last 5 terms of the progression.*

Solution
Given: $S_n = 5500\,,\ a_1 = 100\,, a_n = 1000$
Required: $n = ?\,, d = ?\,, S_{last\ 5-terms} = ?$
Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$
$\Rightarrow 5500 = \frac{n}{2}[100 + 1000] = \frac{1100}{2}n$
$\Rightarrow 5500 = \frac{1100}{2}n\ \therefore n = 10$
$\therefore No.of\ terms\ is\ 10$
Now finding d using $a_n = a_1 + (n-1)d$
$\Rightarrow 1000 = 100 + (10-1)d$
$\therefore d = 100$
The A.P can be stated as; $100\,, 200\,, 300\,, 400, \ldots\ldots\ldots\ldots\ldots\ldots 900\,, 1000$
$Sum\ of\ last\ 5 - terms = Sum\ of\ all\ the\ terms - Sum\ of\ the\ 1^{st}\ 4 - terms$
$\Rightarrow S_{last\ 5-terms} = S_{10} - S_{1^{st}\ five}$
$\Rightarrow S_{last\ 5-terms} = 5500 - \left[\frac{5}{2}(2 \times 100 + (5-1) \times 100)\right]$
$= 5500 - [500 + 1000] = 4000$

Note:
The last five terms are; $600, 700\,, 800\,, 900\,, 1000$

6. *The sum of the last three terms of an A.P having* $n - terms$ *is 150 times the sum of the first three terms of the progressions. If the third term and the second term are 15 and 5. Find the last term and the number of terms of the progression.*

Solution
Given: $S_{3(last)} = 150 \times S_{3(first)}\,,\ a_3 = 15\,, a_2 = 5$
Required: $n = ?\,, d = ?$
For an A.P, the last term is $a_n = a_1 + (n-1)d$
For the $2^{nd}\ last\ tem = a_n - d$
$= a_1 + (n-1)d - d = a_1 + (n-2)d$
For the $3^{rd}\ last\ tem = a_n - 2d$
$= a_1 + (n-1)d - 2d = a_1 + (n-3)d$
Now $S_{3(last)} = [a_1 + (n-1)d] + [a_1 + (n-2)d] + [a_1 + (n-3)d]$
$\therefore S_{3(last)} = 3a_1 + (3n-6)d$
Also $S_{3(first)} = \frac{3}{2}[2a_1 + (3-1)d] = 3a_1 + 3d$
From $S_{3(last)} = 150 \times S_{3(first)}$
$\Rightarrow 3a_1 + (3n-6)d = 150(3a_1 + 3d)$
$\Rightarrow a_1 + (n-2)d = 150(a_1 + d)$
$\therefore 149a_1 + (152 - n)d = 0$ ………………………… (1)
Given: $a_3 = a_1 + (3-1)d = a_1 + 2d$
$\therefore 15 = a_1 + 2d$ ……………………………… (2)
$a_2 = a_1 + d = 5$ ……………………………… (3)
$Eqn.(2) - eqn.(3)$

$\Rightarrow 10 = d$
From $a_1 + d = 5 \Rightarrow a_1 = 5 - 10 = -5$
Now $subst. d = 10\ , a_1 = -5\ in\ to\ eqn. (1)$
$\Rightarrow 149a_1 + (152 - n)d = 0\ yields$
$149(-5) + (152 - n) \times 10 = 0$
$\Rightarrow 152 - n = \frac{149\times5}{10} = 74.5$
$\therefore No. of\ terms\ is\ 78$
$a_{78} = -5 + (78 - 1) \times 10 = 760$

7. *The sum of the first 13-terms of an A.P is 21 and the sum of the first 21 terms is 13. Find the sum of the first 34 – terms.*
Solution
Given: $S_{13} = 21\ , S_{21} = 13$
Required: $S_{34} = ?\ , d = ?\ , a = ?$
Using $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$\Rightarrow S_{13} = \frac{13}{2}[2a + (13 - 1)d] = 21$ .... (1)
$\Rightarrow S_{21} = \frac{21}{2}[2a + (21 - 1)d] = 13$ .... (2)
From (1), $2a + 12d = \frac{42}{13}$ .... (3)
From (2), $2a + 20d = \frac{26}{21}$ .... (4)
$Eqn. (3) - eqn. (4) gives$
$-8d = \frac{544}{273}\ \therefore d = -\frac{68}{273}$
Substitute $d = -\frac{68}{273}$ in to eqn. (3) to find a
$\Rightarrow 2a = \frac{42}{13} - 2 \times \left(-\frac{68}{237}\right)\ \therefore a = \frac{566}{91}$
$\therefore S_{34} = \frac{34}{2}\left[2 \times \frac{566}{91} + (34 - 1)\left(-\frac{68}{237}\right)\right] = -34$

8. *The fifth term of an A.P is 12 and the sum of the first five terms is 80. Determine the first term and the common difference.* (***Uneb* 2007**)
Solution
Given: $a_5 = 12\ , S_5 = 80$
Required: $a_1 = ?\ , d = ?$
Using $a_n = a_1 + (n - 1)d$
$\Rightarrow a_5 = 12 = a_1 + 4d$ .... (1)
Using $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$\Rightarrow 80 = \frac{5}{2}[2a_1 + 4d]$
$\Rightarrow 80 = 5a_1 + 10d\ or\ 16 = a_1 + 2d$ .... (2)
$Eqn. (1) - eqn. (2) gives$
$-4 = 2d\ \therefore d = -2$
Substitute $d = 2$ in to eqn. (2) to find a_1
$a_1 = 16 - 2d = 16 + 4 = 20$
$1^{st} - term\ is\ 20\ , common\ difference = -2$

9. *The* $10^{th} - term$ *of an A.P is 29 and the* $15^{th} - term$ *is 44. Find the value of the common difference and the first term. Hence find the sum of the first 60 terms.*(***Uneb* 2002**)

Solution

Given: $a_{10} = 29\,, a_{15} = 44$

Required: $a_1 = ?\,, d = ?\,, S_{60} = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow a_{10} = 29 = a_1 + 9d$ .... (1)

$\Rightarrow a_{15} = 44 = a_1 + 14d$ .... (2)

$Eqn.(1) - eqn.(2) gives$

$-15 = -5d \;\therefore d = 3$

Substitute $d = 3$ in to eqn. (1) to find a_1

$a_1 = 29 - 9d = 29 - 27 = 2$

Now $S_{60} = \frac{60}{2}[2 \times 2 + (60-1) \times 3] = 5\,430$

$\therefore 1^{st} - term\ is\ 2\,, common\ difference = 3\,, S_{60} = 5\,430$

Other examples on summation of the terms in an A.P

1. a) *Find the sum of all natural numbers between 500 and 1000 which are divisible by 13*

   b) Obtain the sum of all natural numbers up to 1000 which are:

      i) Divisible by 5 but not divisible by 2

      ii) Not divisible by 2 and not divisible by 5

Solution

a) By inspection; $\frac{500}{13} \approx 38\,; \frac{1000}{13} \approx 76.9$

Now multiples of 13 after 38 are;

$507\,, 520\,, 533\,, \ldots\ldots\ldots\ldots\ldots\ldots, 988$

$\boldsymbol{Note}: 13 \times 76 = 988\,, 13 \times 77 = 1001$

This is an A.P with $a_1 = 507\,, a_n = 988\,, d = 13$

Now finding n;

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 988 = 507 + (n-1) \times 13 \;\therefore n = 38$

Now $S_{38} = \frac{38}{2}[2 \times 507 + (38-1) \times 13] = 28\,405$

b) (i) Natural numbers divisible by 5 and not by 2

$\Rightarrow 5\,, 15\,, 25\,, \ldots\ldots\ldots\ldots\ldots\ldots.., 955$

These form an A.P. finding how many terms they are,

$a_1 = 5\,, a_n = 955\,, d = 10\,, n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 955 = 5 + (n-1) \times 10 \;\therefore n = 100$

$\therefore\; S_{100} = \frac{100}{2}[2 \times 5 + (100-1) \times 10] = 50\,000$

ii) Natural numbers not divisible by 2 are;

$\Rightarrow 1\,, 3\,, 5\,, 9\,, \ldots\ldots\ldots\ldots\ldots\ldots., 999$

Using $a_n = a_1 + (n-1)d$

$999 = 1 + (n-1) \times 2 \;\therefore n = 500$

Using $S_n = \frac{n}{2}[a_1 + a_n]$

$\therefore\; S_{500} = \frac{500}{2}[1 + 999] = 250\,000$

Sum of natural numbers divisible by 5 is 50 000 as in above.

Natural numbers not divisible by 2 and not divisible by 5 is;

$S_{500} - S_{100} = 250\,000 - 50\,000 = 200\,000$

2. a) Find the sum of odd numbers between 100 and 200
   b) Find the sum of the even and odd numbers divisible by 3 lying between 400 and 500.
   Solution
a) $101\,, 103\,, 105\,, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,, 199$ are odd numbers. Finding how many they are'
   $a_n = 199\,, a_1 = 101\,, d = 2\,,\ n = ?$
   Using $a_n = a_1 + (n-1)d$
   $\Rightarrow 199 = 101 + (n-1) \times 2\ \therefore n = 50$
   Now $S_{50} = \frac{50}{2}[101 + 199] = 7\,500$
b) Multiples of 3 after 400 and before 500 are:
   $402\,, 405\,, 408\,, \ldots\ldots\ldots\ldots\ldots\ldots.\,, 498$
   $\left[\boldsymbol{HINT}: \frac{400}{3} \approx 133\,, 134 \times 3 = 402\,; \frac{500}{3} \approx 167\,, 166 \times 3 = 498\right]$
   Finding how many they are:
   $a_n = 498\,, a_1 = 402\,, d = 3\,,\ n = ?$
   Using $a_n = a_1 + (n-1)d$
   $\Rightarrow 498 = 402 + (n-1) \times 3\ \therefore n = 33$
   Now $S_{33} = \frac{33}{2}[402 + 498] = 14\,850$
   Also even multiples of 3 are:
   $402\,, 408\,, 414\,, \ldots\ldots\ldots\ldots\ldots\ldots.\,, 498$
   Finding how many they are:
   $a_n = 498\,, a_1 = 402\,, d = 6\,,\ n = ?$
   Using $a_n = a_1 + (n-1)d$
   $\Rightarrow 498 = 402 + (n-1) \times 6\ \therefore n = 17$
   Now $S_{17} = \frac{17}{2}[402 + 498] = 7\,650$
   Now sum of odd numbers divisible by 3 is;
   $S_{33} - S_{17} = 14\,850 - 7\,650 = 7\,200$
   Alternatively, odd numbers are;
   $405\,, 411\,, \ldots\ldots\ldots\ldots\ldots\ldots\,, 495$
   Finding how many they are:
   $a_n = 495\,, a_1 = 405\,, d = 6\,,\ n = ?$
   Using $a_n = a_1 + (n-1)d$
   $\Rightarrow 495 = 405 + (n-1) \times 6\ \therefore n = 16$
   Now $S_{16} = \frac{16}{2}[405 + 495] = 7\,200$

Task
1. *Find the sum of all integers lying between 200 and 800 which are divisible by 9.* **ANSWER: 32 967**
2. *Find the sum of all numbers between 200 and 400 which are divisible by 7.* **ANSWER: 8 729**
3. *In an A.*P consisting of $15-$*terms*, *the middle term is 20. Find the sum of all the terms.* **ANSWER: 300**
4. *An A.P consists of 12 terms. The sum of the two middle terms is 24 and the sum of the first 3-terms is 9. Find the A.P.* **ANSWER: $\mathbf{1\,, 3\,, 5\,, \ldots\ldots\ldots\ldots\ldots\,, 23}$**

PROOFS INVOLVING SUMMATION OF A.P FORMULAE,
$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ *or* $S_n = \frac{n}{2}[a_1 + a_n]$

1. *a) The sum of the first* $m -$ *terms of an A.P is* n *and the sum of the first* $n -$ *terms is* m*. find the sum of the first* $(m+n) -$ *terms.*
b) The sum of the first $m -$ *terms of an A.P is the same as the sum of the first* $n -$ *terms where* $m \neq n$*, show that the sum of the first* $(m+n) -$ *terms is zero.*
Solution

a) *Given:* $S_m = n$, $S_n = m$
Required: $S_{(m+n)} = ?$
$S_m = n = \frac{m}{2}[2a + (m-1)d]$
$\Rightarrow \frac{2n}{m} = 2a + (m-1)d$ .... (1)
$S_n = m = \frac{m}{2}[2a + (n-1)d]$
$\Rightarrow \frac{2m}{n} = 2a + (n-1)d$ .... (2)
In the above equations a and d are unknowns. Calculating for d,
$Eqn. (1) - eqn. (2) gives$
$\Rightarrow \frac{2n}{m} - \frac{2m}{n} = (m-n)d$
$\Rightarrow \frac{2(n^2-m^2)}{mn} = (m-n)d$
$\Rightarrow -\frac{2(m^2-n^2)}{mn} = (m-n)d$ *or* $-\frac{2(m+n)(m-n)}{mn} = (m-n)d$
$\therefore d = -\frac{2(m+n)}{mn}$
$Subst. d = -\frac{2(m+n)}{mn}$ *in to* (1) *or* (2)
From eqn. (1), $\frac{2n}{m} = 2a + (m-1)d$
$\Rightarrow 2a = \frac{2n}{m} - (m-1)d$
$\Rightarrow 2a = \frac{2n}{m} - \frac{(m-1)[-2(m+n)]}{mn}$
$\Rightarrow 2a = \frac{2n}{m} + \frac{2(m-1)(m+n)}{mn}$
$= \frac{2n^2+2(m^2+mn-m-n)}{mn}$
Now $S_{(m+n)} = \frac{m+n}{2}[2a + (m+n-1)d]$
$S_{(m+n)} = \frac{m+n}{2}\left[\frac{2n^2+2(m^2+mn-m-n)}{mn} + (m+n-1) \times -\frac{2(m+n)}{mn}\right]$
$= \frac{m+n}{mn}\left[n^2 + m^2 + mn - m - n - [(m+n)^2 - m - n]\right]$
$= \frac{m+n}{mn}[n^2 + m^2 + mn - m - n - m^2 - 2mn - n^2 + m + n]$
$= m + n$

b) *Given:* $S_m = S_n$; $m \neq n$
Required: $S_{(m+n)} = 0$
$S_m = \frac{m}{2}[2a + (m-1)d] = \frac{2n}{m}[2a + (m-1)d] = S_n$
$\Rightarrow m[2a + (m-1)d] = n[2a + (n-1)d]$
$\Rightarrow 2am + (m^2 - m)d = 2an + (n^2 - n)d$
$\Rightarrow 2a(m-n) = (n^2 - n)d - (m^2 - m)d$
$= [-(m^2 - n^2) + (m-n)]d$
$= [-(m+n)(m-n) + (m-n)]d$

$\Rightarrow 2a = [-(m+n)+1]d$
$\therefore 2a = -(m+n-1)d$
$But\, S_{(m+n)} = \frac{m+n}{2}[2a+(m+n-1)d]$
$\Rightarrow S_{(m+n)} = \frac{m+n}{2}[2a-2a] = 0,\ as\ required.$

2. a) *If the first, second and the last term of an A.P are respectively a, b and x, show that the sum of the n − terms is* $\frac{x+a}{2}+\frac{x^2-a^2}{2(b-a)}$.

b) *If the first ant last terms of an A.P are a and L respectively and if S is the sum of all the n − terms, show that the common difference is* $\frac{L^2-a^2}{2S-(L+a)}$.

Solution

a) *Given:* $a_1 = a\,, a_2 = b\,, a_n = x$
Required: $S_n = ?$
$S_n = \frac{n}{2}[1^{st} + last\ term]$
$S_n = \frac{n}{2}[a+x]$ .... (1)
But $a_2 = a+d = b$ .... (2)
Also $a_n = a+(n-1)d = x$ .... (3)
In equations (2) and (3) d is the unknown.
From (2), $d = b-a$ .... (4)
Equation (4) in to (3)
$\Rightarrow a+(n-1)d = x$, becomes
$n = \frac{x-a}{d}+1$ .... (5)
Equation (4) in to (5)
$n = \frac{x-a}{d}+1 = n = \frac{x-a+b-a}{b-a}$
$\therefore n = \frac{x+b-2a}{b-a}$ .... (6)
Now substitute equation (4) in to (1)
$\Rightarrow S_n = \frac{n}{2}(a+x)$, becomes
$\therefore S_n = \frac{\left[\frac{x-a}{b-a}+1\right]}{2}\times(a+x)$
$= \frac{(x-a)(x+a)}{2(b-a)}+\frac{a+x}{2}$
$\therefore S_n = \frac{x+a}{2}+\frac{x^2-a^2}{2(b-a)}$, as required.

b) *Given:* $a_1 = a\,, a_n = L\,, S_n = S$
Required: $d = ?$
From $a_n = a+(n-1)d = L$........................... (1)
$S_n = \frac{n}{2}[2a+(n-1)d] = S$ .... (2)
From equation (1) and (2), d is the unknown.
From equation (1) , $d = \frac{L-a}{n-1}$ .... (3)
Now solving for n in equation (3)
$\Rightarrow \frac{n}{2}[2a+(n-1)d] = S$, becomes
$\therefore \frac{n}{2}\left[2a+(n-1)\times\frac{(L-a)}{n-1}\right] = S$
$\Rightarrow n[2a+L-a] = 2S$
$\therefore n = \frac{2S}{a+L}$ .... (4)
Equation (4) in to (3)

$\therefore d = \frac{L-a}{\frac{2S}{a+L}-1} = \frac{L-a}{\frac{2S-(a+L)}{(a+L)}}$

$= \frac{(L-a)(L+a)}{2S-(a+L)} = \frac{L^2-a^2}{2S-(a+L)}$

3. *If p, q and r are sums $n, 2n$ and $3n$ terms of an A.P, prove that $r = 3(q-p)$*

Solution

Given: $S_n = p\ ,\ S_{2n} = q\ ,\ S_{3n} = r$

From **R.H.S** $= 3(q-p)$

$\Rightarrow p = \frac{n}{2}[2a+(n-1)d]$ ……………………………… (2)

$\Rightarrow q = \frac{2n}{2}[2a+(2n-1)d]$ …………………………… (3)

$\Rightarrow r = \frac{3n}{2}[2a+(3n-1)d]$ …………………………… (4)

Now $3(q-p) = 3\left[\frac{2n}{2}[2a+(2n-1)d] - \frac{n}{2}[2a+(n-1)d]\right]$

$\Rightarrow 3(q-p) = \frac{3n}{2}\big[2[2a+(2n-1)d] - [2a+(n-1)d]\big]$

$\Rightarrow 3(q-p) = \frac{3n}{2}[2a+(3n-1)d] = r$, as required.

Deducing *last term* of an A.P from summation of $n-terms$ of an A.P

In general if S_n is the sum of n-terms of an A.P and S_{n-1} is the summation of $(n-1)-terms$ of an A.P, then;

$$\boxed{\textbf{Last term} = \mathbf{a_n} = \mathbf{S_n} - \mathbf{S_{n-1}}}$$

Example

1. *The sum of n-terms of a series is n^2+2n , prove that this series is an A.P, and find its first term and common difference.*

Solution

Given: $S_n = n^2+2n$

Also $S_{n-1} = (n-1)^2 + 2(n-1)$

$= n^2 - 2n + 1 + 2n - 2$

$\therefore S_{n-1} = n^2 - 1$

Now $a_n = S_n - S_{n-1}$

$\Rightarrow a_n = n^2 + 2n - (n^2-1) = 2n+1$

$\therefore$ *Last term* $, a_n = 2n+1$

Now for $n = 1\ , a_1 = 2+1 = 3$

$n = 2\ , a_2 = 4+1 = 5$

$n = 3\ , a_2 = 3+1 = 7$

The terms are: $3\ , 5\ , 7\ , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$; which form an A.P of first term 3 and common difference 2.

2. *If the sum up to n-terms of an A.P is given by $\frac{n}{2}(3n+5)$, show that the last term is $3n+1$, and that the series is an A.P.*

Solution

Given: $S_n = \frac{n}{2}(3n+5)$

Also $S_{n-1} = \frac{(n-1)}{2}[3(n-1)+5] = \frac{(n-1)(3n+2)}{2}$

Now $a_n = S_n - S_{n-1}$

$\Rightarrow a_n = \frac{n}{2}(3n+5) - \frac{(n-1)(3n+2)}{2}$

$= \frac{3n^2+5n}{2} - \frac{3n^2+2n-3n-2}{2}$

$= \frac{1}{2}(6n+2) = 3n+1$

$\therefore Last\ term\ , a_n = 3n+1$, as required.

By putting $n = 1, 2, 3, 4, \dots\dots\dots$, we get a series of;

$4, 7, 10, \dots\dots\dots\dots\dots\dots$, which is an A.P.

Task

1. *The sum of the first n-terms of a certain series is* $n^2 + 5n$ *for all integral values of n. find the first three terms and prove that the series is an A.P.*
   ANSWER: $\mathbf{6, 8, 10}$
2. *The sum of the first n-terms of an A.P is* $3n^2 + 2n$. *Find the* $r^{th} - term$. **ANSWER**: $\mathbf{6r-1}$
3. *The second term of an A.*P is -4 and the sixth term is -24. Find the fifteenth term and sum of the first $15 - terms$. **ANSWER**: $\mathbf{-69, 540}$
4. *The eighth term of an A.P is twice the third term, and the sum of the first 8-terms is 39. Find the first three terms of the progression, and show that the sum of n-terms is* $\frac{3}{8}n(n+5)$. **ANSWER**: $\mathbf{\frac{9}{4}, 3, \frac{15}{4}}$
5. *The sum of n-terms of a series* $2, 5, 8, \dots\dots\dots\dots\dots$ *is 950. Find n.* **ANSWER**: $\mathbf{25}$
6. *The number of terms in an A.P is even; the sum of the odd terms is 24, of the even terms 30 and the last term exceeds the first by* $10\frac{1}{2}$. *Find the number of terms.* **ANSWER**: $\mathbf{8}$
7. *If* $p^{th} - term$ *of an A.P is* $\frac{1}{q}$ *and the* $q^{th} - term$ *is* $\frac{1}{p}$, *show that the sum of* $pq - terms$ *is* $\frac{1}{2}(pq+1)$
8. *Find the sum of all natural numbers from 100 up to 300 which are divisible by 4.* **ANSWER**: $\mathbf{10200}$
9. *In an A.P the sum of the first five terms is 30, and the third term is equal to the sum of the first two. Write down the first five terms of the progression.* **ANSWER**: $\mathbf{2, 4, 6, 8, 10}$
10. *The sum of the first three terms of an A.P is 3 and the sum of the first five terms is 20. Find the first five terms of the progression.* **ANSWER**: $\mathbf{-2, 1, 4, 4, 7, 10}$
11. *The sum of the first six terms of an A.P is 21 and the seventh term is three times the sum of the third term and fourth term. Find the first term and the common difference.* **ANSWER**: $\mathbf{-9, 5}$
12. *Sum of n-terms of three A.P's are* S_1 , S_2 *and* S_3 . *The first term of each of them is 1 and the common differences are 1, 2 and 3 respectively. Show that* S_1 , S_2 , S_3 *are in an A.P.*

$$\left[\begin{array}{c} \boldsymbol{HINT}\text{: } S_1 = \frac{n}{2}[2+n-1] = \frac{n(n+1)}{2}\ ,\ S_2 = \frac{n}{2}[2+(n-1)\times 2 = n^2] \\ S_3 = \frac{n}{2}\left[2+(3n-1)\times 2 = \frac{n(3n-1)}{2}; now\ show\ that\ S_2 - S_1 = S_3 - S_2\right] \end{array}\right]$$

WORD PROBLEMS IN A.P'S

1. *A firm produced 1000 sets of T.V during its first year. The total sum of the firm's production at the end of 10 years operation is 14 500 sets.*

i) *Estimate how many units production increased by each year if the increase each year is uniform.*

ii) *Fore cast based on the estimate of annual increments in production the level of output for the* 15^{th} *year.*

Solution

i) *Given:* $a = 1000\,, S = 14\,500\,, n = 10$

Required: $d = ?$

Using $\Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 14\,500 = \frac{10}{2}[2 \times 1000 + (10-1)d]$

$\Rightarrow 14\,500 = 5[2000 + 9d]\;\; \therefore d = 100$

Hence 100 units is the increase per annum.

ii) *Required:* $a_{15} = ?$

$a_{15} = a + (15-1)d = a + 14d$

$= 1000 + 14 \times 100 = 2\,400$

2. *Two posts were offered to a man. In one the starting salary was* £ 120 *per month and the annual increment was* £ 8.

In the other post the salary commenced at £ 85 *per month but the annual increment was* £ 12. *The man decided to accept the post which will give him more earnings in the first twenty years of the service. Which post was acceptable to him? Justify your answer.*

Solution

The total earnings of the man in the first twenty years for the first job are $S_{20} = ?$:

Given: $a = £120\,, d = 8\,, n = 20$

$\Rightarrow S_{20} = \frac{20}{2}[2 \times 120 + (20-1) \times 8] \times 12 = £47\,040$

His total earnings in the first twenty years for the second job are $S_{20} = ?$:

Given: $a = £85\,, d = 12\,, n = 20$

$\Rightarrow S_{20} = \frac{20}{2}[2 \times 85 + (20-1) \times 12] \times 12 = £47\,760$

Total earnings for the second job are greater than the first job hence the second job was accepted by the man.

3. *The annual salary of an employee is in an A.P over the first* $7 - years$, *during which the annual increment is* £ 200. *Over the next* $8 - years$, *the salary is in another A.P during which the annual increment is* £ 300. *After that the salary is constant.*

If the starting salary is £ $14\,000$, *calculate the employee's total earnings in the first 25 years of service.*

Solution

Case I

Given: $a = 14\,000\,, d = 200\,, n = 7$

Required: $S_7 = ?$

$\Rightarrow S_7 = \frac{7}{2}[2 \times 14\,000 + (7-1) \times 200] = £102\,200$

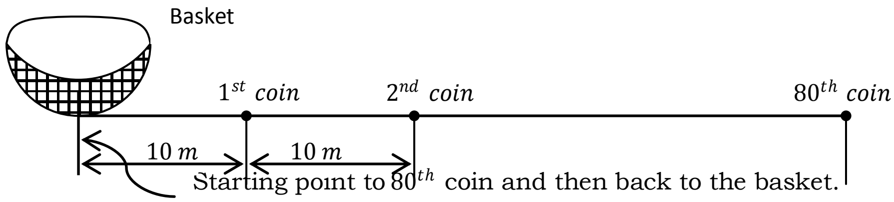
Case II

At the end of the $7^{th} - year$, the annual salary is given from $a_n = a + (n-1)d$

$\Rightarrow a_7 = 14\,000 + (7-1) \times 200 = £15\,200$
Now the next $8 - years$, the sum of all his earnings is given by:
$S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow S_8 = \frac{8}{2}[2 \times 15\,200 + (8-1) \times 300] = £130\,000$
Now at the end of the$15 - years. i.e. 1^{st}\ 7 - years + next\ 8 - years$, the total earnings are given by;
$£130\,000 + £102\,200 = £232\,200$
Remaining years $= 25 - 17 = 10\ years$
The amount in the$15^{th} - year$ is given by $a_n = a + (n-1)d\ ; a = 15\,200\ , n = 8\ , d = 300$
$\Rightarrow a_{15} = 15\,200 + (8-1) \times 300 = £17\,300$
Now for the remaining 10 years, total amount $= 17\,300 \times 10 = £173\,000$
$\therefore$ Total earnings for 25 years is$= 173\,000 + 232\,200 = £405\,200$

4. *Eighty coins are placed in a line on the ground. The distance between any two consecutive coins is 10 meters. How far must a person travel to bring them one by one to a basket placed 10 meters behind the first coin?*

Solution



Given: $a = 10\ , d = 10\ (distance\ between\ 2\ successive\ coins)\ , n = 80\ (No.of\ coins)$
Required: $S_{80} = \frac{80}{2}[2 \times 10 + (80-1) \times 10] = 32\,400\ m$
Total distance$= 32\,400\ (to\ 80^{th}\ coin) + 32\,400\ (back\ to\ basket) = 64\,800\ m$

5. *A man saved £16 500 in 10 years. In each year after the first he saved £100 more than he did in the preceding year. How much did he save in the first year?*

Solution

Given: $S_{10} = £16\,500\ , d = £100\ , n = 10$
Let $a = Amount\ saved\ in\ 1^{st}\ year$
From $S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow 16\,500 = \frac{10}{2}[2a + (10-1) \times 100]$
$\Rightarrow 3\,300 = 2a + 900$
$\therefore d = £1\,200$

6. *Miss Sikyomu Jane takes a loan of £2 000 from Anita and agrees to repay in a number of installment, each installment (beginning with the second) exceeding the previous by £10. If the first installment is £5, find how many installments will be necessary to wipe out the loan completely?*

Solution

Given: $S_{10} = £2000\ , d = £10\ , n = ?\ , a = £5$
Using $S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow 2\,000 = \frac{n}{2}[2 \times 5 + (n-1) \times 10]$

$\Rightarrow 2\,000 = 5n + 5n^2 - 5$
$\Rightarrow n^2 + n - 401 = 0$
$\Rightarrow n = \frac{-1 \pm \sqrt{1+4\times 401}}{2}$
$\therefore n \approx 20$, for a negative n, it is rejected
$\therefore$ Number of installments will be 20.

7. *The monthly salary of a person was* £320 *for each of the first three years. He next got annual increments of* £40 *per month for each of the following successive 12 years. His salary remained stationary till retirement when he found that his average monthly salary during the service period was* £698. *Find the period of his service.*

Solution

Let $n = total\ no.of\ years\ of\ the\ person's\ service$
Given: Average monthly salary $=$ £$698 = \frac{Amount\ \ for\ n-years}{No.of\ \ years}$
$\Rightarrow 698 \times 12 = \frac{Amount\ \ for\ n-years}{n}$
$\therefore Amount\ for\ n - years = 698 \times 12n$
Total salary in the first $3 - years =$ £$320 \times 3 \times 12 =$ £960×12
In the 4^{th} $year$, his monthly salary= £$(320 + 40) =$ £360
In the 5^{th} $year$, his monthly salary= £$(360 + 40) =$ £400
For the next $12 - years$, his total salary is;
$=$ £$12 \times [360 + 400 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots..up\ to\ 12^{th} - term]$
$= 12 \times \left[\frac{12}{2}[2 \times 360 + (12 - 1) \times 40]\right]$
$=$ £$6\,960 \times 12$
At the end of following 12 years, his monthly salary was;
$=$ £$[360 + (12 - 1) \times 40] =$ £800
Now total salary for all the 15 years so far is
$=$ £$6\,960 \times 12 +$ £$960 \times 12 =$ £$7\,920 \times 12$
Amount received after 15 years $.i.e.(n - 15)\ years$ was constant for each year $.i.e.$ amount in the 15^{th} $year =$ £800
$\Rightarrow Total\ amount =$ £$(n - 15) \times 800 \times 12$
Total amount throughout his service
$= (n - 15) \times 800 \times 12 + 7\,920 \times 12 = 698 \times 12n$
$\Rightarrow 800n - 12\,000 + 7\,920 = 698n$
$\therefore n = 40$

8. *Mr. Peter arranges to pay off a debt of* £9 600 *in 48 annual installments which form an arithmetical series. When 40 of these installments are paid, Mr. Peter becomes insolvent and his creditor finds that* £2 400 *still remains unpaid. Find the value of each of the first three installments of Mr. Peter. Ignore the interest.*

Solution

Let $a\,, a + d\,, a + 2d\,, a + 3d\,, \ldots\ldots\ldots$, be the annual installments
Given: $S_{48} =$ £9 600
$\Rightarrow 9\,600 = \frac{48}{2}[2a + (48 - 1)d]\ \ or\ 9\,600 = 24[2a + 47d]$
$\Rightarrow 2a + 47d = 400$ ……………………………………………………………. (1)
After 40 installments are paid, the balance is £$2\,400$ $.i.e.$ the amount paid in 40 installments is;

$= 9\,600 - 2\,400 = £7\,200$
$\Rightarrow$ Sum of 40 terms of an A.P is £7 200
$\Rightarrow 7\,200 = \frac{40}{2}[2a + (20-1)d]$ *or* $7\,200 = 20[2a + 39d]$
$\Rightarrow 2a + 39d = 360$ ………………………………………………………… (2)
Subtracting the equations
$\Rightarrow 8d = 40 \;\therefore d = 5$
From (2) $2a + 39d = 360$, $a = 82.5$
Hence the first installment of Mr. Peter is £82.5 , the second installment is $£(82.5+5)\,.i.e.£87.5$ *and the third installments is* $£(87.5+5)\,.i.e.£92.5$

Task

1. *Phionah secures an interest free loan of £14 500 from a friend and agrees to repay it in 10 installments. She pays £ 1 000 as first installment and then increases each installment by equal amount over the preceding installment. What will be her last installment?*
   $\left[\begin{array}{c}\textbf{HINT}: s_{10} = 14\,500 = 5(2a+9d)\,, a = 1000 \;\therefore d = 100 \\ Required: a_{10} = a + (10-1)d = 1000 + 9 \times 100 = £\,\mathbf{1\,900}\end{array}\right]$

2. *An enterprise produced 600 units in the third year of its existence and 700 units in the seventh year. What was the initial production in the first year?*
   ANSWER: £ **550** $\left[\begin{array}{c}\textbf{HINT}: a_3 = 600 + 2d\,, a_7 = a + 6d, solving\ the\ equations \\ a = 550\,, d = 45 \;\therefore 1^{st}\ production\ is\ £\,\mathbf{550}\end{array}\right]$
3. *A piece of equipment costs£ 600 000. It depreciates in value, 15% the first year, 13 ½ % the next year, 12% the third year and so on. What will be the value at the end of 10 years all percentages applying to the original cost?* **ANSWER** £ **105 000**
4. *A money lender lends £ 1000 and charges an overall interest of £ 140. He recovers the loan and interest by 12 monthly installments each less by £ 10 than the preceding one. Find the amount of the first installment.* **ANSWER**: £ **150**
5. *A man saves £ 20 in the first month, £ 30 in the second month, £ 40 in the third month, and so on. How much has he saved at the end of 5 years?*
   ANSWER: £ **18 900**
6. *A man saves £ 8 250 in 10 years. In each year after the first he saved £ 50 more than he did in previous year. How much did he save the first year?*
   ANSWER: £ **600**
7. *A man borrows £ 5 000 and agrees to repay with a total interest as £ 1 000 in 12 installments, each installment being less than the preceding by £ 50. What should be the first installment?* **ANSWER** : £ **775**
8. *A man borrows £ 4 500 and promises to repay back in 30 installments, each of value £ 10 more than the previous one. Find the first and last installment.*
   ANSWER: £ **5** , £ **295**
9. *A man of 60 years old has 9 children born at equal intervals. The sum of the ages of father and the nine children is known to be 222 years. Calculate the age of the youngest san if the eldest one is 30 years.* **ANSWER**: **6 years**

THE GEOMETRIC PROGRESSION (G.P)

A series in which non-zero quantities $a_1, a_2, a_3, \ldots\ldots\ldots, a_n, \ldots\ldots$ each term of which is equal to the product of the preceding term and a constant number is a **Geometric Progression.** *i.e.* Each term in a G.P is obtained from the preceding one by multiplying or dividing by a constant number. *e.g.*

- $1, 2, 4, 8, 16, \ldots\ldots\ldots\ldots\ldots$ is a G.P
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \ldots\ldots\ldots\ldots\ldots$ is a G.P
- $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots\ldots\ldots\ldots\ldots$ is a G.P
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots\ldots\ldots\ldots\ldots$ is a G.P
- $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots,$ is a G.P

The constant term is called a **common ratio, r** of a G.P

Consider the G.P below;

$$\boxed{a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots\ldots + a_{n-1} + a_n}$$

This will be a G.P if:

$$\boxed{r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots \ldots\ldots\ldots\ldots \frac{a_n}{a_{n-1}}, \text{ for all integers } n > 1}$$

From the above examples of G.P series,

- $1, 2, 4, 8, 16, \ldots\ldots\ldots\ldots\ldots$, has common ratio $r = \frac{2}{1} = \frac{4}{2} = 2$
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \ldots\ldots\ldots\ldots\ldots$, has common ratio $r = -\frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$
- $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots\ldots\ldots\ldots\ldots$, has common ratio $r = \frac{\sqrt{2}}{1} = \frac{2}{\sqrt{2}}$
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots\ldots\ldots\ldots\ldots$, has common ratio $r = \frac{\frac{1}{5}}{1} = \frac{\frac{1}{25}}{\frac{1}{5}} = \frac{1}{5}$
- $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots$, has common ratio $r = \frac{ar}{a} = \frac{ar^2}{ar} = r$

To find the $n^{th}-term$ of a G.P

$(\boldsymbol{General\ term\ of\ a\ G.P})$

Let $a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots\ldots$

Let $r = common\ ratio$

By definition,

$r = \frac{a_2}{a_1} \Rightarrow a_2 = a_1 r$

$r = \frac{a_3}{a_2} \Rightarrow a_3 = a_2 r = a_1 r.r = a_1 r^2$

$r = \frac{a_4}{a_3} \Rightarrow a_4 = a_3 r = a_1 r^2.r = a_1 r^3$

Observation of terms

$a_2 = a_1 r = a_1 r^{2-1}$

$a_3 = a_1 r^2 = a_1 r^{3-1}$

$a_4 = a_1 r^3 = a_1 r^{2-1}$

$\vdots$

$a_n = a_1 r^{n-1}$

Thus the $n^{th} - term$ of a G.P is

$$\boxed{\boldsymbol{a_n = ar^{n-1}}, \text{ where } a = 1^{st} - term\,, n = no.of\ terms\,,\ r = common\ ratio}$$

NOTE:

1. The $n^{th} - term/$ general term of a G.P is $\boldsymbol{a_n = ar^{n-1}}$
   It contains $4 - terms$, is $3 - terms$ are given, the $4^{th} - term$ can be evaluated.
2. From the general term of a G.P, the standard form of a G.P series can be stated as;

$$\boxed{a + ar + ar^2 + ar^3 + ar^4\,, \ldots\ldots\ldots.. + ar^{n-1}}$$

3. **The most convenient terms of a G.P given their product are:**
   a) If *three terms* are in a G.P, then use

   $$\boxed{\boldsymbol{\frac{a}{r}\,, a\,, ar}}$$

   b) If *four terms* are in a G.P, then use

   $$\boxed{\boldsymbol{\frac{a}{r^3}\,, \frac{a}{r}\,, ar\,, ar^3}}$$

NOTICE:

1. In both cases, when terms are multiplied, the common ratio vanishes.
2. In the case of *odd number terms*, the middle terms $\boldsymbol{a}$ and the common ratio is $\boldsymbol{r}$.
3. In the case of *even number terms*, the middle terms are $\boldsymbol{\frac{a}{r}\,, ar}$ and the common ratio is $\boldsymbol{r^2}$.

Properties of a G.P

1. If every term of a G.P is multiplied or divided by a constant, the resulting series is a G.P.
2. If the terms of a G.P are raised to same power, the resulting series is a G.P.
3. If the reciprocals of terms in a G.P are got, the resulting series is a G.P.

Examples

1. *Write down the terms indicated in each of the following series.*
   a) $2 + 4 + 8 + \cdots \ldots\ldots\ldots\ldots..\ldots.11^{th} - term$
   b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \cdots \ldots\ldots\ldots..\ldots\ldots..12^{th} - term$
   c) $\frac{1}{\sqrt{2}}\,, -2\,, \frac{8}{\sqrt{2}}\,, \ldots\ldots\ldots\ldots\ldots\ldots\ldots..8^{th} - term$

Solution

a) $2 + 4 + 8 + \cdots \ldots\ldots\ldots\ldots..\ldots.11^{th} - term$
   This is not an A.P because $4 - 2 \neq 8 - 4$
   This is a G.P because $\frac{4}{2} = \frac{8}{4} = 2$
   Using $\boldsymbol{a_n = ar^{n-1}}$
   Given: $a_n = ?\,, n = 11\,, r = 2\,, a = 2$
   $\therefore a_{11} = 11 \times (2)^{11-1} = 11 \times 2^{10} = 11264$
b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \cdots \ldots\ldots\ldots..\ldots\ldots..12^{th} - term$

Given: $a_n = ?, n = 12, r = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{9}{8}, a = \frac{2}{3}$

Required: $a_{12} = ?$

$\therefore a_{12} = \left(\frac{2}{3}\right) \times \left(\frac{9}{8}\right)^{12-1} = \frac{2}{3} \times \left(\frac{9}{8}\right)^{11} = 11264$

c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 8^{th} - term$

Given: $a_n = ?, n = 8, r = \frac{-2}{\frac{1}{\sqrt{2}}} = \frac{\frac{8}{\sqrt{2}}}{-2} = -\frac{4}{\sqrt{2}}, a = \frac{1}{\sqrt{2}}$

Required: $a_8 = ?$

$\therefore a_8 = \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{4}{\sqrt{2}}\right)^{8-1} = \frac{1}{\sqrt{2}} \times \left(-\frac{4}{\sqrt{2}}\right)^7 = \left(\frac{1}{\sqrt{2}}\right)^8 \times (-4)^7 = -1024$

2. *Find the number of terms in the G.P's below.*
   a) $0.03 + 0.06 + 0.12 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1.92$
   b) $81 + 27 + 9 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{27}$
   c) $a + ar + ar^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1}$

Solution

a) $0.03 + 0.06 + 0.12 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1.92$

Given: $a = 0.03, r = \frac{0.06}{0.03} = 2, a_n = 1.92$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow 1.92 = 0.03 \times (2)^{n-1}$

$\Rightarrow \frac{1.92}{0.03} = 2^{n-1}$ or $2^6 = 2^{n-1}$

$\therefore n - 1 = 6$ or $n = 7$

$\therefore No. of\ terms\ is\ 7$

b) $81 + 27 + 9 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{27}$

Given: $a = 81, r = \frac{27}{81} = \frac{1}{3}, a_n = \frac{1}{27}$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow \frac{1}{27} = 81 \times \left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow \frac{1}{27 \times 81} = \frac{1}{3^{n-1}}$ or $3^3 \times 3^4 = 3^{n-1}$

$\Rightarrow 7 = n - 1 \ \therefore n = 8$

c) $a + ar + ar^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1}$

Using $A_N = A.R^{N-1}$

Given: $A = a, R = \frac{ar}{a} = r, A_N = ar^{n-1}$

$\Rightarrow a.r^{n-1} = a.r^{N-1}$

$\Rightarrow n - 1 = N - 1 \ \therefore n = N$

$\therefore No. of\ terms\ is\ n$

3. *The third term of a G.P is 10, and the sixth term is 80. Find the common ratio, the first term and the twelfth term.*

Solution

Given: $a_3 = 10, a_6 = 80$

Using $a_n = ar^{n-1}$

$\Rightarrow 10 = a.r^{3-1} = a.r^2$ …………………………………………… (1)

$\Rightarrow 80 = a.r^{6-1} = a.r^5$ …………………………………………… (2)

$eqn.(2) \div eqn.(1)$
$\Rightarrow \frac{80}{10} = \frac{a.r^5}{a.r^2} = r^3$
$\Rightarrow 8 = r^3 \ \therefore r = 2$
From eqn. (1)
$a = \frac{10}{r^2} = \frac{10}{4} = \frac{5}{2}$
$Now\ 12^{th} - term = a_{12} = a.r^{12-1} = a.r^{11}$
$\therefore 12^{th} - term = \frac{5}{2} \times 2^{11} = 5 \times 2^{10}$
$\therefore 1^{st} - term = \frac{5}{2}\ , 12^{th} - term = 5(2)^{10}$

4. *The* $3^{rd} - term$ *of a G.P is 2, and the* $5^{th} - term$ *is 18. Find two possible values of common ratio, and the* $2^{nd} - term$ *in each case.*
Solution
Given: $a_3 = 2\ , a_5 = 18$
Required: $r = ?\ , a_2 = ?$
Using $a_n = ar^{n-1}$
$\Rightarrow 2 = a.r^{3-1} = a.r^2$ ………………………………………… (1)
$\Rightarrow 18 = a.r^{5-1} = a.r^4$ ……………………………………… (2)
$eqn.(2) \div eqn.(1)$
$\Rightarrow \frac{18}{2} = \frac{a.r^4}{a.r^2} = r^2$
$\Rightarrow 9 = r^2 \ \therefore r = \pm 3$
Now from eqn. (1)
$a = \frac{2}{r^2} = \frac{2}{(\pm 3)^2} = \frac{2}{9}$
Now $a_2 = ar^{2-1} = ar = \frac{2}{9} \times \pm 3 = \pm \frac{2}{3}$
$\therefore$ For 1^{st} $G.P$ with $r = 3\ , a = 2\ , the\ 2^{nd} - term\ is\ \frac{2}{3}$ and in the 2^{nd} $G.P$ with $r = -3\ , a = 2\ , 2^{nd} - term\ is - \frac{2}{3}$

5. *The ratio of the* 4^{th} *to the* $12^{th} - term$ *of a G.P with a positive common ratio is* $\frac{1}{256}$*. If the sum of the two terms is* 61.68*. find the* $10^{th} - term$*.*
Solution
Given: $\frac{a_4}{a_{12}} = \frac{1}{256} \Rightarrow \frac{a.r^3}{a.r^{11}} = \frac{1}{256}\ \ or\ \frac{1}{r^8} = \frac{1}{256}$
$\Rightarrow r^8 = 256$. let $r^4 = x$
$\Rightarrow x^2 = 256 \ \therefore x = \pm 16$
Now $r^4 = \pm 16\ \ or\ r^4 = 16\ (Taking\ the + ve\ value)$
$\therefore r = \pm 2$
Since r is required as $+ve$, then $r = 2$ is taken, neglecting the negative sign.
Also *given:* $a_4 + a_{12} = 61.68$
$\Rightarrow a.r^3 + a.r^{11} = 61.68$
$\Rightarrow a(r^3 + r^{11}) = 61.68$
$\therefore a = \frac{61.68}{(r^3 + r^{11})} = \frac{61.68}{(2^3 + 2^{11})} = 0.03$
Now $a_{10} = ar^9 = 0.03 \times 2^9 = 15.36$
$\therefore$ The $10^{th} - term\ is\ 15.36$

The Sum and Product of convenient terms of a G.P

Examples:

1. *Find three numbers in a G.P such that*
   a) *Their sum is 19 and their product is 216*
   b) *Their sum is 21 and the sum of their squares is 651*
   c) *The sum of their squares is 819 and their product is 729.*

Solution

Let the numbers be $\frac{a}{r}, a, ar$

a) *Given:* $\frac{a}{r} + a + ar = 19$ or $a\left(\frac{1}{r} + r + 1\right) = 19$.............................. (1)

Also $\frac{a}{r} \cdot a \cdot ar = 216$ or $a^3 = 216$ $\therefore a = \sqrt[3]{216} = 6$ .... (2)

Substituting eqn. (2) in to eqn. (1)

$\Rightarrow \left(\frac{1}{r} + r + 1\right) \cdot 6 = 19$

$\Rightarrow \left(\frac{r^2+r+1}{r}\right) \times 6 = 19$

$\Rightarrow 6r^2 + 6r + 6 = 19r$ or $6r^2 - 13r + 6 = 0$

$\Rightarrow 6r^2 - 9r - 4r + 6 = (3r - 2)(2r - 3) = 0$

$\therefore r = \frac{2}{3}, \frac{3}{2}$

$\therefore$ The numbers are: $\frac{6}{\frac{2}{3}}, 6, 6 \times \frac{2}{3}$ $for\ r = \frac{2}{3}$ or $9, 6, 4$

For $= \frac{3}{2}$, the numbers are in the reverse order of $9, 6, 4$

b) *Given:* $\frac{a}{r} + a + ar = 21$ or $a\left(\frac{1}{r} + r + 1\right) = 21$.............................. (1)

Also $\left(\frac{a}{r}\right)^2 + (a)^2 + (ar)^2 = 651$ or $a^2\left(\frac{1}{r^2} + r^2 + 1\right) = 651$ .... (2)

Squaring equation (1)

$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right)^2 = 441$

$\Rightarrow a^2\left[\left(\frac{1}{r} + r\right)^2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$

$\Rightarrow a^2\left[\frac{1}{r^2} + r^2 + 2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$

$\Rightarrow a^2\left[\left(\frac{1}{r^2} + r^2 + 1\right) + 2\left(\frac{1}{r} + r + 1\right)\right] = 441$ .... (3)

Dividing eqn. (2) by eqn. (3)

$\Rightarrow \frac{a^2\left(\frac{1}{r^2}+r^2+1\right)}{a^2\left[\left(\frac{1}{r^2}+r^2+1\right)+2\left(\frac{1}{r}+r+1\right)\right]} = \frac{651}{441}$ or $441\left(\frac{1}{r^2} + r^2 + 1\right) = 651\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right)$

$\Rightarrow (651 - 441)\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right) = 0$

$\therefore 210\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right)$

Let $y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$

$\Rightarrow 210(y^2 - 2 + 1) + 1302(y + 1) = 0$

$\Rightarrow 210y^2 + 1302y + 1092 = 0$

Solving the equation,

$y = -1, -\frac{26}{5}$

Now for $y = -1, y = \frac{1}{r} + r$

$\Rightarrow \frac{1}{r} + r + 1 = 0$ or $r^2 + r + 1 = 0$

Solving the equation, no real values of r

For $y = -\frac{26}{5}$

$\Rightarrow \frac{1}{r} + r + \frac{26}{5} = 0$ *or* or $5r^2 + 26r + 5 = 0$

Solving the equation, $r = -5, -\frac{1}{5}$

For $r = -5$ in eqn. (1) $.i.e.\, a\left(\frac{1}{r} + r + 1\right) = 21$

$\Rightarrow a\left(-\frac{1}{5} - 5 + 1\right) = 21 \therefore a = -5$

$\therefore$ The numbers are: $\frac{-5}{-\frac{1}{5}}, -5, -5 \times -\frac{1}{5}$ *or* $25, -5, 1$

For $r = -\frac{1}{5}$, the numbers are in the reverse order

c) *Left as an exercise.*

ANSWER: $\boldsymbol{r = \pm 3, r = \pm\frac{1}{3} : No.s\ are: 3, 9, 27\ or\ 27, 9, 3}$

2. *Find five numbers in a G.P such that:*
   a) *Their product is 32 and the product of the last two terms is 108*
   b) *Their sum is 7 ¾ , their product is unity.*

Solution

Let the numbers be: $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

a) *Given:* $\frac{a}{r^2}.\frac{a}{r}.a.ar.ar^2 = 32 \Rightarrow a^5 = 32 \ \therefore a = 2$

Also $ar.ar^2 = 108 \Rightarrow r^3 = \frac{108}{a^2} = \frac{108}{4} = 27$

$\therefore r = 3$

$\Rightarrow$ The numbers are: $\frac{2}{9}, \frac{2}{3}, 2, 6, 18$

b) *Given:* $\frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2 = \frac{31}{4}$

$\Rightarrow a\left(\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2\right) = \frac{34}{4}$ …………………………….. (1)

Also $\frac{a}{r^2}.\frac{a}{r}.a.ar.ar^2 = 1 \Rightarrow a^5 = 1 \ \therefore a = 1$ ………… (2)

Eqn. (2) in to eqn. (2)

$\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2 = \frac{31}{4}$ *or* $\frac{1}{r} + r + \frac{1}{r^2} + r^2 = \frac{27}{4}$

Let $y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$

$\Rightarrow y + y^2 - 2 = \frac{27}{4}$ *or* $y + y^2 = \frac{35}{4}$

$\Rightarrow 4y^2 + 4y - 35 = 0$ *or* $(2y - 5)(2y + 7) = 0$

$\therefore y = \frac{5}{2}, y = -\frac{7}{2}$

For $r = \frac{5}{2}, \Rightarrow r + \frac{1}{r} = \frac{5}{2}$ *or* $2r^2 - 5r + 2 = 0$

$\Rightarrow 2r^2 - 4r - r + 2 = 0$

$\Rightarrow (2r - 1)(r - 2) = 0 \ \therefore r = 2, \frac{1}{2}$

For $r = 2$, the numbers are: $\frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Also for $r = -\frac{7}{2} \Rightarrow r + \frac{1}{r} = -\frac{7}{2}$ *or* $2r^2 + 7r + 2 = 0$

$r = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 2}}{4} = \frac{-7 \pm \sqrt{33}}{4}$

The values of r can be neglected since they are irrational, the numbers are:
$\frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Task

1. *The sum of three numbers in a G.P is 35 and their product is 1000. Find the numbers.* **ANSWER**: $For\ \boldsymbol{r = 2}, \boldsymbol{nos.are\ 5, 10, 20}; for\ \boldsymbol{r = \frac{1}{2}}, \boldsymbol{nos.are\ 20, 10, 5}$
2. *The sum of three numbers in a G.P is* $\frac{124}{5}$ *and their product is 64. Find the numbers.* **ANSWER**: $For\ \boldsymbol{r = 5\ or \frac{1}{5}}, \boldsymbol{nos.are \frac{4}{5}, 4, 20}$
3. *Find three numbers in a G.P such that;*
   i) *Their sum is* $\frac{2}{5}$ *and their product is* -8. **ANSWER**: $\boldsymbol{\frac{2}{5}, -2, 10}$
   ii) *Their sum is* 28 *and their product is* 512. **ANSWER**: $\boldsymbol{4, 8, 16}$ *or* $\boldsymbol{16, 8, 4}$

TO FIND THE SUM OF $\boldsymbol{n - terms}$ OF A G.P

Let $a, ar, ar^2, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. ar^{n-1}$ be the $n - terms$ in a G.P.

Let $S_n = a + ar + ar^2 + \cdots \dots\dots\dots\dots\dots\dots. + ar^{n-2} + ar^{n-1}$ ……………………………… (1)

S_n is the sum of $n - terms$

Multiplying eqn. (1) by r

$\Rightarrow rS_n = ar + ar^2 + ar^3 + \cdots \dots\dots\dots\dots\dots\dots. + ar^{n-1} + ar^n$ ……………………………… (2)

$eqn. (1) - eqn. (2)$

$\Rightarrow S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1 - r) = a(1 - r^n)\ or\ S_n = \frac{a(1-r^n)}{1-r}$

$$\boxed{\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}, \text{where } \boldsymbol{r \neq 1}}$$

NOTE:

1) In case $\boldsymbol{r = 1}, \boldsymbol{S_n} = a + a + a + \cdots \dots\dots\dots\dots\dots\dots + a(n - times) = \boldsymbol{na}$
2) In case $|\boldsymbol{r}| < 1, then\ \boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$
3) In case $|\boldsymbol{r}| > 1, then\ \boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$

Examples

1. *Find the sum of 11 terms of a G.P*
   a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\dots\dots\dots\dots\dots\dots\dots$
   b) $3 + 33 + 333 + \cdots \dots\dots\dots\dots\dots\dots$

Solution

a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\dots\dots\dots\dots\dots\dots\dots$

Common ratio: $r = -\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$

Given: $r = -\frac{1}{2}, a = 1, n = 11$

Using $S_n = \frac{a(1-r^n)}{1-r}$ since $|r| < 1$

$\Rightarrow S_n = \frac{1\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{1+\frac{1}{2^{11}}}{1+\frac{1}{2}}$

$= \frac{2^{11}+1}{2^{11}\times\frac{3}{2}} = \frac{2^{11}+2}{3\times2^{10}} = \frac{2050}{3072} = \frac{1025}{1536}$

b) $3 + 33 + 333 + \cdots \dots\dots\dots\dots\dots\dots$

Common ratio: $r = \frac{33}{3} = \frac{333}{33} = 11$

Given: $r = 11\,, a = 3\,, n = 11$

Using $S_n = \frac{a(r^n - 1)}{r-1}$ since $|r| > 1$

$\Rightarrow S_n = \frac{3(11^{11}-1)}{11-1} = \frac{3}{10}(11^{11} - 1)$

2. *Find how many terms of the G.P* $1 + 3 + 9 + \cdots \ldots\ldots\ldots\ldots$ *are required to make a total of more than 1 million.*

Solution

$1 + 3 + 9 + \cdots \ldots\ldots\ldots\ldots$, has $r = 3$

Using $S_n = \frac{a(r^n - 1)}{r-1}$, $|r| > 1$

$\Rightarrow S_n = \frac{1(3^n-1)}{3-1} = \frac{1}{2} \times (3^n - 1)$

$\Rightarrow 10^6 = \frac{1}{2} \times (3^n - 1)$

$\Rightarrow 2 \times 10^6 + 1 = 3^n$

$\Rightarrow 3^n = 2\,000\,001$

$\Rightarrow \log 3^n = \log 2\,000\,001$

$\therefore n = \frac{\log 2\,000\,001}{\log 3} = 13.206$

Thus the number of terms required to make a total of more than 1 million will be 14.

3. *What is the smallest number of terms of the G.P* $5\,, 10\,, 20\,, \ldots\ldots\ldots\ldots$ *can give a sum greater than* $500\,000$

Solution

$5\,, 10\,, 20\,, \ldots\ldots\ldots\ldots$, has $r = 2$

Using $S_n = \frac{a(r^n - 1)}{r-1}$, $|r| > 1$

$\Rightarrow S_n = \frac{5(2^n-1)}{2-1} = 5 \times (2^n - 1)$

$\Rightarrow 500\,000 = 5 \times (2^n - 1)$

$\Rightarrow 100\,000 + 1 = 2^n$

$\Rightarrow 2^n = 100\,001$

$\Rightarrow \log 2^n = \log 100\,001$

$\therefore n = \frac{\log 100\,001}{\log 2} = 16.6097$

Thus the number of terms required to make a total exceeding 500 000 is 17

4. *The sum of the first eight terms of a G.P (of real terms) is five times the sum of the first four terms. Find the common ratio.*

Solution

Let the G.P be $a\,, ar\,, ar^2\,, ar^3\,, \ldots\ldots\ldots\ldots\ldots$

$S_8 =$ Sum of first $8 -$ $terms = \frac{a(1-r^8)}{1-r}$

$S_4 =$ Sum of first $4 -$ $terms = \frac{a(1-r^4)}{1-r}$

Given: $S_8 = 5S_4$

$\Rightarrow \frac{a(1-r^8)}{1-r} = \frac{5a(1-r^4)}{1-r}$

$\Rightarrow 1 - r^8 = 5(1 - r^4)$

$\Rightarrow 1 - r^8 = 5 - 5r^4$

$\therefore r^8 - 5r^4 + 4 = 0$

Let $r^4 = x$

$\Rightarrow x^2 - 5x + 4 = 0 \;\; or \;\; (x-1)(x-4) = 0$
$\therefore x = 1, 4$
$\Rightarrow r^4 - 1 = 0 \;\; or \; (r^2-1)(r^2+1) = 0$
$\therefore r^2 - 1 = 0 \;\; or \;\; r = \pm 1 \; ; \; r^2 + 1 = 0 \, , has\ no\ real\ roots$
Now for $r = 1$, the series is $a + a + a + a + \cdots \ldots \ldots \ldots$
But then, $S_8 = 8a$ and $S_4 = 4a$
So $S_8 \neq 4S_4$, hence $r \neq 1$
In case $r = -1$, we get $S_8 = 0$ and $S_4 = 0$
Thus $S_8 = 5S_4 \; for \; r = -1$
Again $r^4 - 4 = 0 \;\; or \; (r^2-2)(r^2+2) = 0$
$\therefore r^2 - 2 = 0 \;\; or \;\; r = \pm\sqrt{2}$
$\therefore r = -1 \; ; \; \pm\sqrt{2}$

5. *The sum of the last three terms of a G.P having $n -$ terms is 1024 times the sum of the first $3 -$ terms of the progression. If the third term is 5, find the last term.*

Solution

Let the G.P be $a \, , ar \, , ar^2 \, , ar^3 \, , \ldots \ldots \ldots \ldots \ldots . ar^{n-3} \, , ar^{n-2} \, , \; ar^{n-1}$
Given: $ar^{n-3} + \; ar^{n-2} + ar^{n-1} = 1024 \times S_3$
$a_3 = ar^2 = 5$
Also $\quad S_3 = \frac{a(1-r^3)}{1-r}$
$\Rightarrow ar^{n-3} + \; ar^{n-2} + ar^{n-1} = 1024 \times \frac{a(1-r^3)}{1-r}$
$\Rightarrow r^{n-3} + r^{n-2} + r^{n-1} = 1024 \times \frac{(1+r+r^2)(1-r)}{1-r}$
$\Rightarrow r^{n-3}[1 + r + r^2] = 1024(1 + r + r^2)$
$\Rightarrow r^{n-3} = 1024$ ………………………………………………… (1)
Also $ar^2 = 5$ ……………………………………………………… (2)
Now last term, $a_n = ar^{n-1}$ …………………………………… (3)
$Eqn. (2) \div eqn. (3) to\ eliminate\ a$
$\Rightarrow \frac{ar^2}{ar^{n-1}} = \frac{5}{a_n}$
$\therefore a_n = 5 \times r^{n-3}$ ………………………………………………… (4)
$Eqn. (1)\ in\ to\ eqn. (4)$
$\Rightarrow a_n = 5 \times r^{n-3} = 5 \times 1024 = 5120$
$\therefore$ *Last term is* 5120

6. *The second term and the third term of a G.P are* 24 *and* $12(b+1)$ *respectively. Find b if the sum of the first* $3 -$ *terms of progression is 76.*

Solution

Given: $a_2 = 24 \, , a_3 = 12(b+1), S_3 = 76$
$a_2 = ar = 24$ …………………………………………………… (1)
$a_3 = ar^2 = 12(b+1)$ ………………………………………… (2)
$S_3 = \frac{a(1-r^3)}{1-r} = 76$ ………………………………………………… (3)
$Eqn. (1) \div eqn. (2)\ to\ eliminate\ a$
$\Rightarrow \frac{ar}{ar^2} = \frac{24}{12(b+1)} \;\; or \;\; \frac{1}{r} = \frac{2}{b+1}$
$\Rightarrow r = \frac{b+1}{2}$ ………………………………………………………… (4)
From $eqn. (3)$

$\frac{a(1-r^3)}{1-r} = 76 \ or \ \frac{a(1+r+r^2)(1-r)}{1-r} = 76$

$\Rightarrow a(1 + r + r^2) = 76$ .... (5)

$Eqn. (5) \div eqn. (1) \ to \ eliminate \ a$

$\Rightarrow \frac{a(1+r+r^2)}{ar} = \frac{76}{24} = \frac{19}{6}$

$\Rightarrow 6 + 6r + 6r^2 = 19r$

$\Rightarrow 6r^2 - 13r + 6 = 0 \ or \ (2r - 3)(3r - 2) = 0 \ \therefore r = \frac{2}{3}, \frac{3}{2}$

From eqn. (4) $r = \frac{b+1}{2}$

For $r = \frac{b+1}{2}, r = \frac{2}{3}$

$\Rightarrow \frac{b+1}{2} = \frac{2}{3} \ or \ 4 = 3b + 3 \ \therefore b = \frac{1}{3}$

For $r = \frac{b+1}{2}, r = \frac{3}{2}$

$\Rightarrow \frac{b+1}{2} = \frac{3}{2} \ or \ 1 = b + 1 \ \therefore b = 2$

$\therefore b = \frac{1}{3}, 2$

7. *The* $n^{th} - term$ *of a G.P is* a_n. *Given that* $a_1 = 1, a_2 = 1 + x, \ a_3 = 5 + x^2, a_4 = a + x^4$. *Find a and x, hence determine the sum of the first* $10 - terms$ *of a G.P.*

Solution

Given: $a_1 = 1, a_2 = 1 + x, a_3 = 5 + x^2, a_4 = a + x^4$

$\Rightarrow r = \frac{1+x}{1} = \frac{5+x^2}{1+x} = \frac{a+x^4}{5+x^2}$

$\Rightarrow \frac{1+x}{1} = \frac{5+x^2}{1+x} \ or \ (1 + x)^2 = 5 + x^2$

$\Rightarrow 1 + 2x + x^2 = 5 + x^2$

$\Rightarrow 1 + 2x = 5 \ \therefore x = 2$

$\therefore r = 1 + x = 1 + 2 = 3$

From $\frac{a+x^4}{5+x^2} = r$

$\Rightarrow \frac{a+x^4}{5+x^2} = 3 \ \therefore \frac{a+2^4}{5+2^2} = 3$

$\Rightarrow a + 16 = 27 \ \therefore a = 11$

$\therefore a = 11, x = 2$

$S_{10} = \frac{1\times(3^{10}-1)}{3-1} = \frac{1}{2} \times (3^{10} - 1) = 29\,524$

Task

1. *The ratio of the* $9^{th} - term$ *of a G.P to the* $6^{th} - term$ *is* -8, *and the* $5^{th} - term$ *is 16. Ding the G.P.*

   ANSWER: 1, −2, 4, −8, 16

2. *The* $10^{th} - term$ *of a G.P is double the* $12^{th} - term$. *If the* $3^{rd} - term$ *is 6. Find the*

   $5^{th} - term$.

   ANSWER: 3

3. *The sum of the first three terms of a G.P is 7 and the fourth term exceeds the first by 7. Find the seventh term.*

   ANSWER: 64

PROOFS INVOLVING SUMMATION OF A G.P $.i.e.\ S_n = \frac{a(r^n-1)}{r-1}$ **FORMULAE AND** $n^{th} - TERM\ a_n = ar^{n-1}$

Examples

1. *The* 4^{th}, 7^{th} *and* 10^{th} *terms of a G.P are* l, m *and* n *respectively. Prove that* $m^2 = ln$.

Solution

Given: $a_4 = l$, $a_7 = m$, $a_{10} = n$

$\Rightarrow l = ar^3$ ………………………………………………… (1)

$\Rightarrow m = ar^6$ ………………………………………………… (2)

$\Rightarrow n = ar^9$ ………………………………………………… (3)

$Eqn.(1) \div eqn.(2)$ *to eliminate* a

$\Rightarrow \frac{l}{m} = \frac{ar^3}{ar^6} = \frac{1}{r^3}\ \therefore r^3 = \frac{m}{l}$ ……………………………… (4)

$Eqn.(2) \div eqn.(3)$ *to eliminate* a

$\Rightarrow \frac{m}{n} = \frac{ar^6}{ar^9} = \frac{1}{r^3}\ \therefore r^3 = \frac{n}{m}$ ……………………………… (5)

$Eqn.(4) = eqn.(5)$

$\Rightarrow \frac{m}{l} = \frac{n}{m}\ \therefore\ m^2 = ln$

2. *If* p^{th}, q^{th}, r^{th} *term of a G.P are respectively equal to* a, b, c, *then prove that* $a^{q-r}.b^{r-p}.c^{p-q} = 1$

Solution

Let the $1^{st} - term = A$ and $common\ ratio = R$

Using $A_n = AR^{\,n-1}$

Given: $A_p = a = AR^{\,p-1}$ ………………………………………………… (1)

$A_q = b = AR^{\,q-1}$ ………………………………………………… (2)

$A_r = c = AR^{\,r-1}$ ………………………………………………… (3)

Now $a^{q-r}.b^{r-p}.c^{p-q} = [AR^{\,p-1}]^{q-r}.[AR^{\,q-1}]^{r-p}.[AR^{\,r-1}]^{p-q}$

$= A^{(q-r)}.R^{(p-1)(q-r)}.A^{(r-p)}.R^{(q-1)(r-p)}.A^{(p-q)}.R^{(r-1)(p-q)}$

$= A^{(q-r+r-p+p-q)}.R^{(p-1)(q-r)(q-1)(r-p)+(r-1)(p-q)}$

$= A^0.R^{pq-pr-q+r+qr-pq-r+p+pr-qr-p+q}$

$= A^0.R^0 = 1$

3. *If* x, y *and* z *are* n^{th}, $(2n)^{th}$ *and* $(3n)^{th}$ *terms of a G.P, show that* $y^2 = xz$.

Solution

Given: $a_n = x$, $a_{2n} = y$ *and* $a_{3n} = z$

$\Rightarrow x = ar^{n-1}, y = ar^{2n-1}, z = ar^{3n-1}$

$\Rightarrow \frac{x}{y} = \frac{ar^{n-1}}{ar^{2n-1}} = r^{n-1-2n+1} = r^{-n}$

$\therefore \frac{y}{x} = r^n$ ……………………………………………………………… (1)

$\frac{y}{z} = \frac{ar^{2n-1}}{ar^{3n-1}} = r^{2n-1-3n+1} = r^{-n}$

$\therefore \frac{z}{y} = r^n$ ……………………………………………………………… (2)

$eqn.(1) = eqn.(2)$

$\Rightarrow \frac{z}{y} = \frac{y}{x}\ \therefore\ y^2 = xz$

4. *The p^{th} term of a G.P is p and the q^{th} term is q. show that the n^{th} term is given as* $p.\left[\frac{p^{n-q}}{q^{n-p}}\right]^{\frac{1}{p-q}}$

Solution

Given: $a_p = p = ar^{p-1}$.. (1)

$a_q = q = ar^{q-1}$.. (2)

In these two equations, a and r are unknowns. Eliminating a from the equations,

$\Rightarrow \frac{p}{q} = \frac{ar^{p-1}}{ar^{q-1}} = r^{p-q}$

$\therefore r = \left[\frac{p}{q}\right]^{\frac{1}{p-q}}$ .... (3)

$eqn.(3) in\ to\ eqn.(1)$ to find a,

From $p = ar^{p-1} \Rightarrow a = \frac{p}{r^{p-1}} = \frac{p}{\left(\left[\frac{p}{q}\right]^{\frac{1}{p-q}}\right)^{p-1}} = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}}$

$\therefore a = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}}$ .... (4)

Now $a_n = ar^{n-1}$

$\Rightarrow a_n = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}} \times \left[\left[\frac{p}{q}\right]^{\frac{1}{p-q}}\right]^{n-1}$

$= \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}} \times \left[\frac{p}{q}\right]^{\frac{n-1}{p-q}} = p \times \left[\frac{p}{q}\right]^{\frac{n-1}{p-q}-\frac{p-1}{p-q}} = p.\left[\frac{p}{q}\right]^{\frac{n-p}{p-q}} = p \times \left[\frac{p^{n-p}}{q^{n-p}}\right]^{\frac{1}{p-q}}$

Task:

Prove that the $(n+1)^{th}$ term of a G.P of which the first term is a and the third term is b, is equal to $(2n+1)^{th}$ term of a G.P of which the first term is a and the fifth term is b.

5. *If S be the sum, P the product and R be sum of reciprocal of $n-$ terms in a G.P. prove that $P^2\ R^n = S^n$.*

Solution

Let $a, ar, ar^2, \dots\dots\dots\dots\dots\dots\dots\dots\dots . ar^{n-1}$ be $n - terms$ in a G.P with $|r| < 1$

$\Rightarrow S = a + ar + ar^2 + \cdots \dots\dots\dots\dots\dots . + ar^{n-1}$

$\therefore S = \frac{a(1-r^n)}{1-r}$ .... (1)

$\Rightarrow P = a\,.\,ar.\,ar^2. \dots\dots\dots\dots\dots\dots\dots\dots\dots . ar^{n-1}$

$P = a^{1+1+1+1+\cdots\dots\dots+n}.\,r^{1+2+3+\cdots\dots\dots\dots\dots+n-1}$

$= a^n.r^{\frac{n(n-1)}{2}}$ $\left[\begin{array}{l}\because 1+2+3+\cdots\dots\dots..+n-1\ is\ an\ A.P\ with\ a=1 \\ d=1, n=n-1\ \therefore S_n = \frac{n-1}{2}\left(2+(n-1+1)\right) = \frac{n(n-1)}{2}\end{array}\right]$

$\therefore P = a^n.r^{\frac{n(n-1)}{2}}$ .... (2)

$\Rightarrow R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \cdots \dots\dots\dots\dots\dots + \frac{1}{ar^{n-1}}$

$= \frac{1}{a}\left[\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots + \frac{1}{r^{n-1}}\right]$

$= \frac{1}{a}\left[\frac{1.\left[\left(\frac{1}{r}\right)^n - 1\right]}{\frac{1}{r}-1}\right]$ $\left[\because Since\ |r| > 1, then\ using\ S_n = \frac{a(r^n-1)}{r-1}\right]$

$\therefore R = \frac{1}{a}\left[\frac{(1-r^n)r}{(1-r)r^n}\right]$ .... (3)

$eqn.(1) in\ to\ eqn.(3)$

$\Rightarrow R = \frac{1}{a}.\left[\frac{S}{a}.\frac{r}{r^n}\right] = \frac{S.r}{a^2.r^n} = \frac{S}{a^2.r^{n-1}}$

Now $R^n = \left[\frac{S}{a^2.r^{n-1}}\right]^n = \frac{S^n}{a^{2n}.r^{n(n-1)}}$

$\Rightarrow a^{2n}.r^{n(n-1)}R^n = S^n$ .... (4)

But from $eqn.(1)$

$\Rightarrow P = a^n.r^{\frac{n(n-1)}{2}}$ $\therefore \frac{P}{a^n} = r^{\frac{n(n-1)}{2}}$

Squaring

$\Rightarrow \frac{P^2}{a^{2n}} = r^{n(n-1)}$ .... (5)

$eqn.(5) in\ to\ eqn.(4)$

$\Rightarrow S^n = a^{2n}.r^{n(n-1)}R^n = a^{2n} \times \frac{P^2}{a^{2n}} \times R^n$

$\therefore\ P^2\, R^n = S^n$

6. *If S_1, S_2, S_3 be respectively the sum of n, $2n$, $3n$ terms of a G.P, prove that* $S_1(S_3 - S_2) = (S_2 - S_1)^2$

Solution

Given: $S_1 = S_n = \frac{a(1-r^n)}{1-r}$.. (1)

$S_2 = S_{2n} = \frac{a(1-r^{2n})}{1-r}$.. (2)

$S_3 = S_{3n} = \frac{a(1-r^{3n})}{1-r}$.. (3)

For all $|r| < 1$

Now $S_3 - S_2 = \frac{a(1-r^{3n})}{1-r} - \frac{a(1-r^{2n})}{1-r}$

$= \frac{a}{1-r}.(r^{2n} - r^{3n}) = \frac{ar^{2n}}{1-r}.(1 - r^n)$ .... (4)

Again $S_1(S_3 - S_2) = \frac{a(1-r^n)}{1-r} \times \frac{ar^{2n}}{1-r}.(1 - r^n)$

$\Rightarrow S_1(S_3 - S_2) = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2}$ .... (5)

$\boldsymbol{R.H.S} = (S_2 - S_1)^2 = \left[\frac{a(1-r^{2n})}{1-r} - \frac{a(1-r^n)}{1-r}\right]^2$

$= a^2\left[\frac{r^n - r^{2n}}{1-r}\right]^2 = a^2\left[\frac{r^n(1-r^n)}{1-r}\right]^2 = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2}$

$\therefore$ **R. H. S = L. H. S**

7. *If S_n represent sum of $n-$ terms of a G.P whose first term and common ratio are a and r respectively. Prove that* $S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

Solution

Given: $S_n = \frac{a(1-r^n)}{1-r}$

$S_1 = \frac{a(1-r^1)}{1-r} = a$; $S_2 = \frac{a(1-r^2)}{1-r}$; $S_3 = \frac{a(1-r^3)}{1-r}$

$S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + S_n$

$= a + \frac{a(1-r^2)}{1-r} + \frac{a(1-r^3)}{1-r} + \cdots \ldots\ldots.. + \frac{a(1-r^n)}{1-r}$

$= \frac{a}{1-r}[(1-r) + (1-r^2) + (1-r^3) + \cdots \ldots\ldots\ldots\ldots\ldots + (1-r^n)]$

$= \frac{a}{1-r}[(1 + 1 + 1 + 1 + \cdots ..n - times) - (r + r^2 + r^3 + \cdots \ldots. + r^n)]$

$= \frac{a}{1-r}\left[n - \frac{r(1-r^n)}{1-r}\right] = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

$\therefore\ S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

8. *The sum of $2n-$ terms of a G.P whose first term is a and the common ratio r is equal to the sum of $n-$ terms of a G.P whose first term is b and common ratio r^2. Prove that b is equal to the sum of the first two terms of the series.*

Solution

Given: $S_{2n} = S_n$

1^{st} *serries*: $a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots\ldots, ar^{2n-1}$

2^{nd} *serries*: $b, br^2, br^4, br^6, \ldots\ldots\ldots\ldots\ldots, br^{n-1}$

Now $S_{2n} = \frac{a(1-r^{2n})}{1-r}$, $S_n = \frac{b[1-(r^2)^n]}{1-r^2}$

From $S_{2n} = S_n$

$\Rightarrow \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{1-r^2}\ \ \therefore \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{(1+r)(1-r)}$

$\Rightarrow a = \frac{b}{1+r}\ \therefore b = a + ar$

Thus b is the sum of the first two terms of the series.

Task

1) *If in a G.P, $(p+q)^{th}$ term is m and $(p-q)^{th}$ is n. show that p^{th} term is $\sqrt{mn}$ and the q^{th} term is $m.\frac{2q-p}{2q}.n.\frac{p}{2q}$*

2) *There are $n-$ terms in a G.P, show that the n^{th} root of their product is equal to the square root of the product of its first and last terms.*

Other forms of proofs in G.Ps

1. *if a, b, c are in a G.P and $a^x = b^y = c^z$, prove that $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$*

Solution

a, b, c are in a G.P if

$\frac{b}{a} = \frac{c}{b}$ *or* $b^2 = ac$ ……………………………………………… (1)

Expressing ac in terms of b

$a^x = b^y \Rightarrow a = b^{y/x}$ …………………………………………… (2)

$b^y = c^z \Rightarrow c = b^{y/z}$ …………………………………………… (3)

$eqn.(2),(3)\ in\ to\ (1)$

$\Rightarrow b^2 = b^{y/x}.b^{y/z} = b^{\frac{y}{z}+\frac{y}{x}}$

$\therefore 2 = \frac{y}{z}+\frac{y}{x}$ *or* $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$

2. *if a, b, c, d are in a G.P, show that:*
   i) $(a+b), (b+c), (c+d)$ are in a G.P
   ii) $(a-b)^2, (b-c)^2, (c-d)^2$ are in a G.P
   iii) $a^2-b^2,\ b^2-c^2, c^2-d^2$

Solution

i) a, b, c, d are in a G.P if:

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ *or* $b = ar, c = br = ar^2, d = cr = ar^3$

Now $(a+b), (b+c), (c+d)$ are in a G.P if:

$\frac{b+c}{a+b} = \frac{ar+ar^2}{a+ar} = \frac{r(1+r)}{1+r} = r$

$\frac{c+d}{b+c} = \frac{ar^2+ar^3}{ar+ar^2} = \frac{r^2(1+r)}{r(1+r)} = r$

Since $\frac{b+c}{a+b} = \frac{c+d}{b+c} = r$, then $(a+b)\,,(b+c)\,,(c+d)$ are in a G.P

ii) $(a-b)^2\,,(b-c)^2\,,(c-d)^2$ are in a G.P if:

$\frac{(b-c)^2}{(a-b)^2} = \frac{(c-d)^2}{(b-c)^2}$

Now $\frac{(b-c)^2}{(a-b)^2} = \frac{(ar-ar^2)^2}{(a-ar)^2} = \frac{a^2r^2(1-r)^2}{a^2(1-r)^2} = r^2$

Also $\frac{(c-d)^2}{(b-c)^2} = \frac{(ar^2-ar^3)^2}{(ar-ar^2)^2} = \frac{a^2r^4(1-r)^2}{a^{2r^2}(1-r)^2} = r^2$

Since $\frac{(b-c)^2}{(a-b)^2} = \frac{(c-d)^2}{(b-c)^2} = r^2$, then $(a-b)^2\,,(b-c)^2\,,(c-d)^2$ are in a G.P

iii) $a^2-b^2\,,\ b^2-c^2\,,c^2-d^2$ are in a G.P if:

$\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$

Now $\frac{b^2-c^2}{a^2-b^2} = \frac{(ar)^2-(ar^2)^2}{a^2-(ar)^2} = \frac{a^2r^2(1-r^2)}{a^2(1-r^2)} = r^2$

Also $\frac{c^2-d^2}{b^2-c^2} = \frac{(ar^2)^2-(ar^3)^2}{(ar)^2-(ar^2)^2} = \frac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)} = r^2$

Thus $a^2-b^2\,,\ b^2-c^2\,,c^2-d^2$ are in a G.P since $\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$

3. *If $a\,,b\,,c\,,d$ are in a G.P, show that:*
   i) $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$
   ii) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$

Solution

i) $a\,,b\,,c\,,d$ are in a G.P if:

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ *or* $b = ar\,,c = br = ar^2\,,d = cr = ar^3$

Now $(b-c)^2 + (c-a)^2 + (d-b)^2 = (ar-ar^2)^2 + (ar^2-a)^2 + (ar^3-ar)^2$

$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$

$= a^2[r^2(1-2r+r^2) + r^4 - 2r^2 + 1 + r^2(r^4-2r^2+1)]$

$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$

$= a^2[r^6 - 2r^3 + 1] = a^2(r^3-1)^2$

Now $\mathbf{R.H.S} = (a-d)^2 = (a-ar^3)^2 = a^2(1-r^3 1)^2 = a^2(r^3-1)^2$

Thus $\mathbf{R.H.S} = \mathbf{L.H.S}$

ii) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$

$\frac{ab-cd}{b^2-c^2} = \frac{a.ar-ar^2.ar^3}{a^2r^2-(ar^2)^2} = \frac{1-r^4}{r^2-r^4} = \frac{1-r^4}{r(1-r^2)}$

$= \frac{(1-r^2)(1+r^2)}{r(1-r^2)} = \frac{1+r^2}{r}$

Now $\frac{a+c}{b} = \frac{a+ar^2}{ar} = \frac{1+r^2}{r}$

Thus $\mathbf{R.H.S} = \mathbf{L.H.S}$

Task

If $a\,,b\,,c\,,d$ are in a G.P, show that

i) $a^2b^2c^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = a^3 + b^3 + c^3$

ii) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$

iii) $(ab+bc+cd)^2 = (a^2+b^2+c^2)(b^2+c^2+d^2)$

iv) $a^2+b^2\,,ab+bc\,,b^2+c^2$, are in a G.P

4. *If $a\,,b\,,c$ form an A.P, $b\,,c\,,a$ form a G.P, show that $\frac{1}{c}\,,\frac{1}{a}\,,\frac{1}{b}$ form an A.P.*

Solution

If a, b, c *form an A.P, then*

$b - a = c - b$ *or* $a + c = 2b$ .... (1)

b, c, a *, form a G.P if;*

$\frac{c}{b} = \frac{a}{c}$ *or* $ab = c^2$ .... (2)

Now for $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ to be in an A.P, then

$\frac{1}{a} - \frac{1}{c} = \frac{1}{b} - \frac{1}{a} \Rightarrow \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$

Generating $\frac{2}{a} = \frac{1}{b} + \frac{1}{c}$ from eqn. (1) and eqn. (2)

Eqn. (1) $\div$ eqn. (2)

$\Rightarrow \frac{a+c}{ab} = \frac{2b}{c^2}$

But $ab = c^2$,

$\Rightarrow \frac{1}{b} + \frac{c}{c^2} = \frac{2b}{ab}$ $\therefore \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$

5. *If* a, b, c *are in A.P,* a, x, b *and* b, y, c *are in a G.P, show that* x^2 , b^2 , y^2 *are in an A.P.*

Solution

a, b, c are in an A.P if;

$b - a = c - b$ *or* $2b = a + c$ .... (1)

a, x, b are in a G.P if;

$\frac{x}{a} = \frac{b}{x}$ *or* $x^2 = ab$ .... (2)

b, y, c are in a G.P if;

$\frac{y}{b} = \frac{c}{y}$ *or* $y^2 = bc$ .... (3)

Now x^2 , b^2 , y^2 are in an A.P if;

$b^2 - x^2 = y^2 - b^2$ *or* $2b^2 = x^2 + y^2$ .... (4)

Generating eqn. (4) from the other three equations

Now eqn. (2) + eqn. (3)

$\Rightarrow x^2 + y^2 = ab + bc = b(a + c)$

From eqn. (1), $2b = a + c$

$\therefore x^2 + y^2 = 2b^2$, which is eqn. (4). Hence x^2 , b^2 , y^2 are in an A.P.

Task

1. *If* a, b, c *are in G.P, prove that* $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ *are in an A.P.*
2. *If* b, a, c *are in an A.P and* c, b, a *are in a G.P, show that* $\frac{a+b}{b^2} = \frac{2}{c}$

Combination of A.P and G.P questions

1) Three numbers are in an A.P and their sum is 21, if $1, 5, 15$ *be added to them respectively, they form a G.P. find the numbers.*

Solution

Let the 3 numbers in an A.P be $a - d, a, a + d$

Given: $a - d + a + a + d = 21 \therefore a = 7$

Also: $(a - d + 1), (a + 5), (a + d + 15)$ *are in a G.P*

$\Rightarrow \frac{a+5}{a-d+1} = \frac{a+d+15}{a+5}$, but $a = 7$

$\Rightarrow \frac{7+5}{7-d+1} = \frac{7+d+15}{7+5}$

$\Rightarrow \frac{12}{8-d} = \frac{22+d}{12}$ or $144 = (8-d)(22+d) = 176 + 8d - 22d - d^2$
$\Rightarrow d^2 + 14d - 32 = 0$
$\Rightarrow d^2 + 16d - 2d - 32 = 0$ or $(d-2)(d+16) = 0$
$\therefore d = 2\,, -16$
Thus for $a = 7\,, d = 2,$ the numbers are: $6\,, 12\,, 24$
For $a = 7\,, d = -16,$ the numbers are: $6\,, 12\,, 24$

2) *The sequence of three numbers $a\,, b\,, c$ is an A.P whose sum is 18. If a and b are each increased by 4 and c increased by 36, the new numbers form a G.P. find $a\,, b\,, c$.*
Solution
$a\,, b\,, c$ are in an A.P if;
$b - a = c - b$ or $2b = a + c$ .... (1)
$a + b + c = 18$ .... (2)
Given: $a+4\,, b+4\,, c+36$ *are in a G.P if;*
$\frac{b+4}{a+4} = \frac{c+36}{b+4} \Rightarrow (b+4)^2 = (a+4)(c+36)$ .... (3)
Solving the equations simultaneously
Eqn. (1) in to eqn. (2)
$\Rightarrow b + 2b = 18\,, b = 6$
Put $b = 6$ in to eqn. (1) and (3)
$\Rightarrow a + c = 12$ .... (4)
$\Rightarrow 100 = (a+4)(c+36)$ .... (5)
From eqn. (4) $a = 12 - c$.. (6)
Eqn. (6) in to (5)
$\Rightarrow 100 = (12 - c + 4)(c + 36) = (16 - c)(c + 36)$
$\Rightarrow 100 = -c^2 - 20c + 576$
$\therefore c^2 + 20c + 476 = 0$ $\therefore c = 14\,, -34$
From eqn. (6), $a = 12 - c$
For $c = 14\,, a = 12 - 14 = -2$
For $c = -34\,, a = 12 + 34 = 46$
Hence numbers are: $-2\,, 6\,, 14$ or $46\,, 6\,, -34$

3) *The 1^{st} , 10^{th} and 28^{th} terms of an A.P are three successive terms of a G.P. find the common ratio of a G.P. given that the sum of the first 28 terms of an A.P is 210, find its first term.*
Solution
Given: $a_1 = a\,, a_{10} = a + 9d\,, a_{28} = a + 27d$
Now $a\,, a + 9d\,, a + 27d$ are in a G.P
$\Rightarrow \frac{a+9d}{a} = \frac{a+27d}{a+9d}$ .... (1)
Let the three consecutive terms of a G.P be: $a\,, ar\,, ar^2$
Note: the first term of an A.P is equal to the first term of a G.P
$\Rightarrow ar = a + 9d$ or $ar - a = 9d$ .... (2)
$\Rightarrow ar^2 = a + 27d$ or $ar^2 - a = 27d$ .... (3)
Given: $S_{28} = \frac{28}{2}(2a + 27d) = 210$
$\Rightarrow 2a + 27d = 15$ .... (4)
$eqn.(2) \div eqn.(3)$
$\Rightarrow \frac{ar-a}{ar^2-a} = \frac{r-1}{r^2-1} = \frac{1}{3}$

$\Rightarrow \frac{r-1}{(r+1)(r-1)} = \frac{1}{3}$ *or* $3 = r + 1 \quad \therefore r = 2$

Substitute $r = 2$ in to eqn. (2)

$\Rightarrow 2a - a = 9d \therefore a = 9d$ .... (5)

eqn. (5) in to eqn. (4)

$\Rightarrow 2(9d) + 27d = 15 \quad \therefore d = \frac{15}{45} = \frac{1}{3}$

$\therefore 1^{st}\ term = 9d = 9 \times \frac{1}{3} = 3$

4) *The sum of the first three terms of two series: one an A.P and the other a G.P is the same. If the first term of each of these is* $\frac{2}{3}$ *and the common difference of an A.P is equal to the common ratio of the G.P, find the sum of each series to 2o terms.*

Solution

In an A.P, $S_3 = \frac{3}{2}(2a + 2d) = 3a + 3d$

In a G.P, $S_3 = \frac{a(1-r^3)}{1-r} = \frac{a(1-r)(1+r+r^2)}{1-r} = a(1 + r + r^2)$

Given: $a = \frac{2}{3}, r = d$

$\Rightarrow 3a + 3d = a(1 + r + r^2)$

$\Rightarrow 3 \times \frac{2}{3} + 3r = \frac{2}{3}(1 + r + r^2)$ *or* $6 + 9r = 2 + 2r + 2r^2$

$\Rightarrow 2r^2 - 7r - 4 = 0$ *or* $(2r + 1)(r - 4) = 0$

$\therefore r = 4, -\frac{1}{2}$

Now when $r = d = 4$, for an A.P,

$S_{20} = \frac{20}{2}\left(2 \times \frac{2}{3} + 19 \times 4\right) = 773\frac{1}{3}$

For a G.P, $S_{20} = \frac{\frac{2}{3}(4^{20}-1)}{4-1} = \frac{2}{9}(4^{20} - 1)$

Also when $r = d = -\frac{1}{2}$, for an A.P

$S_{20} = \frac{20}{2}\left(2 \times \frac{2}{3} + 19 \times -\frac{1}{2}\right) = -\frac{245}{3}$

For a G.P, $S_{20} = \frac{\frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{20}\right)}{1 - -\frac{1}{2}} = \frac{1}{9}\frac{(2^{20}-1)}{2^{18}}$

5) *Find three numbers* a, b, c *between 2 and 18 such that their sum is 25, the numbers* $2, a, b$ *are consecutive terms of an A.P and the numbers* $b, c, 18$ *are the consecutive terms of a G.P.*

Solution

Given: $a + b + c = 18$ .... (1)

$2, a, b$ form an A.P

$\Rightarrow a - 2 = b - a$ *or* $2a = b + 2$ .... (2)

Also $b, c, 18$ form a G.P

$\Rightarrow \frac{c}{b} = \frac{18}{c}$ *or* $c^2 = 18b$ .... (3)

Solving the equations simultaneously

Eqn. (3) in to eqn. (2)

$\Rightarrow 2a = \frac{c^2}{18} + 2$ *or* $36a = c^2 + 36$ .... (4)

Eliminating b from eqn. (1) and eqn. (2)

From eqn. (2), $b = 2a - 2$

Substituting In equation (1)

$\Rightarrow a + 2a - 2 + c = 25$

$\Rightarrow 3a + c = 27$ ……………………………………………………………… (5)

From eqn. (5), $c = 27 - 3a$

Substituting In equation (4)

$\Rightarrow 36a = (27 - 3a)^2 + 36$

$\Rightarrow 36a = 9(9 - a)^2 + 36$ ………………………………($\div$ 9)

$\Rightarrow 4a = 81 - 18a + a^2 + 4 \;\; or \; a^2 - 22a + 85 = 0$

$\Rightarrow (a - 5)(a - 17) = 0 \;\; \therefore a = 5\,, 17$

From $c = 27 - 3a$

For $a = 5\,, c = 27 - 15 = 12$

For $a = 17\,, c = 27 - 51 = -24\;(Rejected)$

Thus $a = 5\,, c = 12$ are used to fond b.

From eqn. (3), $b = \frac{c^2}{18}$, for $c = 12$

$\Rightarrow b = \frac{144}{18} = 8$

$\therefore a = 5\,, b = 8\,. c = 12$

Hence the numbers are: $5\,, 8\,, 12$

6) *Three integers form an increasing G.P. if the third is decreased by 16 an A.P is formed. If the second number is decreased by 2, a G.P is formed. Find the numbers.*

Solution

Let the integers be: $a\,, ar\,, ar^2$

Decreasing the 3^{rd} number by 16 gives $a\,, ar\,, ar^2 - 16$ are in an A.P

$\Rightarrow ar - a = ar^2 - 16 - ar$

$\Rightarrow 2ar - a = ar^2 - 16$ ………………………………………… (1)

Decreasing the 2^{nd} number by 2 from the new set of numbers give $a\,, ar - 2\,, ar^2 - 16$ are in a G.P

$\Rightarrow \frac{ar-2}{a} = \frac{ar^2-16}{ar-2}$

$\Rightarrow (ar - 2)^2 = a(ar^2 - 16) \;\; or \; a^2r^2 - 4ar + 4 = a^2r^2 - 16a$

$\Rightarrow -4ar + 4 = -16a \;\; or \; 4 = a(4r - 16)$

$\therefore a = \frac{1}{r-4}$ ……………………………………………………………… (2)

$eqn.\,(2) in\; to\; eqn.\,(1)$

$\Rightarrow 2\left(\frac{1}{r-4}\right)r - \frac{1}{r-4} = \frac{1}{r-4} \times r^2 - 16$

$\Rightarrow \frac{2r-1}{r-4} = \frac{r^2-16r+64}{r-4} \;\; or \;\; 2r - 1 = r^2 - 16r + 64$

$\Rightarrow r^2 - 18r + 64 = 0 \;\; or \; (r - 5)(r - 13) = 0$

$\therefore r = 5\,, 13$

From eqn. (2), $a = \frac{1}{r-4}$

For $r = 5\,, a = 1$

For $r = 13\,, a = \frac{1}{9}$

Thus the numbers are formed by $a = 1\,, r = 5\,. i.e.\, 1\,, 5\,, 25$

Task

1. *The* 2^{nd} , 4^{th} *and* 8^{th} *terms of an A.P series are in a G.P series. The sum of the* 3^{rd} *and* 5^{th} *terms is 20. Find the sum of the first 20 terms of the progression.*
ANSWER: 525

2. *An arithmetic series and geometric series have r as the common difference and the common ratio respectively. The first term of the arithmetic series is 1 and the first term of the geometric series is 2. If the* 4^{th} *term of the arithmetic series is equal to the sum of the third and fourth terms of the geometric series, find three possible values of r when* $|r| < 1$*, find in the form* $p + q\sqrt{2}$ *the sum of the first* $10-$ *terms of the arithmetic series.* **ANSWER**: $\frac{-70+45\sqrt{2}}{2}$

3. *A G.P and an A.P have the same first term. The sums of their first, second and third terms are* 6 , 10 , 10.5 *and 18 respectively. Calculate the sum of their fifth terms.* (***Uneb*** **1995**)

$$\left[\begin{array}{l} \boldsymbol{HINT}: \\ For\ an\ A.P: a = 1^{st}\ term\,, a_2 = a + d\,, a_3 = a + 2d\,, a_5 = a + 4d \\ For\ a\ G.P: a = 1^{st}\ term,, a_2 = ar\,, a_3 = ar^2\,, a_5 = ar^4 \\ Now\ a + a = 6\ \therefore a = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1) \\ \Rightarrow a + d + ar = 10.5\ \therefore 3 + d + 3r = 10.5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \\ \Rightarrow a + 2d + ar^2 = 18\ \therefore 3 + 2d + 3r^2 = \cdots. \ldots\ldots\ldots\ldots. \ldots (3) \\ Solving\ eqn. (2) and\ eqn. (3) gives; 3 + 6r - 3r^2 = 3\ \therefore 2r - r^2 = 0 \\ \quad \therefore r = 0\,, r = 2 \\ From\ eqn. (2)\ d = 10.5 - 3 - 6 = 1.5\ \ \therefore d = 1.5 \\ Now\ a_5 = a + 4d + ar^4 = Sum\ of\ their\ fifth\ terms \\ \quad \therefore 3 + 4 \times 1.5 + 3 \times 2^4 = 57 \end{array}\right]$$

4. *The* 2^{nd} *term*, 5^{th} *term and* 11^{th} *term of an A.P are in a G.P. if the* 11^{th} *term is 4, find the difference between the sums of the* $8-$ *terms of each progressions.*

Solution

Given: $a_2 = a + d\,, a_5 = a + 4d\,, a_{11} = a + 10d = 4$ *are in a G.P;*

$Let\ a = 1^{st}\ term\ d = common\ difference$

$\Rightarrow \frac{a+4d}{a+d} = \frac{a+10d}{a+4d} = r = common\ ratio$

$\Rightarrow (a + 4d)^2 = (a + 10d)(a + d)$ …………………………………… (1)

Also $a + 10d = 4$ ………………………………………………………… (2)

Solving eqn. (2) and eqn. (1)

$\Rightarrow (4 - 10d + 4d)^2 = 4(4 - 10d)$ *or* $4(2 - 3d)^2 = 4(4 - 10d)$

$\Rightarrow 4 - 12d + 9d^2 = 4 - 9d\ \therefore 9d^2 - 3d = 0$

$\therefore d = 0\,, \frac{1}{3}$

From eqn. (2) and for $d = \frac{1}{3}\,, a = 4 - \frac{10}{3} = \frac{2}{3}$

For an A.P, $S_8 = \frac{8}{2}\left[2 \times \frac{2}{3} + 7 \times \frac{1}{3}\right] = 4 \times \frac{11}{3} = \frac{44}{3}$

Now the common ratio, $r = \frac{a+4d}{a+d} = \frac{\frac{2}{3}+\frac{4}{3}}{\frac{2}{3}+\frac{1}{3}} = 2$

Now the G.P is; $(a + d)\,, (a + 4d)\,, (a + 10d)$

$1^{st}\ term = a + d = \frac{2}{3} + \frac{1}{3} = 1$

Using $S_n = \frac{a(r^n-1)}{r-1}$

$\Longrightarrow S_8 = \frac{2^8-1}{2-1} = 255$

$\therefore S_{8(G.P)} - S_{8(A.P)} = 255 = \frac{44}{3} = \frac{721}{3}$

5. *If a, b, c are in a G.P, show that $loga$, $logb$, $logc$ are in an A.P.*

TO FIND THE SUM TO INFINITY OF A G.P WHOSE COMMON RATIO IS LESS THAN ONE

Let $a, ar, ar^2, ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. ar^{n-1}$ be a G.P with $|r| < 1$

Sum of $n - terms$ is $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

Since $|r| < 1$, then $r^2 < r, r^3 < r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. r^n < r^{n-1}$

As the power of r goes on increasing, the corresponding term in a G.P decrease in value. So assuming that n becomes indefinitely large, r^n becomes indefinitely small. $1. e. r^n \longrightarrow 0$

Thus from $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

As $n \longrightarrow \infty, r^n \longrightarrow 0$

$$\boxed{\therefore \boldsymbol{S_\infty} = \frac{\boldsymbol{a}}{\boldsymbol{1-r}} \text{, where } |r| < 1 \;\; or \;\; -1 < r < 1}$$

This is the sum to infinity. *i.e.* the limiting value to which the sum approaches/tends to.

$i.e.\, a + ar + ar^2 + ar^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. = \frac{a}{1-r} = \boldsymbol{S_\infty}$

Examples

1. *Express the recurring decimals to fraction form in lowest term.*
   a) $0.1\dot{7}$
   b) $0.0\dot{7}$
   c) $1.\overline{45}$

Solution

a) $0.1\dot{7} = 0.17777 \ldots\ldots\ldots\ldots.$

$0.1\dot{7} = 0.1 + 0.07 + 0.007 + 0.0007 + \cdots \ldots\ldots\ldots$

$= \frac{1}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots.$

$= \frac{1}{10} + \left\{\frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots.\right\}$

The bracketed series is a G.P with first term as $\frac{7}{100}$ and common ratio as $\frac{1}{10}$

$\Longrightarrow \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. = \boldsymbol{S_\infty} = \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{7}{90}$

$\therefore 0.1\dot{7} = \frac{1}{10} + \frac{7}{90} = \frac{8}{45}$

b) $0.0\dot{7} = 0.07777 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$= 0.07 + 0.007 + 0.0007 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$

$= \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots \ldots\ldots\ldots\ldots\ldots.. \ldots\ldots.. = \boldsymbol{S_\infty} = \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{7}{90}$

$\therefore 0.0\dot{7} = \frac{7}{90}$

c) $1.\overline{45} = 1.45454545 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$

$= 1 + 0.45 + 0.0045 + 0.00045 + \cdots \ldots\ldots\ldots.$

$= 1 + \left\{\frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\right\}$

$= 1 + \frac{\frac{45}{100}}{1-\frac{1}{100}} = \frac{45}{99} + 1 = \frac{16}{11}$

$\therefore 1.\overline{45} = \frac{16}{11}$

2. a) *Find the first term of a G.P whose sum to infinite terms is 8 and the second term is 2*

b) The sum of infinite terms of a G.P is 15 and the sum of their squares is 45. Find the series.

Solution

a) *Given:* $S_\infty = 8 = \frac{a}{1-r}$, $ar = 2$

Dividing the equations

$\frac{8}{2} = \frac{\frac{a}{1-r}}{ar} = \frac{1}{r-r^2}$ or $4r - 4r^2 = 1$ $\therefore 4r^2 - 4r + 1 = 0$

$\therefore r = \frac{1}{2}$

$\therefore 1^{st}$ term $= a = \frac{2}{r} = \frac{2}{\frac{1}{2}} = 4$

b) Let $a , ar , ar^2 , ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ be the G.P with $|r| < 1$

Given: $S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 15$ .... (1)

Sum to infinity of squares = 45

$\Rightarrow S_\infty = a^2 + a^2r^2 + a^2r^4 + a^2r^6 \ldots\ldots\ldots\ldots\ldots\ldots = \frac{a^2}{1-r^2} = 45$ .... (2)

From eqn. (1), $a = 15(1-r)$, substituting in eqn. (2)

$\Rightarrow \frac{[15(1-r)]^2}{1-r^2} = 45$ *or* $\frac{225(1-r)^2}{(1+r)(1-r)} = 45$

$\Rightarrow \frac{225(1-r)}{1+r} = 45$ $\therefore 5 - 5r = 1 + r$ $\therefore r = \frac{2}{3}$

But $a = 15(1-r) \Rightarrow a = 15\left(1 - \frac{2}{3}\right) = 5$

Thus the series is $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

3. *The second term of a G.P is 24 and its sum to infinity is 100. Find two possible values of the common ratio and the corresponding first terms.*

Solution

Given: $ar = 24$.. (1)

$S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 100$

$\therefore a = 100 - 100r$ .... (2)

Eqn. (2) in to eqn. (1) to eliminate a

$(100 - 100r)r = 24$ *or* $25r - 25r^2 - 6 = 0$

$\therefore 25r^2 - 25r + 6 = 0$

$\Rightarrow r = \frac{25 \pm \sqrt{(25)^2 - 4\times25\times6}}{2\times25} = \frac{25\pm5}{50}$

$\therefore r = \frac{3}{5} , \frac{2}{5}$

From eqn. (1), for $r = \frac{3}{5}$, $a = \frac{24}{\frac{3}{5}} = 40$

For $r = \frac{2}{5}$, $a = \frac{24}{\frac{2}{5}} = 60$

$\therefore \frac{3}{5} , 40 ; \frac{2}{5} , 60$

4. *The sum of infinite terms of a G.P is 4 and the sum of their cubes is 192. Find the series.*

Solution

Let the series be: $a\,,ar\,,ar^2\,,ar^3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Cubing gives $a^3\,,a^3r^3\,,a^3r^6\,,a^3r^9\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Given: $S_\infty = a^3 + a^3r^3 + a^3r^6 + a^3r^9 \ldots\ldots\ldots\ldots\ldots = \frac{a^3}{1-r^3} = 192$

$\frac{a^3}{1-r^3} = \frac{a^3}{(1-r)(1+r+r^2)} = 192$ ………………………………………………………………… (1)

Also $S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 4$

$\frac{a}{1-r} = 4\ \ or\ a = 4(1-r)$ …………………………………………… (2)

$eqn.(2) in\ to\ eqn.(1)$

$\Longrightarrow \frac{[4(1-r)]^3}{(1-r)(1+r+r^2)} = 192$

$\Longrightarrow \frac{64(1-r)^3}{(1-r)(1+r+r^2)} = 192\ \ or\ \ \frac{(1-r)^2}{(1+r+r^2)} = 3$

$\Longrightarrow 1 - 2r + r^2 = 3 + 3r + 3r^2$

$\therefore 2r^2 + 5r + 2 = 0\ \ or\ (2r+1)(r+1) = 0$

$\therefore r = -\frac{1}{2}\,,-1$, rejecting $r = -1$ since $|r| < 1$

For $r = -\frac{1}{2}\,, a = 4 - 4 \times -\frac{1}{2} = 6$

$\therefore$ The series is $6\,,-3\,,\frac{3}{2}\,,-\frac{3}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Task

1. *Show that there are two possible G.Ps, in each of which the first term is 8, and the sum of three terms is 14. Find the sum to infinity in each.*
2. *The sum infinity of a G.P with positive common ratio is 9 and the sum of the first two terms is 5. Find the first four terms of the G.P.*

   ANSWER: $\mathbf{3\,,2\,,\frac{4}{3}\,,\frac{8}{9}}$
3. *Show that there are two G.P's in which the second term is* $-\frac{4}{3}$ *and the sum of the first* $3-terms$ *is* $\frac{28}{9}$. *Show that one of these progressions converges to a sum of 3.*
4. *If* $x = 1 + a + a^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + \infty\,, y = 1 + b + b^2 + \cdots \ldots\ldots\ldots\ldots + \infty$, *prove that if* $|a| < 1\ and\ |b| < 1\ then\ 1 + ab + a^2b^2 + \cdots \ldots\ldots\ldots\ldots = \frac{xy}{x+y-1}$
5. *Show that the sum of the series* $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \frac{1}{2^7} + \cdots \ldots to\ \infty = \frac{19}{24}$
6. a) If $S_1\,,S_2\,,S_3\,,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots S_n$ are sums of infinite G.P series whose first terms are $1\,,2\,,3 \ldots\ldots\ldots\ldots\ldots, n$ and the common ratios are $\frac{1}{2}\,,\frac{1}{3}\,,\frac{1}{4}\,,\ldots\ldots\ldots,\frac{1}{n+1}$ respectively. Show that $S_1 + S_2 + S_3 + \cdots \ldots\ldots + S_n = \frac{n(n+3)}{2}$

b) If $S_1\,,S_2\ and\ S$ are sums of $n-terms, 2n-terms$ and to infinity of a G.P respectively, show that $S_1(S_1 - S) = S(S_1 - S_2)$

Solution

a) $S_\infty = \frac{a}{1-r}$

$S_1 = \frac{1}{1-\frac{1}{2}} = 2\,, S_2 = \frac{2}{1-\frac{1}{3}} = 3\,, S_3 = \frac{3}{1-\frac{1}{4}} = 4\,, S_n = \frac{n}{1-\frac{1}{n+1}} = n + 1$

$\mathbf{L.H.S}$: $S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + S_n = 2 + 3 + 4 + \cdots \ldots\ldots\ldots + n + 1$

This is an A.P with $1^{st}\ term = 2$, $common\ diffreerence = 1$ and $last\ term = n + 1$

Using $S_N = \frac{N}{2}(1^{st}\ term + last\ term)$. Finding N, using

Using $a_N = A + (N-1)D\ \ or\ N = \frac{a_N - A}{D} + 1$

$\Longrightarrow N = \frac{n+1-2}{1} + 1 = n$

$\therefore S_n = \frac{n}{2}(2 + n + 1) = \frac{n}{2}(n+3)$

b) *Given:* $S_n = S_1$, $S_{2n} = S_2$, $S_\infty = S_3$; $let\ 1^{st}\ term = a, common\ ratio = r$

Using $S_n = \frac{a(1-r^n)}{1-r} = S_1$ ………………………………………… (1)

$S_{2n} = \frac{a(1-r^{2n})}{1-r} = S_2$ ………………………………………… (2)

$S_\infty = \frac{a}{1-r} = S$ …………………………………………………… (3)

Now $S_1(S_1 - S) = \frac{a(1-r^n)}{1-r}\left[\frac{a(1-r^n)}{1-r} - \frac{a}{1-r}\right]$

$= \frac{a^2(1-r^n)}{(1-r)^2}[1 - r^n - 1] = \frac{-a^2r^n(1-r^n)}{(1-r)^2}$

$\therefore S_1(S_1 - S) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$

Also $S(S_1 - S_2) = \frac{a}{1-r}\left[\frac{a(1-r^n)}{1-r} - \frac{a(1-r^{2n})}{1-r}\right]$

$= \frac{a^2}{(1-r)^2}[1 - r^n - 1 + r^{2n}]$

$\therefore S(S_1 - S_2) = \frac{a^2}{(1-r)^2}(r^{2n} - r^n) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$

$\therefore S_1(S_1 - S) = S(S_1 - S_2) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$

5. *Find the sum to infinity of the G.P* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$ *and show that is* S *denotes* S_∞ *and* S_n *denotes sum of* $n-terms$ *then* $S - S_n < \frac{1}{1000}$ *if* $n \geq 11$.

Solution

$S = \frac{1}{1-\frac{1}{2}} = 2$, $S_n = \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$

Now $S - S_n < \frac{1}{1000} \Longrightarrow 2 - 2\left(1 - \frac{1}{2^n} < \frac{1}{1000}\right)$

$\Longrightarrow \frac{2}{2^n} < \frac{1}{1000}\ \ or\ 2^{n-1} < 10^{-3}$

Introducing logarithms

$\Longrightarrow \log(2^{n-1}) < \log 10^{-3}$

$\Longrightarrow (1-n)\log 2 < -3\log 10$

$\Longrightarrow 1 - n < -\frac{3}{\log 2}$

$\therefore n > 1 + \frac{3}{\log 2}\ \ or\ n > 10.96578425$

$\therefore n \geq 11$

PROOF BY INDUCTION

Proof by induction is a mathematical technique that uses reasoning that if "proposition P_n" is true for a particular value of n, say $n = 1$ then it must be true for the next value of $n. i. e.$ $n = 2$, and now if it is true for $n = 2$, then it must be also true for $n = 3$ and hence true for all positive integral values of n.

Characteristics of positive integral values (Natural numbers)

Axiom of Induction:
If T is a set of natural numbers with properties;
i) $1 \in T$ and
ii) $k \in T$ implies $k + 1 \in T$
, then T is a set of Natural numbers N.

Note:
Both conditions must be satisfied for the Axiom of induction to hold.

Illustration

Consider $T = \{1 , 2 , 3 \}$
Clearly$1 \in T$, but is not true that $k + 1 \in T$ every time $k \in T$ $e. g$ $3 \in T$ but $4 \notin T$. Thus $T \neq N$
Also let $T = \{10 , 11 , 12 , \dots \dots \dots . . \}$
Clearly if$k \in T$, then $k + 1 \in T$ but $1 \notin T$. Thus again $T \neq N$
Now let $T = \{1 , 2 , 3 , \dots \dots \dots . . \}$. Both $1 , k , k + 1 \in T$, hence $T = N$.

Statement of Principle of Proof by Induction

For every $n \in N$, let P_n be a statement which is either true of false. If
i) P_1 is true and
ii) Whenever this statement is true for $n = k$, it is true for $n = k + 1$, then P_n is true for all $n \in N$

In general, to apply the statement of proof by induction, two things must be done.

i) First verify that P_1 is true and
ii) Verify that P_{k+1} is true whenever P_k is true.

Examples

Prove by induction that

a) $n^3 + 2n$ is divisible by 3
b) $n^3 - n$ is divisible by 6
c) $3^{2n} - 1$ is a multiple of 8
d) $9^n - 1$ is a multiple of 8
e) $8^n - 7n + 6$ is divisible by 7

Solution

a) Let $P_n = n^3 + 2n$
Verify that P_1 is true
$\Rightarrow P_1 = 1^3 + 2(1) = 3$, which is divisible by $3 . i. e. P_1 = 3(1)$

Verify that whenever P_n is true for a natural number, say $n = k$, then it is true for the next natural number $n = k + 1$. To this you must assume that P_k is true for some$k \in N$. This assumption is the *induction hypothesis.*
Assume P_k is true $.i.e. k^3 + 2k$ is divisible by 3.
Next show thatP_k implies P_{k+1} $.i.e..i.e. (k + 1)^3 + 2(k + 1)$ is divisible by 3.
$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= (k^3 + 2k) + 3k^2 + 3k + 3$
$= P(k) + 3(k^2 + k + 1)$
Since $P(k)$ is divisible by 3 and $3(k^2 + k + 1)$ is also divisible by 3, $P(k + 1)$ is divisible by 3. Thus both conditions of principle of proof by induction are satisfied. You must conclude that$n^3 + 2n$ is divisible by 3 for$n \in N$.

NOTE:
The symbol $P(k)$ in the working stand for "$k^3 + 2k$". Don't confuse it with the symbol P_k which stands for the sentence "$k^3 + 2k$ is divisible by 3".
Similarly $P(k + 1)$ stands for "$(k + 1)^3 + 2(k + 1)$"

b) Let $P_n = n^3 - n$
For $n = 1, P_1 = 1^3 - 1 = 0 = 6(0)$. Hence it is divisible by 6
Now for $n = k$, let P_k be true $.i.e. k^3 - k$ is divisible by 6.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e. (k + 1)^3 - (k + 1)$ is divisible by 6
But $(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1$
$$= (k^3 - k) + 3k^2 + 3k = P(k) + 3(k^2 + k)$$
Since $P(k)$ is divisible by 6 and $3(k^2 + k)$ is divisible by 6 for any k≥ 1 and $k \in N$ then $P(k + 1)$ is divisible by 6. Hence $n^3 - n$ is divisible by 6for$n \in N$.

c) Let $P_n = 3^{2n} - 1$
For $n = 1, P_1 = 3^{2(1)} - 1 = 8 = 8(1)$. Hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.$ $3^{2k} - 1$ be a multiple of 8.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e. 3^{2(k+1)} - 1$a multiple of 8.
But $3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k}.3^2 - 1 = 9.3^{2k} - 1$
$$= 9\left[3^{2k} - \frac{1}{9}\right] = 9\left[3^{2k} - 1 + 1 - \frac{1}{9}\right]$$
$$= 9\left[(3^{2k} - 1) + \frac{8}{9}\right] = 9(3^{2k} - 1) + 8 = 9.P(k) + 8$$
Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then$P(k + 1)$ is a multiple of 8. Thus $3^{2n} - 1$ is a multiple of 8 for $n \in N$.

d) Let $P_n = 9^n - 1$ be a multiple of 8.
For $n = 1, P_1 = 9 - 1 = 8 = 8(1)$,hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.$ $9^k - 1$ be a multiple of 8.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e. 9^{k+1} - 1$a multiple of 8
But $9^{k+1} - 1 = 9^k.9 - 1 = 9.[9^k - 1 + 1] - 1$
$$= 9(9^k - 1) + 9 - 1 = 9.P(k) + 8$$
Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then$P(k + 1)$ is a multiple of 8. Thus $9^n - 1$ is a multiple of 8 for$n \in N$.

e) Let $P_n = 8^n - 7n + 6$
For $n = 1, P_1 = 8^1 - 7.1 + 6 = 7 = 7(1)$, hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.$ $8^k - 7k + 6$ be divisible by 7.

Now for $n = k + 1$, P_{k+1} must also be true $.i.e.\, 8^{k+1} - 7(k+1) + 6$ be divisible by 7

But $8^{k+1} - 7(k+1) + 6 = 8^k.8 - 7k - 7 + 6 = 8^k.8 - 7k - 1$

$= 8[8^k - 7k + 6 + 7k - 6] - 7k - 1$

$= 8(8^k - 7k + 6) + 8(7k) - 48 - 1 = 8.P(k) + 7(k - 7)$

Since $8.P(k)$ is divisible by 7 and $7(k - 7)$ is divisible by 7, then $P(k + 1)$ is divisible by 7. Thus $8^n - 7n + 6$ is divisible by 7 for $n \in N$

Task

Prove by induction that for positive integer of n,

i) $n^3 + 3n^2 - 10n$ is divisible by 3
ii) $7^n - 4^n + 1^n$ is divisible by 6
iii) $10^n - 1$ is divisible by 9
iv) $7^n - 3^n$ is a multiple of 4
v) $10^{2n-1} + 1$ is divisible by 11
vi) $3^{2n} + 7$ is a multiple of 8
vii) $n(n + 1)(n + 2)$ is a multiple of 6
viii) $2^{4n-1} - 1$ is a multiple of 15
ix) $n^3 + 6n^2 + 8n$ is divisible by 3

Extension of Proof by Induction Principle

Suppose that for some proposition P_n where n is a positive integer and that P_1 is false/meaningless but true for $P_k \Rightarrow P_{k+1}$, clearly P_n cannot be proved for all positive integral values of n. however, provided that there is some positive integer m such that P_m is true, then the induction P_n holds for all integral values of n greater than or equal to m.

Examples

Prove by induction that $2^n + 3^{2n-3}$ *for* $n \geq 2$ *is always divisible by 7.*

Solution

Let $P_n = 2^n + 3^{2n-3}$

For $n = 2$, $P_2 = 2^2 + 3^{2\times2-3} = 7$, hence proof holds for $n = 2$

Now for $= k$, let P_k be true i.e. $2^k + 3^{2k-3}$ be divisible by 7

Now for $n = k + 1, P_{k+1}$ must be true *i.e.* $2^{k+1} + 3^{2(k+1)-3}$ is divisible by 7.

But $2^{k+1} + 3^{2(k+1)-3} = 2^k.2 + 3^{2k-1}$

$= 2^k.2 + 3^{2k}.3^{-1} = 2^k.2 + \frac{3^{2k}}{3}$

$= 2[2^k + 3^{2k-3} - 3^{2k-3}] + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2.3^{2k-3} + \frac{3^{2k}}{3}$

$= 2(2^k + 3^{2k-3}) - 2.3^{2k}.3^{-3} + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2.\frac{3^{2k}}{27} + \frac{3^{2k}}{3}$

$= 2(2^k + 3^{2k-3}) + 3^{2k}\left(-\frac{2}{27} + \frac{1}{3}\right)$

$= 2(2^k + 3^{2k-3}) + 3^{2k}.\frac{7}{27} = 2(2^k + 3^{2k-3}) + 7.3^{2k-3}$

$\therefore\ 2^{k+1} + 3^{2k-1} = 2P(k) + 7(3^{2k-3})$

Since $2P(k)$ is divisible by 7 and $7(3^{2k-3})$ is also divisible by 7, then $2^n + 3^{2n-3}$ is divisible by 7 for $n \geq 2$.

Proof by Induction of a finite series

Here a formula of $n-terms$ of a sum of finite series and a general term($n^{th}-term$) of the series must be known to prove by induction.

Examples

1. *Prove by induction that*

a) $1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots.+n=\frac{n(n+1)}{2}$

b) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots..+\frac{1}{n(n+1)}=\frac{n}{n+1}$

c) $1^2+2^2+3^2+\cdots\ldots\ldots\ldots\ldots\ldots\ldots.+n^2=\frac{n(n+1)(2n+1)}{6}$

d) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$

Solution

a) Let $P_n=1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots.+n=\frac{n(n+1)}{2}$

For $n=1$, $P_1=1=\frac{1(1+1)}{2}=1$, thus $LHS=RHS$. Hence P_1 holds.

Now for $n=k$

Let P_k be true. $i.e.$ $1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots.+k=\frac{k(k+1)}{2}$

Now for $n=k+1, P_{k+1}$ must be true . $i.e$

LHS $=+2+3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots.+k+(k+1)$

RHS $=\frac{k(k+1)}{2}+k+1$

NOTE: On LHS of P_{k+1} , $(k+1)$ has been added, hence this proof is done by adding the general term of the series after replacing n by $k+1$ on either side of P_k.

$\therefore$ **RHS** $=\frac{k(k+1)}{2}+k+1=(k+1)\left[\frac{k}{2}+1\right]$

$=\frac{(k+1)(k+2)}{2}=\frac{(k+1)[(k+1)+1]}{2}$

Hence proof holds for $n=k+1$. Since the proof holds for $n=1, n=k$ and $n=k+1$, then it is true for all integral values of n.

b) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots..+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Let $P_n=\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots..+\frac{1}{n(n+1)}=\frac{n}{n+1}$

For $n=1$, $P_1=\frac{1}{1\times2}=\frac{1}{1+1}=\frac{1}{2}$. Hence P_1 is true.

Now for $n=k$

Let P_k be true. $i.e.$ $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots..+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Now for $n=k+1, P_{k+1}$ must be true . $i.e$

$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots.+\frac{1}{k(k+1)}+\frac{1}{k(k+1)(k+1+1)}=\frac{k}{k+1}+\frac{1}{k(k+1)(k+1+1)}$

$=\frac{1}{k(k+1)(k+2)}$

$=\frac{1}{(k+1)}\left[k+\frac{1}{k+2}\right]$

$=\frac{1}{(k+1)}\left[\frac{k^2+2k+1}{k+2}\right]=\frac{1}{(k+1)}.\frac{(k+1)^2}{(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{[(k+1)+1]}$

Hence proof holds for $n=k+1$

Note:

LHS and **RHS** for P_{k+1} has been increased by $\frac{1}{n(n+1)}$ for $n=k+1$.

$i.e.\frac{1}{k(k+1)(k+1+1)} = \frac{1}{k(k+1)(k+2)}$

Since proof holds for $n = 1, k, k+1$, the proof holds for all integral values of n.

c) Let $P_n = 1^2 + 2^2 + 3^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. + n^2 = \frac{n(n+1)(2n+1)}{6}$

For $n = 1, P_1 = 1^2 = 1 = \frac{1(1+1)(2\times1+1)}{6} = 1$. Hence P_1 is true.

Now for $n = k$

Let P_k be true. $i.e.$ $1^2 + 2^2 + 3^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. + k^2 = \frac{k(k+1)(2k+1)}{6}$

Now for $n = k+1, P_{k+1}$ must be true . $i.e$

$1^2 + 2^2 + 3^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)]$

$= \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$

$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k^2+3k+4k+6)}{6}$

$= \frac{(k+1)(2k+3)(k+2)}{6}$

$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$, hence proof holds.

Since the proof holds for $n = 1, n = k$ and $n = k+1$, then it is true for all integral values of n

d) Let $P_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^n} = \frac{2^n-1}{2^n}$

For $n = 1, P_1 = \frac{1}{2} = \frac{2^1-1}{2^1} = \frac{1}{2}$.Hence P_1 is true.

Now for $n = k$

Let P_k be true. $i.e.\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k} = \frac{2^k-1}{2^k}$

Now for $n = k+1, P_{k+1}$ must be true . $i.e$

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^k-1}{2^k} + \frac{1}{2^{k+1}}$

$= \frac{1}{2^{k+1}}\left[\frac{2^k-1}{2^{-1}} + 1\right] = \frac{1}{2^{k+1}}[2(2^k - 1) + 1]$

$= \frac{1}{2^{k+1}}[2^{k+1} - 1] = \frac{2^{k+1}-1}{2^{k+1}}$, hence proof holds.

Since the proof holds for $n = 1, n = k$ and $n = k+1$, then it is true for all integral values of n

2. *Prove by induction that*

   a) $\sum_{r=2}^{n}\frac{1}{r^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$

   b) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

   c) $\sum_{r=1}^{n}\frac{r}{2^r} = 2 - \frac{n+2}{2^n}$

   d) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$

Solution

a) $\sum_{r=2}^{n}\frac{1}{r^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$; $r = 2, 3, 4, \ldots\ldots\ldots\ldots\ldots., n$

Generating the series;

$\Rightarrow \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \cdots \ldots\ldots\ldots\ldots. + \frac{1}{n^2-1}$

$\Rightarrow \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{1}{n^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$

Let $P_n = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{1}{n^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$

For $n = 2$, $P_2 = \frac{1}{3} = \frac{3}{4} - \frac{2.2+1}{2.2(2+1)} = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$. Hence P_2 is true.

For $n = k$, let P_k be true . $i.e.$ $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots \dots\dots\dots\dots\dots\dots + \frac{1}{k^2-1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)}$

Now for $n = k+1, P_{k+1}$ must be true . $i.e$

$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots \dots\dots\dots\dots\dots\dots + \frac{1}{k^2-1} + \frac{1}{(k+1)^2-1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)} + \frac{1}{(k+1)^2-1}$

$= \frac{3}{4} + \frac{1}{(k+1)^2-1} - \frac{2k+1}{2k(k+1)} = \frac{3}{4} + \frac{1}{k(k+2)} - \frac{2k+1}{2k(k+1)}$

$= \frac{3}{4} + \frac{2(k+1)-(2k+1)(k+2)}{2k(k+1)(k+2)}$

$= \frac{3}{4} + \frac{2k+2-(2k^2+4k+k+2)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2k^2+3k}{2k(k+1)(k+2)}$

$= \frac{3}{4} - \frac{k(2k+3)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2(k+1)+1}{2k(k+1)[(k+1)+1]}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all $n \geq 2$

b) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$; $r = 1, 2, 3, 4, \dots\dots\dots\dots\dots\dots, n$

Generating the series;

$\Rightarrow 1.(1+1) + 2.(2+1) + 3.(3+1) + \cdots \dots\dots\dots + n(n+1)$

$= \frac{1}{3}n(n+1)(n+2)$

$\Rightarrow 2 + 6 + 12 + \cdots \dots\dots\dots\dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

Let $P_n = 2 + 6 + 12 + \cdots \dots\dots\dots\dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

For $= 1$, $P_1 = 2 = \frac{1}{3}.1.(1+1)(1+2) = 2$. Hence P_2 is true.

For $n = k$, let proof be true . $i.e.$ $2 + 6 + 12 + \cdots \dots\dots\dots\dots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$

Now for $n = k+1, P_{k+1}$ must be true . $i.e$

$2 + 6 + 12 + \cdots \dots\dots + k(k+1) + (k+1)(k+2)$

$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$

$= (k+1)(k+2)\left[\frac{k}{3} + 3\right] = \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all integral values of n.

c) $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$; $r = 1, 2, 3, \dots\dots\dots\dots\dots\dots, n$

Generating the series;

$\Rightarrow \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \dots\dots\dots\dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

Let $P_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \dots\dots\dots\dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

For $n = 1$, $P_1 = \frac{1}{2} = 2 - \frac{1+2}{2^1} = \frac{1}{2}$. Hence P_1 is true.

For $n = k$, let the proof be true . $i.e.$ $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \dots\dots\dots\dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$

Now for $n = k+1, P_{k+1}$ must be true . $i.e$

$\Rightarrow \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \dots\dots\dots\dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

$= 2 + \frac{k+1}{2^{k+1}} - \frac{k+2}{2^k} = 2 + \frac{1}{2^k}\left[\frac{k+1}{2} - k + 2\right]$

$= 2 + \frac{1}{2^{k+1}}[-k-3] = 2 - \frac{k+3}{2^{k+1}}$

$= 2 - \frac{[(k+1)+2]}{2^{k+1}}$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all integral values of n.

d) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n - 1}{2}$; $r = 1, 2, 3, \dots \dots \dots \dots \dots \dots, n$

Generating the series;

$\Rightarrow 3^{1-1} + 3^{2-1} + 3^{3-1} + \cdots \dots \dots \dots \dots \dots + 3^{n-1} = \frac{3^n - 1}{2}$

Let $P_n = 1 + 3 + 3^2 + \cdots \dots \dots \dots \dots \dots + 3^{n-1} = \frac{3^n - 1}{2}$

For $n = 1, P_1 = 1 = \frac{3^1 - 1}{2} = 1$. Hence P_1 is true.

For $n = k$, let the proof be true $.i.e.$ $1 + 3 + 3^2 + \cdots \dots \dots \dots \dots \dots + 3^{k-1} = \frac{3^k - 1}{2}$

Now for $n = k + 1, P_{k+1}$ must be true $.i.e$

$\Rightarrow 1 + 3 + 3^2 + \cdots \dots \dots \dots \dots \dots + 3^{k-1} + 3^{k+1-1} = \frac{3^k - 1}{2} + 3^{k+1-1}$

$= \frac{3^k - 1}{2} + 3^k = \frac{3^k + 2.3^k - 1}{2}$

$= \frac{3.3^k - 1}{2} = \frac{3^{k+1} - 1}{2}$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all integral values of n.

Task

1. *Prove by induction that*
   a) $1^3 + 2^3 + 3^3 + \cdots \dots \dots \dots \dots \dots \dots . + n^3 = \frac{1}{4}n^2(n + 1)^2$
   b) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$
   c) $\sum_{r=1}^{n} r(r + 1)(r + 4) = \frac{1}{4}n(n + 1)(n + 4)(n + 5)$
   d) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \cdots \dots \dots \dots \dots \dots \dots + \frac{1}{n(n+3)} = \frac{11}{18} - \frac{1}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)$
   e) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \dots \dots \dots \dots \dots . . + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
   f) $\sum_{r=1}^{n} r(r + 3) = \frac{1}{3}n(n + 1)(n + 5)$
   g) $\sum_{r=1}^{n} 2r(2r - 1) = \frac{1}{3}n(n + 1)(4n - 1)$
2. *Prove that* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all positive integral values of n.*
3. *Show that* $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ *and prove by induction that*
   $$(1 + x)^n = 1 + nx + \cdots \dots \dots . . + \binom{n}{r}x^r + \cdots \dots \dots \dots + x^n$$
4. $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

SUMMATION OF A FINITE SERIES

There are various standard results that can be used to find the sum of finite series.

The two formulae of sums of A.P's and G. P's are:

1) The sum of n-terms of the A.P with first term a and common difference d is:

$$\boxed{S_n = \frac{1}{2}n(2a + (n - 1)d)}$$

2) The sum of $n - terms$ of a G.P with first term a and common ratio r is:

$$S_n = \frac{a(1-r^n)}{1-r} \quad for\ |r| < 1$$

The standard series are usually given as:

i) $\mathbf{1 + 2 + 3 + \cdots \ldots \ldots \ldots . + n = \sum_r^n r = \frac{1}{2} n(n+1)}$
ii) $\mathbf{1^2 + 2^2 + 3^2 + \cdots \ldots . + n^2 = \sum_r^n r^2 = \frac{1}{6} n(n+1)(2n+1)}$
iii) $\mathbf{1^3 + 2^3 + 3^3 + \cdots \ldots . + n^3 = \sum_r^n r^3 = \frac{1}{4} n^2(n+1)^2 = [\sum_r^n r]^2}$

The standard series above can be used in summation of finite number of terms.

Examples

1. *Find the sum of the series:*
   a) $1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)$
   b) $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + (n-1)^2$
   c) $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$

Solution

NOTE: in all the above series, the last term is not a general term for $n \in N$. The series exercise the form of a standard series, hence the result of standard series will hold for the last term.

a) $1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)$

Using $1 + 2 + 3 + \cdots \ldots . + N = \frac{1}{2}N(N+1)$

$\Rightarrow N = 2n + 1$

$\therefore 1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1) = \frac{1}{2}(2n+1)[(2n+1)+1]$

$= (2n+1)(n+1)$

b) $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + (n-1)^2$

Using $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + N^2 = \frac{1}{6}N(N+1)(2N+1)$

Taking $N = (n-1)$

$\Rightarrow 1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots + (n-1)^2 = \frac{(n-1)(n-1+1)[2(n-1)+1]}{6} = \frac{n(nn-1)(2n-1)}{6}$

c) $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$

Using $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + N^3 = \frac{N^2(N+1)^2}{4}$

Taking $N = 2n$

$1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3 = \frac{(2n)^2((2n)+1)^2}{4} = n^2(2n-1)^2$

2. *Find the sum of the series;*
   a) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)^3$
   b) $2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$
   c) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n-1)^3$

Solution

a) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)^3$

(Summation of the cubes of odd terms of natural numbers)

Using $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + N^3 = \frac{N^2(N+1)^2}{4}$

Now $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots + (2n)^3 + (2n+1)^3 = \frac{(2n+1)^2((2n+1)+1)^2}{4}$

$= (2n+1)^2(n+1)^2$

Finding the sum of the series; $2^3 + 4^3 + 6^3 + \cdots \ldots\ldots\ldots\ldots\ldots + (2n)^3$

$2^3 + 4^3 + 6^3 + \cdots \ldots\ldots\ldots\ldots\ldots + (2n)^3 = 2^3[\,1^3 + 2^3 + 3^3 + \cdots \ldots\ldots + n^3]$

$= 8\left[\frac{n^2(n+1)^2}{4}\right] = 2n^2(n+1)^2$

$\therefore 1^3 + 3^3 + 5^3 + \cdots \ldots\ldots + (2n+1)^3 = (2n+1)^2(n+1)^2 - 2n^2(n+1)^2$

$= (n+1)^2[(2n+1)^2 - 2n^2]$

$= (n+1)^2[4n^2 + 4n + 1 - 2n^2] = (n+1)^2(2n^2 + 4n + 1)$

b) $2^3 + 4^3 + 6^3 + \cdots \ldots\ldots + (2n)^3 = 2^3[\,1^3 + 2^3 + 3^3 + \cdots \ldots\ldots + n^3]$

$= 8\left[\frac{n^2(n+1)^2}{4}\right] = 2n^2(n+1)^2$

c) $1^3 + 3^3 + 5^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + (2n-1)^3$

First summing the natural numbers to the $(2n)^{th}$ term,

$\Rightarrow 1^3 + 2^3 + 3^3 + \cdots \ldots\ldots\ldots + (2n-1)^3 + (2n)^3 = \frac{(2n)^2(2n+1)^2}{4} = n^2(2n+1)^2$

Now summing the series $2^3 + 4^3 + 6^3 + \cdots \ldots\ldots\ldots\ldots\ldots + (2n)^3$

$2^3 + 4^3 + 6^3 + \cdots \ldots\ldots\ldots\ldots\ldots + (2n)^3 = 2n^2(n+1)^2(\mathit{See\ previous\ number\ above})$

Now subtracting the series,

$\Rightarrow 1^3 + 3^3 + 5^3 + \cdots \ldots\ldots\ldots + (2n-1)^3 = n^2(2n+1)^2 - 2n^2(n+1)^2$

$= n^2[(2n+1)^2 - (n+1)^2]$

$= n^2[4n^2 + 4n + 1 - n^2 - 2n - 1]$

$= n^2(2n^2 - 1)$

NOTE:

In this last example the series $1^3 + 3^3 + 5^3 + \cdots \ldots\ldots\ldots + (2n-1)^3$ has the last term as $(2n-1)^3$ and is the general term of the series for all $n \in N$

Alternative summation of the series

$$1^3 + 3^3 + 5^3 + \cdots \ldots\ldots\ldots + (2n-1)^3 = \sum_{r=1}^{n}(2r-1)^3$$

$$= \sum_{r=1}^{n}(8r^3 - 12r^2 + 6r - 1) = 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8\left[\frac{n^2(n+1)^2}{4}\right] - 12\left[\frac{n(n+1)(2n+1)}{6}\right] + 6\left[\frac{n(n+1)}{2}\right] - n$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1) + 3n(n+1) - n$$

$$= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n$$

$$= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n$$

$$= n(n+1)(2n^2 - 2n + 1) - n$$

$$= n[2n^3 - 2n^2 + n + 2n^2 - 2n + 1 - 1]$$

$$= n[2n^3 - n] = n^2(2n^2 - 1)$$

3. a) *It can be proved by induction that for positive n,*

$1^3 + 2^3 + 3^3 + \cdots \ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

From this result deduce that

$(n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \ldots\ldots\ldots + (2n)^3 = \frac{1}{4}n^2(3n+1)(5n+3)$

b) *Show that the sum of the series*

$(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots \ldots\ldots\ldots + (2n)^2 = \frac{1}{6}n(2n+1)(7n+1)$

Solution

a) $1^3 + 2^3 + 3^3 + \cdots \ldots\ldots\ldots + n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \ldots\ldots + (2n)^3$

$$= \frac{(2n)^2(2n+1)^2}{4} = n^2(2n^2+1)$$

Also $1^3 + 2^3 + 3^3 + \cdots \dots \dots . \dots \dots \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Subtracting the series,

$(n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \dots \dots \dots + (2n)^3 = n^2(2n^2+1) - \frac{n^2(n+1)^2}{4}$

$= \frac{1}{4}n^2[4(4n^2 - 4n + 1) - (n^2 + 2n + 1)]$

$= \frac{1}{4}n^2[16n^2 + 16n + 4 - n^2 - 2n - 1] = \frac{1}{4}n^2[15n^2 + 14n + 3]$

$= \frac{1}{4}n^2[15n^2 + 9n + 5n + 3] = \frac{1}{4}n^2[3n(5n+3) + (5n+3)]$

$= \frac{1}{4}n^2(3n+1)(5n+3)$

b) From $1^2 + 2^2 + 3^2 + \cdots \dots \dots \dots \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

$\Longrightarrow \; 1^2 + 2^2 + 3^2 + \cdots \dots \dots + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots .. + (2n)^2$

$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) = \frac{2n(2n+1)(4n+1)}{6}$

Subtracting the series,

$(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots \dots \dots \dots + (2n)^2$

$= \frac{2n(2n+1)(4n+1)}{6} - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(2n+1)(7n+1)$

$= \frac{n(2n+1)}{6}[2(4n+1) - (n+1)]$

$= \frac{n(2n+1)}{6}(8n + 2 - n - 1) = \frac{1}{6}n(2n+1)(7n+1)$

4. *Show that the sum of the series*

$3 + 6 + 11 + \cdots \dots \dots \dots \dots \dots \dots + (n^2+2) = \frac{1}{6}n(2n^2 + 3n + 13)$

Solution

For this series, (n^2+2) is the general term of the series for all$n \in N$

$\therefore 3 + 6 + 11 + \cdots \dots \dots \dots \dots \dots \dots + (n^2+2) = \sum_{r=1}^{n}(r^2+2) \;\; = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2$

$= \frac{1}{6}n(n+1)(2n+1) + 2n$

$= \frac{n}{6}[(n+1)(2n+1) + 12] = \frac{n}{6}[2n^2 + 3n + 1 + 12]$

$= \frac{1}{6}n(2n^2 + 3n + 13)$

Task

1. *Show that the sum of the series;*

   a) $2 + 10 + 30 + \cdots \dots \dots \dots \dots \dots \dots \dots + (n^3+n) = \frac{n(n+1)(n^2+n+2)}{4}$

   b) $2 + 12 + 36 + \cdots \dots \dots \dots \dots \dots \dots .. + (n^3+n^2) = \frac{n(n+1)(n+2)(3n+1)}{12}$

   c) $2 + 6 + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots + (n^2-n) = \frac{n(n+1)(n-1)}{3}$

   d) $1.4 + 2.5 + 3.6 + \cdots \dots \dots \dots \dots \dots .. + n(n+3) = \frac{n(n+1)(n+5)}{3}$

   e) $2^2 + 5^2 + 8^2 + \cdots \dots \dots \dots \dots ... \dots \dots + (3n-1)^2 = \frac{n(6n^2+3n-1)}{6}$

   f) $1.2 + 3.4 + 5.6 + \cdots \dots \dots \dots \dots . \dots .. + 2n(2n-1) = \frac{n(n+1)(4n-1)}{3}$

   g) $1.2.5 + 2.3.6 + 3.4.7 + \cdots \dots \dots .. \dots + n(n+1)(n+4) = \frac{n(n+1)(n+2)(3n+12)}{12}$

2. *Find the sum of the series*

   a) $21^3 + 22^3 + 23^3 \dots \dots \dots \dots . \dots \dots \dots + 35^3$

   b) $1^3 + 3^3 + 5^3 + \cdots \dots \dots \dots \dots \dots \dots ... + 25^3$

   c) $1^3 - 2^3 + 3^3 - 3^3 + \cdots \dots \dots \dots .... + (2n-1)^3 - (2n)^3$

   ANSWER: $\boldsymbol{a})\ \mathbf{352\,800}\quad \boldsymbol{b})\ \mathbf{56\,953}\quad \boldsymbol{c})\ -\boldsymbol{n}(\mathbf{4}\boldsymbol{n}+\mathbf{3})$

INTEREST AND ANNUITIES

Simple and Compound Interest

When person A lends money to person B he makes a charge for the use of his money. The amount of money lent is called *Principal (P)* and the charge made for lending is called *Interest (I).* The sum of interest and principal is the *Amount* or *Total amount (A)*, which is the amount B will pay back to A after any interval of time. $i.e.$ $(A = P + I)$

The *Rate per cent per annum (R %)* is the interest payable on principal for one year.

Types of Interest

- Simple interest
- Compound interest

Simple Interest (S.I)

Simple Interest is the interest that is paid on the principal only.

S.I is paid on a given principal, and is the same after equal intervals of time. The principal is always constant.

To find the Interest and Amount of a given sum in a given time at Simple Interest

When money is put out at S.I, the interest is payable for each year, but is not added to the principal.

Let P be the given principal and n the number of years for which the principal is lent, R the rate of interest per annum.

The interest of P for$1 - year = P.\frac{R}{100}$ and therefore for $n - years$ is $P.\frac{R}{100}.n$

$\therefore$ Amount at the end of $1 - year = A_1 = P + P.\frac{R}{100} = P\left(1 + \frac{R}{100}\right)$

Since the principal is lent at S.I, the interest $P.\frac{R}{100}$ will be added to the amount at the end of each year.

Thus the amount at the end of $n - years$ will be;

$$A_n = P\left(1 + \frac{R}{100}\right) + \frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots. + \frac{PR}{100}$$

$$= P + \underbrace{\frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots. + \frac{PR}{100}}_{n - times}$$

$A_n = P + \frac{PRn}{100} = P\left(1 + \frac{nR}{100}\right)$, which is the amount at the end of n$-years$.

$$\boxed{\therefore Amount\ for\ n - years, \boldsymbol{A_n} = \boldsymbol{P}\left(\mathbf{1} + \frac{\boldsymbol{nR}}{\mathbf{100}}\right)}$$

, which is the formula for amount at the end of $n - years$, where;

$P = Principal$

$n = Number\ of\ years\ principal\ is\ lent$

$R = Rate\ per\ cent\ per\ annum$

Note:

1. The amount at the end of each year constitute a series in an A.P with a common difference $\frac{PR}{100}$
2. Simple interest can be paid at different intervals . $i.e.$ monthly, quarterly, half yearly or yearly.

3. In $A = P\left(1 + \frac{nR}{100}\right)$, formulae contain four quantities $.i.e. P, A, R, n$ and if three are given, the fourth can be evaluated.
4. When working out the interest due on a sum of money for a period between two specific dates, always take a year as containing 365 days.
5. For purposes of interest, the day a deposit is first made does not count. However the day on which money is withdrawn does count. Thus if someone deposits money on an account on 15^{th} March and withdraws it on17^{th} April, he is not paid interest for 15^{th} March but he is paid interest for 17^{th} April.

Examples

1. *Find the simple interest on and the amount at simple interest of;*
   a) $Sh. 8000$ for $2\frac{1}{4}$ years at $8\frac{1}{5}\%\ p.a.$
   b) $Sh. 480$ for 2 months at $3\frac{3}{4}\%\ p.a.$

Solution

a) Simple Interest $= \frac{P.R.n}{100}$; $P = 8000, R = 8.2\%, n = 2.25\ years$

$$= \frac{8000 \times 8.2 \times 2.25}{100} = 1\ 476$$

$Amount = P + I = 8000 + 1476 = 9\ 476\ /=$

b) $P = 480, R = 3.75\%, n = \frac{20}{12}\ years$

$\therefore Simple\ Interest = \frac{480 \times 3.75 \times \frac{20}{12}}{100} = 30$

$Amount = 30 + 480 = 510\ /=$

2. *Calculate to the nearest shilling, the simple interest on;*
   a) $Sh. 550$ from $March$ 6 to May 6 at $11\%\ p.a.$
   b) $Sh. 990$ from $August$ 20 to $December$ 13 at $9\frac{1}{2}\%\ p.a.$

Solution

a) Simple Interest $= \frac{P.R.n}{100}$, $P = 550, R = 11\%$,

$n = (31 - 6)days + (30 + 6)\ days = 61\ days = \frac{61}{365}\ years$

$[April = 30\ days, May\ 6 = 6\ days, March\ 6 = (31 - 6)days]$

$\therefore Simple\ Interest = \frac{550 \times 11 \times \frac{61}{365}}{100} = 10\ /=$

b) $P = 990, R = 9\frac{1}{2}\%$,

$n = (31 - 20)days + (30 + 31 + 30 + 13)days = 115days = \frac{115}{365}\ years$

$\therefore Simple\ Interest = \frac{990 \times 9\frac{1}{2} \times \frac{115}{365}}{100} = 30\ /=$

3. a) In what time will $Sh. 3\ 200$ amount to $Sh. 4\ 188$ at $9\frac{1}{2}\%\ p.a.$ simple interest?
   b) A sum of money invested at $9\%\ p.a.$ simple interest amounts after 3 years to $sh. 7\ 620$. Find the sum invested.

Solution

a) *Given:* $P = 3\ 200, A = 4\ 188, R = 9\frac{1}{2}\%,\ n = ?$

From Simple Interest $= \frac{P.R.n}{100} \Longrightarrow n = \frac{100I}{P.R}$

But $Interest = Amount - Principal$

$\therefore I = 4188 - 3200 = 988$

$\therefore n = \frac{100\times 988}{3200\times 9\frac{1}{2}} = 3\frac{1}{4}$years

b) $Given{:}\ Amount = 7\,620 = Principal + Interest$

$Interest = \frac{P.R.n}{100} = \frac{P\times 9\times 3}{100}$

Now $7\,620 = P + \frac{27P}{100} = \frac{127P}{100}\ \therefore P = 6\,000/=$

Task

1. *Find the un known quantities in the table below*

Principal	**Interest**	**Amount**	**Time**	**Rate** % $\boldsymbol{p.a.}$
$sh.650$	$sh.156$	………………	$3\ years$	………………
………………	$sh.1\,942.25$	………………	$1\frac{1}{4}\ years$	$9\frac{1}{4}$
$sh.8\,450$	…………………		$9\ months$	………………
………………	$sh.170.40$		$16\ months$	$11\frac{1}{4}$
$sh.1\,818$	…………………	$sh.1\,908.90$	……………	$8\frac{1}{3}$

ANSWER:

a) $sh.806\,;8\%$ b) $sh.16\,800\,;\ sh.18\,742.50$ c) $sh.549.25\,;8\frac{2}{3}\%$ d) $sh.1\,136\,;\ sh.1\,306.40$

e) $sh.90.90\,;219\ days$

2. *Two men both invest the same sum, one at* $9\frac{3}{4}\%\ p.a.$ *if the difference in the interest they receive is* $sh.100$ *, what sum each man invested is.*
ANSWER: $\boldsymbol{sh.20\,000}$

3. *Two men both invest* $sh.800$ *at* 10% *and* $8\frac{1}{2}\%\ p.a$ *respectively. What is the difference in the interest they earn after* $9\ months$*?*
ANSWER: $\boldsymbol{sh.9}$

4. *A club of 320 members has to meet a bill of* $sh.9\,500$ *in 5 months time. Each member subscribes* $sh.30$ *and the money accumulates at* $8\frac{1}{4}\%\ p.a$ *simple interest. What balance remains when the bill has been paid?*
ANSWER: $\boldsymbol{sh.430}$

$$\left[\begin{array}{l}\boldsymbol{Hint}{:}Total\ principal = 30\times 320 = 9\underset{=}{600}, n = \frac{5}{12}\ , I = \frac{9600\times 8\frac{1}{4}\times\frac{5}{12}}{100} = 330\\ Now\ New\ amount = 330 + 9600 = Sh.9\,930\ \therefore Balance = 9930 - 9500 = 430/=\end{array}\right]$$

5. *A sum of money amounts at simple interest to* $sh.819$ *in 2 years at* $8\frac{1}{2}\%\ p.a.$ *to what would it amount in 1 year 9 months at* $8\%\ p.a$*?* **ANSWER**: $\boldsymbol{sh.798}$

$$\left[\begin{array}{l}\boldsymbol{Hint}{:}Let\ sum = P = Principal, A = P + I\ , I = \frac{P\times 8\frac{1}{2}\times 2}{100} = 0.17P\\ Now\ 819 = P + 0.17P\ , P = 700\\ Now\ at\ 8\%\ p.a, amount, A = 700 + \frac{700\times 8\frac{1}{2}\times 1\frac{9}{12}}{100} = 798/=\end{array}\right]$$

6. *Diana invests* $\frac{2}{5}$ *of her capital at* $15\%\ p.a$ *and the remainder at* $10\%\ p.a$ *simple interest. After 25 cents in the shillings has been deducted for tax, she receives* $sh.720$ *interest each year. What is the size of her capital?*
ANSWER: $\boldsymbol{sh.8000}$

7. *A man borrows* $sh.5000$ *at* $10\ \frac{1}{2}\%$ $p.a$ *payable yearly. After one year he pays back* $sh.2500$. *How much will he need to clear his date completely at the end of another one year?* **ANSWER**: $\boldsymbol{sh.3\ 343}$

$$\begin{bmatrix} \boldsymbol{Hint}: P = 5000\ , R = 10.5\%, Amount\ after\ 1\ year = 5000 + \frac{10.5}{100} \times 5000 = 5\ 525 \\ Balance\ after\ paying\ off\ 2500 = 5525 - 2500 = 3025 \\ Now\ amount\ after\ another\ 1\ year = 3025 + 3025 \times \frac{10.5}{100} = 3\ 343\ /= \end{bmatrix}$$

8. *Godfrey borrows* $sh.20\ 000$ *from a finance corporation and pays* 10% *interest p.a on an amount still owing at the beginning of that year. After one year he pays* $sh.5\ 000$ *back,* $sh.6\ 000$ *a year later and* $sh.7\ 000$ *at the end of the third year. How much does he still need to clear the debt completely?* **ANSWER**: $\boldsymbol{sh.6\ 970}$

$$\begin{bmatrix} \boldsymbol{Hint}: \quad P = 20\ 000\ , R = 10\%\ p.a. Amount\ \ after\ 1\ year = 20000 + \frac{10}{100} \times 20000 = 22\ 000; \\ Amount\ remaining\ after\ paying\ 5000 = 22\ 000 - 5\ 000 = 17\ 000 \\ Expected\ amount\ after\ another\ year = 17000 + \frac{10}{100} \times 17000 = 18\ 700; \\ Balance\ after\ paying\ sh.6000 = 18\ 700 - 6000 = 12\ 700/= \\ Expected\ amount\ after\ another\ 3^{rd}\ year = 12\ 700 + 12\ 700 \times \frac{10}{100} = 13\ 970/= \\ Now\ balannce\ after\ clearing\ sh.7000 = 13\ 970 - 7000 = 6\ 970/= \\ \boldsymbol{NOTE}: Principal\ is\ not\ constant\ every\ year\ but\ interest\ is\ the\ same. \end{bmatrix}$$

COMPOUND INTEREST (C.I)

Compound interest is the interest calculated on the principal and the accrued interest.

The interest unpaid is added to the principal and for the next period the interest is computed. $i.e.$ Compound interest is the interest on the growing principal.

NOTE

1. Both principal and compound interest change from period to period since the principal changes.
2. Interest earned on the principal is not withdrawn.
3. Compound interest can be computed annually, half yearly, quarterly or monthly.
4. When a given amount is due after a specified period of time, its present value (principal for the next period) is that sum together with the interest for that period produces the given amount.

To find Interest and Amount of a given sum at Compound Interest

If the money is lent at compound interest the interest is added each year to the principal and for the following year the interest is calculated on their sum.

Let P be the principal and n the number or years for which the principal is lent at $r\%$ per annum compound interest.

The amount at the end of one year will be $= P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right) = PR$

The amount at the end of two years will be $= PR + PR.\frac{r}{100} = PR\left(1 + \frac{r}{100}\right) = PR^2$

The amount at the end of three years will be $= PR^2 + PR^2.\frac{r}{100} = PR^2\left(1 + \frac{r}{100}\right) = PR^3$

Then arguing in this way, the amount at the end of n years will be= PR^n

Thus the amount, A at the end of n years for principal P lent at $r\%$ per annum is

$$\boxed{A = P\left(1 + \frac{r}{100}\right)^n \text{ or } A = PR^n}$$

, where $A = Amount$

$P = Principal$

$r = rate\ per\ cent\ per\ annum$

$R = \left(1 + \frac{r}{100}\right)$

Deduction

From $A = PR^n$, $\boxed{P = \frac{A}{R^n}}$ this enables us to find the sum which will produce amount A in n years.

Also $\boxed{R = \sqrt[n]{\frac{A}{P}}}$, from which interest can be calculated.

Again from $R^n = \frac{A}{P}$, $\log(R^n) = \log\left(\frac{A}{P}\right)$

$\Rightarrow n\log R = \log A - \log P$ $\boxed{\therefore n = \frac{\log A - \log P}{\log R}}$, from which time taken for P to amount to A can be evaluated.

Note

The amount in each year constitute terms in a G.P with first term PR and common ratio $\left(1 + \frac{r}{100}\right)$

Examples

1. *Find the compound interest on;*
   a) *Sh.* 10 000 *for* 2 *years at* 10% *p.a paid annually*
   b) *Sh.* 6 000 *for* 3 *years at* 10% *p.a paid annually*

Solution

a) $P = 10\ 000, n = 2, r = 10\%$

$Interest, I = Amount - Principal$

$I = P\left(1 + \frac{r}{100}\right)^n - P = 10\ 000\left(1 + \frac{10}{100}\right)^2 - 10\ 000 = Sh.\ 2\ 100$

b) $P = 6\ 000, n = 3, r = 10\%$

$A = 6000\left(1 + \frac{10}{100}\right)^3 = 7\ 986$

$\therefore Interest = 7\ 986 - 6\ 000 = sh.\ 1\ 986$

2. *Prove that in order that a sum of money may double itself in 10 years by investment at compound interest payable annually, the interest should be approximately 7.2 per cent per annum.*

Solution

Let $P = Amount\ invested\ for$

$n = 10\ years, which\ amounts\ to$

$A = 2P\ at\ a\ rate\ of$

$r = r\%\ per\ annum$

Using $A = P\left(1 + \frac{r}{100}\right)^n$,

$\Rightarrow 2P = P(1 + 0.01r)^{10}\ \ or\ \ 2 = (1 + 0.01r)^{10}$

Introducing logarithms

$\Rightarrow \log 2 = \log(1 + 0.01r)^{10} = 10\log(1 + 0.01r)$

$\Rightarrow \frac{\log 2}{10} = \log(1 + 0.01r)$

$\Rightarrow 2^{0.1} = 1 + 0.01r\ \therefore r = \frac{2^{0.1} - 1}{0.01} \approx 7.2\%$

3. *How many years (correct to the nearest whole year) does it take a sum of money to double in value if compound interest is paid at a rate of* $9\%\ p.a$*?*

Solution

Let the sum be P be invested for $n - years$ to amount to $2P$ at a rate of $9\%\ p.a$

Using $A = P\left(1 + \frac{r}{100}\right)^n$,

$\Rightarrow 2P = P\left(1 + \frac{9}{100}\right)^n\ \ or\ \ 2 = 1.09^n$

Introducing logarithms

$\Rightarrow \log 2 = \log 1.09^n = n\log 1.09$

$\Rightarrow \therefore n = \frac{\log 2}{\log 1.09} \approx 8\ years$

To find Amount of a given sum for $\left(n + \frac{1}{m}\right)$ *years* at Compound Interest

In business transactions when the time contains a fraction of a year, allow Simple Interest for the fraction of a year, and Compound Interest for the whole years.

Let P be invested for $\left(n + \frac{1}{m}\right)$ $years$ at a rate per cent per annum ($r\%$) Compound Interest.

$Amount,\ \ A \qquad = Amoun\_after\ n - years +\ Interest\ for \frac{1}{m}\ years$

$after\ \left(n + \frac{1}{m}\right)\ years \qquad\qquad (principal)\ due\ to\ C.I$

$on\ principal\ (Amount\ due\ to\ C.I)$

Amount after $n - years = P(1 + r)^n = PR^n$

Now principal for $\frac{1}{m}\ years = PR^n$

Interest for $\frac{1}{m}\ years = PR^n \times r \times \frac{1}{m} = PR^n \times \frac{r}{m}$

Now Total Amount $= PR^n + PR^n \times \frac{r}{m} = PR^n\left(1 + \frac{r}{m}\right)$

$$\boxed{\boldsymbol{A = PR^n\left(1 + \frac{r}{m}\right)}}$$

Example

1. *Find Compound Interest on;*
   a) $Sh. 6\ 000\ for\ 2\frac{1}{2}\ years\ at\ 10\%\ p.a\ paid\ annually$
   b) $Sh. 8\ 000\ for\ 1\frac{3}{4}\ years\ at\ 7.5\%\ p.a\ paid\ annually$

Solution

a) $P = 6\ 000, n = 2, \frac{1}{m} = \frac{1}{2}, r = 10\%\ p.a.$

Using $A = PR^n\left(1 + \frac{r}{m}\right), R = 1 + r = 1 + \frac{10}{100} = 1.1$

Amount after $2\frac{1}{2}\ years = A = 6\,000 \times (1.1)^2 \times \left(1 + \frac{10}{100} \times \frac{1}{2}\right) = 7\,623$

Now $Compound\ Interest = Amount - Principal$

$= 7\,623 - 6\,000 = sh.1\,623$

b) $P = 8\,000\,, n = 1\,, \frac{1}{m} = \frac{3}{4}\,, r = 7.5\%\,p.a.$

Amount after $1\frac{3}{4}\ years = A = 8\,000 \times \left(1 + \frac{7.5}{100}\right)^1 \times \left(1 + \frac{7.5}{100} \times \frac{3}{4}\right) = Sh.9\,083.75$

$\therefore Compound\ Interest = 9\,083.75 - 8\,000 = Sh.1\,083.75$

2. *A person owes a merchant $Sh.5\,702.27$ to be paid at the end of $2\frac{1}{4}$ years from now. How much the merchant ought to accept now in order to clear off the accounts taking the money to be worth $6\%\ p.a$ compound interest?*

Solution

Let $P = Sum\ to\ be\ invested\,, A = Sh.5\,702.27\,, r = 6\%\ p.a, n = 2\,, \frac{1}{m} = \frac{1}{4}$

Using $A = PR^n\left(1 + \frac{r}{m}\right)\,, R = 1 + r = 1 + \frac{6}{100}$

$\Rightarrow 5\,702.27 = P \times \left(1 + \frac{6}{100}\right)^2 \times \left(1 + \frac{6}{100} \times \frac{1}{4}\right)$

$\Rightarrow 5\,702.27 = P(1.06)^2 \times (1.015)$

$\therefore P = \frac{5\,702.27}{(1.06)^2 \times (1.015)} = Sh.5\,000$

$\therefore$ The merchant has to accept now $Sh.5\,000$ in order to get $Sh.5\,702.27$ after $2\frac{1}{4}\ years$ at a rate of $6\%\ p.a\ C.I$

Note

- The rate r% must be in per annum
- If the investment is for only n years, then $\frac{1}{m} = 0$. Hence the formula $A = P(1 + r)^n$

3. *A sum of money is put out at compound interest amounts in 2 years to $R.s\ 2\,809$ and in 3 years to $R.s\ 2\,977.54$. Find the rate of interest and the original sum.*

Solution

Amount for 2 years = $R.s\ 2\,809$

Amount for 3 years = $R.s\ 2\,977.54$

$\Rightarrow A = P(1 + r)^2 = 2\,809$ …………………………………………………… (1)

$\Rightarrow A = P(1 + r)^3 = 2\,977.54$ ………………………………………………… (2)

$Dividing\ the\ equations$

$\Rightarrow \frac{P(1+r)^2}{P(1+r)^3} = \frac{2\,809}{2\,977.54}$

$\therefore 1 + r = 1.06\,0\ \ or\ \ r = 0.06$

From eqn. (1)

$P(1 + r)^2 = 2\,809\ \Rightarrow P(1 + 0.06)^2 = 2\,809$

$\therefore P = \frac{2\,809}{(1.06)^2} = R.s\ 2\,500$

4. *A person wishes to divide Rs. 65 050 between his two sons who are 13 and 15 years old respectively in such a way that their shares if invested at 4% compound interest should produce the same amount when they become 18 years of age. Find the share of each.*

Solution

Let the share of the young son = $Rs.x$

$\therefore$ The share of the older son= $65\,050 - x$

The young son has to go for 5 years and the elder son for 3 years to become 18 years old.

Case I: younger son
$P = x\,, n = 5\,, r = 4\%\,, A = ?$
$A = x(1 + 0.04)^5 = x.(1.04)^5$
Case II: Elder son
$P = 65\,050 - x\,, n = 3\,, r = 4\%\,, A = ?$
$A = (65\,050 - x)(1 + 0.04)^3 = (65\,050 - x).(1.04)^3$
Now the amounts must be the same
$\Rightarrow x.(1.04)^5 = (65\,050 - x).(1.04)^3$
$\Rightarrow x.(1.04)^2 = 65\,050 - x$
$\therefore 1.0816x = 65\,050 \;\Rightarrow= Rs.31\,250$
$\therefore$ The share of the younger son is Rs. 31 250 and the share of the elder son is $Rs.65\,050 - 31\,250 \; = Rs.33\,800$

Find the sum if the difference between the simple interest and compound interest on the sum is £30 at 9% p.a for 3 years.
Solution
Let $P = sum\,, r = 9\%\; p.a\,, n = 3\; years$
At Simple Interest, simple interest $= \frac{P\times 9\times 3}{100} = 0{,}27P$
At Compound Interest, compound interest $= Amount - Principal$
$compound\ interest = P\left(1 + \frac{9}{100}\right)^3 - P = P(1.09)^3 - P$
$= 0.295029P$
Now $0.295029P - 0.27P = 30\;\; or\;\; 0.025029P = 30$
$\therefore P = £1198.6$
Task
1. *Find the compound interest on;*
   a) $Sh.12\,000\; for\; 2\frac{1}{2}\; years\; at\; 10\%\; p.a.$
   b) $Sh.10\,000\; for\; 1\frac{1}{4}\; years\; at\; 5\%\; p.a.$

   ANSWER: $a)\; \boldsymbol{Sh.3\,246}\;\; b)\;\; \boldsymbol{Sh.631.25}$
2. *Find the compound interest on the sums, giving your answers to 3 significant figures*
   a) $Sh.500\; for\; 4\; years\; at\; 10\%\; p.a.$
   b) $Sh.18\,765\; for\; 4\; years\; at\; 9\%\; p.a.$
   c) $Sh.15\,000\; for\; 3\; years\; at\; 20\%\; p.a.$

   ANSWER: $a)\; \boldsymbol{Sh.232}\;\; b)\;\; \boldsymbol{Sh.7\,690}$ c) $\boldsymbol{Sh.10\,920}$
3. *On a certain sum, the compound interest for 2 years is Sh.91.35 and the simple interest is Sh. 90. Fond the sum and the rate of interest per annum.*
   ANSWER: $\boldsymbol{Sh.1\,500\,, 3\%}$
4. A sum of money invested at compound interest, payable yearly, amounts to Sh. 2 704 at the end of the second year and to Sh. 2 812.16 at the end of the third year. Find the rate of interest and the sum.
   ANSWER: $\boldsymbol{Sh.25\,000\,, 4\%}$
5. *A person deposited in a bank £1000 at 4% p.a interest being added annually. After 5 years the rate of interest was raised to 5% and after four more years,*

the rate was further increased to 6%. The money was withdrawn at the end of 12 years. Find the amount.

ANSWER: $\boldsymbol{Sh.1\,761.40}$

6. *A person has two sons aged 12 and 15 years respectively. He wants to divide* $Sh.51\,783$ *between them and deposit the share of each separately in the bank so that each gets the same amount when he is 18 years old. If the bank pays compound interest at the rate of* $5\%\ p.a$, *what should be the share of each?*

ANSWER: $\boldsymbol{Sh.24\,000\,for\ young\ son;\ Sh.27\,783\,for\ elder\ son}$

To find Amount on a sum (P) in given n-years, interest payable q-times a year at a rate r% per annum

Suppose you deposit $\$\,1\,000$ in a savings and a loan that pays 8% compounded semi annually, how much will the savings and the loan owe you at the end of 2 years?

Solution

Compounded semi annually means "Interest is paid to your account at the end of each 6 month period, and the interest will in turn earn interest"

Annual rate $r = 8\%$

Semi annual rate$=\frac{8}{2} = 4\%$

Now amount after 6 months $= 1000 + \frac{4}{100} \times 1000 = 1000(1 + 0.04)$

Principal for the next 6 months $= 1000(1 + 0.04)$

Now amount for 1 year $= 1000(1 + 0.04) + 1000(1 + 0.04) \times \frac{4}{100}$

$$= 1000(1 + 0.04)^2$$

Again amount for another 6 months

$$= 1000(1 + 0.04)^2 + 1000(1 + 0.04)^2 \times \frac{4}{100} = 1000(1 + 0.04)^3$$

This amount for 2 years $= 1000(1 + 0.04)^3 + 1000(1 + 0.04)^3 \times \frac{4}{100}$

$$= 1000(1 + 0.04)^4$$

Now the amount that would be expected at the end of 6 years will be

$$= 1000(1 + 0.04)^{12}$$

In general,

"If the interest is payable $q - times$ a year and r is the nominal rate, the amount of P in $n - years$ or $q.n$ intervals is;

$$\boxed{\boldsymbol{A = P\left(1 + \frac{r}{q}\right)^{n.q}}}$$

NOTE

1. Since the principal P represents the initial amount in the account and A represents the amount in the account n-years later, P is called *present value* of the account and A is called the *future value* of the account.
2. $r\%\ p.a$ is the annual rate/nominal rate and $\frac{r}{q}\%\ per \frac{1}{q^{th}}\ annum$ is the rate for compounding period/interval.

Example

1. *Find how much Peter will have on the account at the end of 2 years if he invests a sum of* $\$\,100$ *in an account paying* $8\%\ p.a$ *interest compounded;*
   a) *Annually*

b) *Semi annually*
c) *Quarterly*
d) *Weekly*
e) *Daily*
f) *Hourly*

Solution

Using $\boldsymbol{A = P\left(1+\frac{r}{q}\right)^{n.q}}$		
Compounding frequency	$\boldsymbol{q}$	$\boldsymbol{A = 100\left(1+\frac{0.08}{q}\right)^{2q}}$
Annually	1	\$116.64
Semi annually	2	\$116.9859
Quarterly	4	\$117.1659
Weekly	52	\$117.3367
Daily	365	\$117.3490
Hourly	8 760	\$117.3501

NOTE:

- $1\,year = 12\,months = 365\,days = 52\,weeks = 8760\,hours$
- Semi annually $= \frac{1}{2} \times 12 = 6\,months$; Now in $1\,year, q = 2$
- Quarterly $= \frac{1}{4} \times 12 = 3\,months$; now in $1\,year, q = 4$
- Weekly $= \frac{1}{7} \times 365 \approx 52\,weeks\ \therefore q = 52$

2. *Find compound interest on;*
   a) $Sh.5\,000$ *for* $1\frac{1}{2}$ *years at* $5\%\ p.a$ *compound interest is paid half yearly*
   b) $Sh.1\,000$ *for* 1 *year at* $10\%\ p.a$ *compound interest is paid quarterly*
   c) $Sh.10\,000$ *for* $2\frac{1}{2}$ *years at* $15\%\ p.a$ *compound interest is paid weekly*

Solution

Using $A = P\left(1+\frac{r}{q}\right)^{n.q}$ $, A = P + I$

a) $P = 5000\,, q = 2\,, n = 1\frac{1}{2}\ years = \frac{3}{2} years\,, r = 5\% = 0.05$

$A = 5000\left(1+\frac{0.05}{2}\right)^{\frac{3}{2}\times 2} = 5000(1.025)^3 = Sh.5\,384.45$

$\therefore Compound\ Interest = A - P = 5\,384.45 - 5\,000 = Sh.384.45$

b) $P = 1000\,, q = 4\,, n = 1\,, r = 10\% = 0.1$

$A = 1000\left(1+\frac{0.1}{4}\right)^{4\times 1} = Sh.1\,103.83$

$\therefore Compound\ Interest = 1\,103.83 - 1\,000 = Sh.103.83$

c) $P = 10\,000\,, q = 52\,, n = 2\,, r = 15\% = 0.15$

$A = 10000\left(1+\frac{0.15}{52}\right)^{52\times 2} = Sh.13\,492.76$

$\therefore Compound\ Interest = 13\,492.76 - 10\,000 = Sh.3\,492.76$

Task

1. *Find the amount the principal* $Sh.10\,000$ *will accumulate at a rate of* $5\%\ p.a$ *compound interest for* 2 *years if compound interest is paid;*
   i) Annually **ANSWER**: $\boldsymbol{Sh.1\,025}$
   ii) Half yearly **ANSWER**: $\boldsymbol{Sh.1\,038.13}$
   iii) Quarterly **ANSWER**: $\boldsymbol{Sh.1\,044.86}$

2. *How many years to the nearest year will it take a sum of money to double if it is invested at 15% compounded annually.*
   ANSWER: $\mathbf{5}$ ***years***. $\left[Use\ A = P\left(1+\frac{r}{q}\right)^{n.q}\right]$
3. *How many years to the nearest year will it take money to quadruple if it is invested at 20% compounded annually.* **ANSWER**: $\mathbf{8}$ ***years***. $\left[Use\ A = P\left(1+\frac{r}{q}\right)^{n.q}\right]$
4. *If you deposit* \$5 000 *in an account paying 9% compounded daily, how much will you have in an account in 5 years. Compute your answer to the nearest cent.* **ANSWER**: **\$7 841.12**

Other problems related to compound interest

Problems relating

- Depreciation
- Change of population growth
- Annuity calculations

, are majorly on the principle of compound interest.

DEPRECIATION

This happens when the value of a new item decreases with passage of time. Depreciation can be expressed in terms of an *annual percentage rate*, so as to calculate the decrease in value after a given period of time.

Let P = price of a new commodity, which depreciates at $r\%\ p.a$ for $n - years$. The price after n-years will be calculated as below.

Value after 1 year $= P \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)$

Value after 2 years $= P\left(1-\frac{r}{100}\right) \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)^2$

Value after 3 years $= P\left(1-\frac{r}{100}\right)^2 \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)^3$

$$\boxed{\text{Value after n years , } \boldsymbol{A} = \boldsymbol{P}\left(\mathbf{1}-\frac{\boldsymbol{r}}{\mathbf{100}}\right)^{\boldsymbol{n}}}$$

Example

1. *A machine costing* £15 000 *is depreciated at 10% p.a. how long will it take to reduce the value to* £1 400*?*

Solution

The cost of machine, P = £15 000

Rate of depreciation, $r = 10\% = 0.1$

Value of the machine after n years, A = £1 400

But value of the machine after n years is $A = P\left(1-\frac{r}{100}\right)^n$

$\Rightarrow 1400 = 15000(1-0.1)^n$

$\Rightarrow \frac{1400}{15000} = (0.9)^n$

$\Rightarrow \log\left(\frac{14}{150}\right) = \log(0.9)^n = n\log 0.9$

$\therefore n = \frac{\log\left(\frac{14}{150}\right)}{\log\ 0.9} = \frac{0.8451-1.8751}{-1.9542} = 22.5\ years$

2. *A machine costing* $£30\ 000$ *has an estimated life time of 15 years. If the depreciation is at a rate of* $20\%\ p.a$*, find the scrap value of the machine.*
Solution
The cost of the machine $= P = £30\ 000$
Life of the machine $= n = 15\ years$
Rate of depreciation $= r = 20\% = 0.2$
$\therefore$ The value of the machine at the end of 15^{th} year $= P\left(1-\frac{r}{100}\right)^n$
$= 30\ 000(1-0.2)^{15} = 30\ 000 \times (0.8)^{15} = £1\ 055$
3. *A machine was bought at* $Rs.\ 25\ 000$ *on* $Jan.\ 1, 1981$*. It was depreciated at* $10\%\ p.a.$ *It was sold on* $31.12.1983$ *at* $Rs.\ 20\ 047.50$*. Find out the percentage of profit on the reduced cost.*
Solution
The cost of the machine $= P = Rs.\ 25\ 000$
Life of the machine $= n = 3\ years\ (i.e.\ Period\ from\ Jan.\ 1, 1981\ to\ 31.12.1983)$
Rate of depreciation $= r = 10\%$
Expected value after $n = 3\ years = P\left(1-\frac{r}{100}\right)^3$
$= 25\ 000(1-0.1)^3 = 25\ 000(0.9)^3$
$= Rs.\ 18\ 225$
Selling price $= Rs.\ 20\ 047.50$
Profit realized$= 20\ 047.50 - 18\ 225 = Rs.\ 1\ 822.5$
$\%\ profit = \frac{1822.5}{18225} \times 100 = 10\%$

Task
1. A machine depreciates in value each year at a rate of 10% of its value at the beginning of a year. The machinery was purchased for Rs. 10 000. Obtain to the nearest rupee its value at the end of the tenth year. $\boldsymbol{Answer: Rs.\ 3\ 483}$
2. *A machine depreciates at the rate of 7% of its value at the beginning of the year. If the machine was purchased for Rs. 8 500, what is the minimum number of complete years at the end of which the worth of the machine will be less that or equal to half of its original price?*
ANSWER: $\boldsymbol{10\ years}$
3. *John bought a new car for Sh. 650 000. The car depreciated in value by 25%, 15%, 10% and 5% of its value at the beginning of each year for four successive years. Find its value at the end of the fourth year.*
If after the fourth year it depreciated at a uniform annual rate of 4%, find its value at the end of 10 years (Give your answers to 3 significant figures).
ANSWER: $\boldsymbol{Sh.\ 354\ 000\ , Sh.\ 277\ 000}$
4. *A woman opened a hair dressing saloon but went out of business three years later. She sold off various pieces of equipment for Sh. 12 000. This represented a uniform rate of depreciation of 15% p.a. estimate the cost price of the equipment giving your answers to* $3\ s.f.$ **ANSWER:** $\boldsymbol{Sh.\ 19\ 500}$
5. *A man bought a sewing machine for Sh. 10 800. After 12 years he sold it for Sh. 4 000. At what annual rate did the machine depreciate? (Answer to the nearest integer)*
What was the value of the machine when it was 6 years old? (Answer to the $3\ s.f$*)*
ANSWER: $\boldsymbol{8\ \%\ , Sh.\ 65\ 500}$

POPULATION GROWTH

When the population P increases at a uniform rate $r\%\ p.a$ for $n-\ years$, at the end of the $n^{th}\ year$, the population can be calculated as below.

Initial (present) population $= P$

Rate on increase $= r\%$

Population after 1 year $= P + P \times \frac{r}{100} = P\left(1+\frac{r}{100}\right)$

Population after 2 years $= P\left(1+\frac{r}{100}\right) + P\left(1+\frac{r}{100}\right) \times \frac{r}{100}$

$$= P\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right) = P\left(1+\frac{r}{100}\right)^2$$

Population after 3 years $= P\left(1+\frac{r}{100}\right)^2 + P\left(1+\frac{r}{100}\right)^2 \times \frac{r}{100}$

$$= P\left(1+\frac{r}{100}\right)^2\left(1+\frac{r}{100}\right) = P\left(1+\frac{r}{100}\right)^3$$

$$\boxed{\therefore \text{Population after n years } = \boldsymbol{P\left(1+\frac{r}{100}\right)^n}}$$

Note

If the population is decreasing at a rate r% with passage of time, then the population after n years can be evaluated as $\boldsymbol{P\left(1-\frac{r}{100}\right)^n}$

Example

1. *The population of a town increases yearly at a rate of 25 per 1000. It is now 378 530, what it be 3 years hence?*

Solution

Present population, $P = 378\ 530$

Time, $n = 3\ years$

Rate of growth, $r = \frac{25}{1000} = 2.5\%$

$\therefore$ Population after 3 years $= P\left(1+\frac{r}{100}\right)^n$

$$= 378\ 530(1+0.025)^3$$
$$= 378\ 530(1.025)^3 = 407\ 635$$

2. *In a certain town, population raised at 20 per 1000. Find the number of years in which the population will be doubled assuming there is neither immigration nor migration.*

Solution s

Let present population $= P$

Rate of growth $= \frac{20}{1000} = 2\% = 0.02$

Time of growth $= n - years$

Population after n-years $= 2P$

But population after n years $= P(1+0.02)^n = P(1.02)^n$

$\Rightarrow 2P = P(1.02)^n\ \ or\ 2 = (1.02)^n$

$\log 2 = \log(1.02)^n = n\log 1.02$

$\therefore n = \frac{\log 2}{\log 1.02} = 35\ years$

3. *In a Military Academy, 150 trainees were recruited in 2000. The recruiting team planned to increase the number of trainees each subsequent year at a rate of 20% of the number for the previous year. Determine the number expected to be in 2010.*

Solution

In 2000 there were 150

In 2001 there were $150 + 0.2(150) = 150(1 + 0.2) = 150(1.2)$

In 2002 there were $150(1.2) + (150)(1.2)(0.2) = 150(1.2)(1 + 0.2) = 150(1.2)^2$

In 2003 the number will be $150(1.2)^2 + 150(1.2)^2(0.2)$

$= 150(1.2)^2(1 + 0.2) = 150(1.2)^2$

$\therefore$ In 2010, after 10 years, the number is $150(1.2)^{10} \approx 929$ trainees.

Task

1. *The population of a country increases every year by 2.4% of the population at the beginning of that year. In what time will the population double itself?*
   ANSWER: $\mathbf{29}\ \boldsymbol{years}$
2. *In Jinja district, 6500 pupils registered for PLE 1996.* In 1997 the number of students increased by 1250 and in each subsequent year the number of pupils increased by 1/5 more than the previous year. Find the number of pupils that will be registered in 2006. **ANSWER**: $\mathbf{40\,000}$ *to* 3 *s.f*
3. *If Kenya has a population of 23 million people and a doubling time of 19 years, and growth continues at the same rate, find the population in;*
   a) 10 years. **ANSWER**: $\mathbf{33}\ \boldsymbol{million}$
   b) 30 years. **ANSWER**: $\mathbf{69}\ \boldsymbol{million}$
4. *If the population in a particular third world country is growing at 4% compounded annually how long will it take the population to double?*
   ANSWER: $\mathbf{18}\ \boldsymbol{years}$

ANNUITY CALCULATIONS

FUTURE VALUE OF ANNUITY

Bankers call a series of equal annual payment made at equal intervals of time an *Annuity.*

Terms associated with Annuities

i) **Immediate/Ordinary Annuity**

If each of the annuity payment is made at the end of each period (time interval), is called *immediate annuity.*

ii) **Future value of an annuity**

Is the sum of all the annuity payments plus any accumulated interest.

To find the Amount of Annuity left unpaid for a given number of years, allowing Compound Interest.

Let R be the periodic deposit, i the rate per period and n the number of periods in the given years.

Case I: Payment at the end of each period.

At the end of the 1^{st} year, R is due and the amount of this sum in the remaining $(n-1)$ year is $R(1+i)^{n-1}$

At the end of 2^{nd} year, another R is due and the amount of this sum in the remaining $(n-2)$ years is $R(1+i)^{n-2}$

$\therefore$ Future value $= R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots \ldots \ldots \ldots + R(1+i)^2 + R(1+i)^1 + R$

$$= R[1 + (1+i) + (1+i)^2 + \cdots \ldots \ldots \ldots + (1+i)^{n-2} + (1+i)^{n-1}]$$

$$= R[1 + (1+i) + (1+i)^2 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \, to\ n - terms]$$

$$= \frac{R[(1+i)^n - 1]}{(1+i)-1} = \frac{R[(1+i)^n - 1]}{i}$$

In formula form,

Future value of Immediate Annuity is:

$$\boldsymbol{FV = PMT.\frac{(1+i)^n - 1}{i}}$$

$PMT =$ Periodic payment

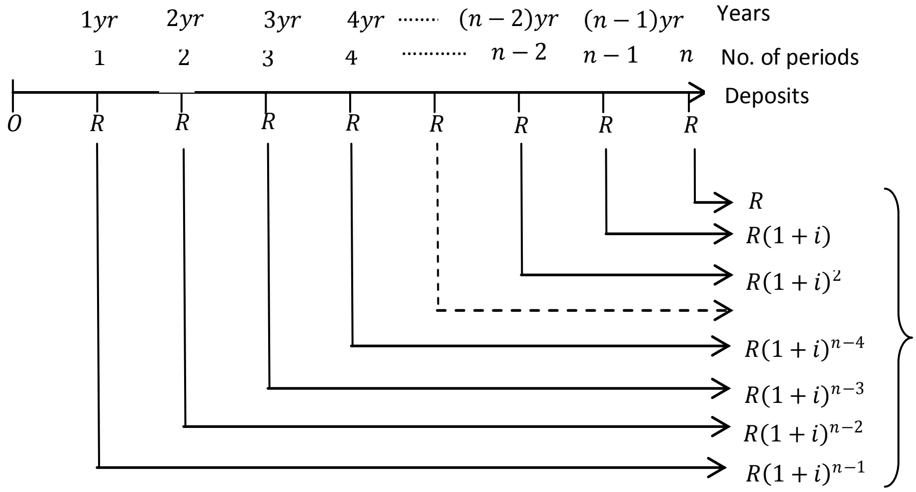
$i =$ Rate per period

$n =$ Number of payments (Periods)

$FV =$ Future Value (Amount)

[Payments are made at the end of each period]

Generalization of the above series using time line



Hence the sum of all these annuities plus any interest is;
$= R + R(1+i) + R(1+i)^2 + \cdots \ldots\ldots\ldots.. + R(1+i)^{n-4} + R(1+i)^{n-3} + R(1+i)^{n-2} + R(1+i)^{n-1}$

NOTE:

1. The future value of annuity is considered when payments are made at the end of each period (Immediate Annuity)
2. Banks usually state interest rates as *Annual Percentage Rates*(APRS)

Case II: Payment at the beginning of every period (payment for every period)

At the beginning of 1^{st} year, R is due and amount of this sum for n years is $R(1+i)^n$

At the beginning of 2^{nd} year, another R is due and the amount of the remaining $(n-1)$ year is $R(1+i)^{n-1}$

$\therefore$ Future value $= R(1+i)^n + R(1+i)^{n-1} + \cdots \ldots\ldots\ldots\ldots. + R(1+i)^2 + R(1+i)$

$$= R[(1+i) + (1+i)^2 + \cdots \ldots\ldots.. + (1+i)^{n-1} + (1+i)^n]$$
$$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots \ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}]$$
$$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots \ldots\ldots\ldots to\ n - terms]$$
$$= \frac{R(1+i)[(1+i)^n - 1]}{i}$$

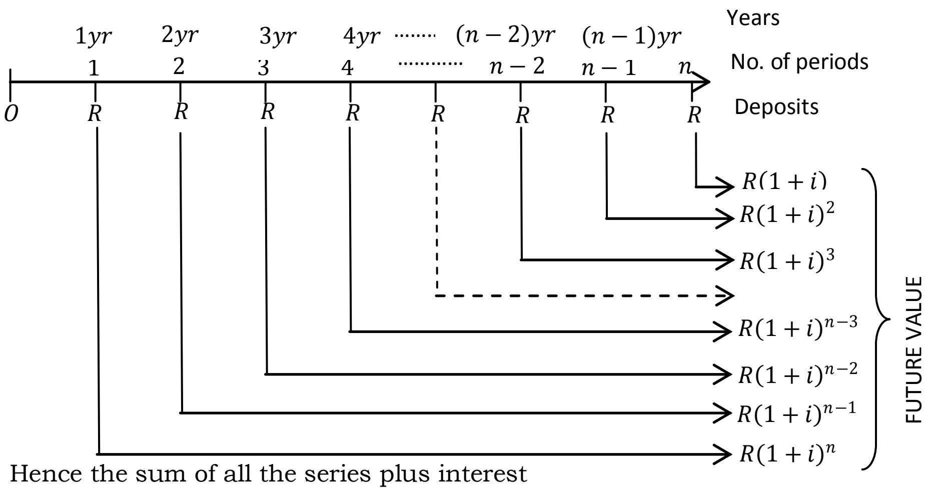
In formula form,

Future value of an Annuity due is:
$\boldsymbol{FV = PMT.\frac{(1+i)}{i}.[(1+i)^n - 1]}$
$PMT =$ Periodic payment
$i =$ Rate per period
$n =$ Number of payments (Periods)
$FV =$ Future Value (Amount)
[Payments are made at the beginning of each period/payments are made for every period]

Case III: Annuity due

Is an annuity where the first payment falls due in the beginning of the first period or interval, hence Case II.

Generation of Case II series above using time line.



Hence the sum of all the series plus interest

$\therefore$ Future value $= R(1+i)^n + R(1+i)^{n-1} + \cdots \ldots\ldots\ldots\ldots\ldots . + R(1+i)^2 + R(1+i)$

$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots \ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}]$

General examples:

(Immediate Annuity)

1. *A person wishes to deposit* $\$\,1000$ *at the end of each year in a bank which pays* 10% *p.a Compound Interest. If installments are allowed to accumulate, what will be the total accumulation at the end of 10 years?*

Solution

At the end of $1yr, 1^{st}$ installment of $\$\,1000$ amounts in 9 years to:

$1000(1+0.1)^9 = 1000(1.1)^9$

The 2^{nd} installment of $\$\,1000$ amounts in 8 years to: $1000(1+0.1)^8 = 1000(1.1)^8$

The last installment of $\$\,1000$ amounts to: $1000(1+0.1)^0 = 1000$

So the total accumulated at the end of 10 years is:

$Total = 1000 + 1000(1.1) + 1000(1.1)^2 + \cdots \ldots\ldots\ldots .. + 1000(1.1)^8 + 1000(1.1)^9$, which is a G.P of 10 terms.

$= \frac{1000\,[(1.1)^{10}-1]}{1.1-1} = \frac{1000\,[(1.1)^{10}-1]}{0.1} = \$\,15\,937$

Alternatively, using the formula,

$P = \$\,1000\,, i = 0.1\,, n = 10\,, A = ?$

$A = \frac{P}{i}[(1+i)^n - 1]$

$= \frac{1000}{0.1}[(1+0.1)^{10} - 1] = \$\,15\,937$

(Immediate Annuity)

2. *Jessica deposits* $\$\,100$ *every 6 months in an account that pays 6% Compound Interest compounded semi annually. If she made 6 deposits one at the end of each interest payment period over 3 years, how much money will be on the account after the last deposit is made?*

Solution

$\%\, rate\ p.a = 6, \%\, rate\ per\ 6 - months = \frac{6}{2} = 3\%\ .i.e.i = 0.03, no.of\ periods = 6$

At the end of the 1^{st} $6\, months,$ \$100 is due, and this amounts to:

$\$\,100(1+0.03)^5$

The 2^{nd} installment of \$ 100 amounts to: $\$\,100(1+0.03)^4 = \$\,100(1.03)^4$

The 3^{rd} installment of \$ 100 amounts to: $\$\,100(1+0.03)^3 = \$\,100(1.03)^3$

The *last* installment of \$ 100 amounts to: $\$\,100(1+0.03)^0 = \$\,100(1.03)^0 =$ $\$\,100$

Thus the total accumulated at the end of the 6^{th} deposit (at the end of 3 years) is:

$= 100 + 100(1.03) + 100(1.03)^2 + 100(1.03)^3 + \cdots \ldots. + 100(1.03)^5$

$= 100[1 + 1.03 + (1.03)^2 + \cdots \ldots. +(1.03)^5]$

$= \frac{100\,[(1.03)^6 - 1]}{1.03 - 1} = \$\,646.84$

Alternatively, using the formula

$P = \$\,100\,, i = \frac{6}{2} = 3\% = 0.03\,, n = 3\, yrs = 6\, periods, A = ?$

$A = 100\left[\frac{(1.03)^6 - 1}{0.03}\right] = \$\,646.84$

(Immediate Annuity)

3. *When Jim started his first job after he finished college, he opened an Individual Retirement Account (IRA). He plans to contribute\$ 2 500 per year for 38 years until he reaches age 62. He hopes to earn an average annual percentage rate (APR) of 8% over the 38 year period.*
   a) *If Mr. Jim contributes to his IRA at the rate he plans, how much will his account be worth when he is 62 years old?*
   b) *How much interest will be earned on the account?*

Solution

Given: $P = \$\,2\,500\,, i = 8\% = 0.08\,, n = 38\, yrs$

a) At the end of 1^{st} year \$ 2 500 is due, and this amounts to:
$\$\,2\,500(1+0.08)^{37} = \$\,2\,500(1.08)^{37}$ for the remaining years.
The 2^{nd} installment of \$ 2 500 amounts to:
$\$\,2\,500(1+0.08)^{36} = \$\,2\,500(1.08)^{36}$
The 2^{nd} installment of \$ 2 500 amounts to:
$\$\,2\,500(1+0.08)^{35} = \$\,2\,500(1.08)^{35}$
$\therefore$ the *last* deposit amounts to:
$\$\,2\,500(1+0.08)^0 = \$\,2\,500(1.08)^0 = \$\,2\,500$
$\therefore$ the accumulated amount after 38 years is:
$= 2500 + 2500(1.08) + 2500(1.08)^2 + \cdots \ldots \ldots + 2500(1.08)^{37}$
$= 2500[1 + 1.08 + (1.08)^2 + \cdots \ldots \ldots + (1.08)^{37}]$
$= \frac{2500\,[(1.08)^{38} - 1]}{1.08 - 1} = \$\,550\,789.86$

b) Mr. Jim will pay\$ $2500 \times 38 = \$95\,000$ in to his IRA, so the account will earn:
$\$\,(550\,789.86 - 95\,000) = \$\,455\,789.86$

4. *John, procrastinating, does not make his first \$ 2000 deposit in to an IRA account until he is 32, but then he continues to deposit \$ 2000 on every birthday until he is 65. If his account also earns 9% compounded annually, how much (to the nearest dollar) will he have on his account when he makes the last deposit on his 65^{th} birth day?*

Solution

The 1^{st} deposit is at when John is 32 years.

The *last* deposit is at when John is 65 years.

∴ number of deposits in all= 34, at the end of every year $i = 9\%\ p.a, P =$ \$ 2 000

The 1^{st} deposit (at 32 years) grows to $2000(1+0.09)^{33}$

The 2^{nd} deposit (at 33 years) grows to $2000(1+0.09)^{32}$

∴ the last deposit (at 65 years) grows to $2000(1+0.09)^{0}$

∴ the total amount accumulated at the end of the 65^{th} birthday is;

$$= 2000 + 2000(1+0.09) + 2000(1+0.09)^2 + \cdots \dots \dots \dots + 2000(1+0.09)^{33}$$
$$= 2000[1 + 1.09 + (1.09)^2 + \cdots \dots \dots \dots \dots + (1.09)^{33}]$$
$$= \frac{2000\,[(1.09)^{34}-1]}{1.09-1} = \$\ 393\ 965$$

5. *A man puts* $Rs. 10$ *at the end of every year in the savings Bank at 2 ½ % compound interest. How much will his savings amount to in 15 years?*

Solution

At the end of 1^{st} year, $Rs. 10$ is due, and amounts to $10(1+0.025)^{14}$

At the end of the 2^{nd} year, $Rs. 10$ is again due, and amounts to $10(1+0.025)^{13}$

The last deposit grows to $10(1+0.025)^{0}$

∴ total amount $= 10 + 10(1.025) + 10(1.025)^2 + \cdots \dots \dots \dots \dots + 10(1.025)^{14}$
$$= 10[1 + 1.025 + (1.025)^2 + \cdots \dots \dots \dots \dots + (1.025)^{14}]$$
$$= \frac{10[(1.025)^{15}-1]}{1.025-1} = Rs. 179.32$$

6. *A man deposits* \$ 100 *per month in to an account earning 9% compounded monthly for 5 years. How much interest does he receive?*

Solution

At the end of the 1^{st} month, 1^{st} deposit grows to \$ $100\left(1+\frac{0.09}{12}\right)^{59}$ in the remaining 59 months.

At the end of the 2^{nd} month, 2^{nd} deposit grows to \$ $100\left(1+\frac{0.09}{12}\right)^{58}$ in the remaining 58 months.

∴ at the end of the last month of deposit, deposit grows to \$ $100\left(1+\frac{0.09}{12}\right)^{0} =$ \$ 100

∴ total accumulation at the end of 5 years($60\ months$) is:

$$= 100 + 100\left(1+\frac{0.09}{12}\right)^1 + 100\left(1+\frac{0.09}{12}\right)^2 + \cdots \dots \dots \dots \dots + 100\left(1+\frac{0.09}{12}\right)^{59}$$
$$= 100\left[1 + \left(1+\frac{0.09}{12}\right) + \left(1+\frac{0.09}{12}\right)^2 + \cdots \dots \dots .. + \left(1+\frac{0.09}{12}\right)^{59}\right]$$
$$= \frac{100\left[\left(1+\frac{0.09}{12}\right)^{60}-1\right]}{\left(1+\frac{0.09}{12}\right)-1} = \frac{100\left[\left(1+\frac{0.09}{12}\right)^{60}-1\right]}{0.0075} = \$\ 7\ 542.41$$

Now total amount deposited= $60 \times 100 = 6\ 000$

$Interest = Value - Deposits$

interest= $7\ 542.41 - 6\ 000 = \$\ 1\ 542.41$

Note:

Since 9% is compounded monthly, then $i = \frac{0.09}{12} = 0.0075$

7. *Jane deposits* $\$ 2000$ *a year in to an IRA account earning 9% compounded annually. She makes her first deposit on her* 24^{th} *birthday and her last deposit on her* 31^{st} *birthday (8 deposits in all). Making no additional deposits, she leaves the accumulated amount from the 8 deposits in an account, earning interest at 9% compounded annually until her* 65^{th} *birthday. How much to the nearest dollar will be in her account on her* 65^{th} *birthday?*

Solution

Given: $P = \ \$ 2000\,, i = 9\% = 0.09\,, n = 8$

At the end of the 1^{st} year, $\$ 2000$ is due and amounts to $2000(1 + 0.09)^7$ in the remaining 7 years.

At the end of the 2^{nd} year, $\$ 2000$ is again due and amounts to $2000(1 + 0.09)^8$ in the remaining 6 years.

$\therefore$ the last deposit of $\$ 2000$ grows to $2000(1 + 0.09)^0 = 2000$

$\therefore$ the total amount for the 8 years of depositing is:

$$= 2000 + 2000(1.09) + 20002000(1.09)^2 + \cdots \ldots \ldots \ldots \ldots + 2000(1.09)^7$$
$$= 2000[1 + 1.09 + (1.09)^2 + \cdots \ldots \ldots \ldots \ldots + (1.09)^7]$$
$$= \frac{2000\,[(1.09)^8 - 1]}{1.09 - 1} = \$\ 22\ 056.94759$$

Now since no other deposit after the 8 −deposits, the money grows at compound interest at a rate of 9% $p.a$ for 34 years after the 31^{st} *birthday.*

Using $A = P(1 + i)^n$

$$A = 22\ 056.94759(1 + 0.09)^{34} = \$\ 413\ 092$$

(Annuity due)

8. *A person invests* £ 2000 *every year with a company which pays interest 10% p.a. he allows his deposit to accumulate at compound interest. Find the amount to the credit of the person at the end of the* 5^{th} *year.*

Solution

At the beginning of the 1^{st} year, £ 2000 is due and his sum amounts in 5 years to

£ $2000(1 + 0.1)^5$

At the beginning of the 2^{nd} year, another £ 2000 is due and his sum amounts in 4 years to

£ $2000(1 + 0.1)^4$

The last deposit (5^{th} year) grows to £ $2000(1 + 0.1)^1$

$\therefore$ total amount accumulated

$$= 2000(1 + 0.1) + 2000(1 + 0.1)^2 + \cdots \ldots \ldots . + 2000(1 + 0.1)^5$$
$$= 2000(1.1)[1 + 1.1 + (1.1)^2 + \cdots \ldots \ldots \ldots + (1.1)^4]$$
$$= \frac{2000\,(1.1)[(1.1)^5 - 1]}{1.1 - 1} = \frac{2000\,(1.1)[(1.1)^5 - 1]}{0.1} = £13\ 431.22$$

Alternatively by formula method

$P = £\,2000\,, n = 5\,, i = 0.1\,, A = \frac{P(1+i)}{i}[(1 + i)^n - 1]$

$A = £\,2000 \times \frac{1.1}{0.1} \times [(1.1)^5 - 1] = £13\ 431.22$

(Annuity due)

9. *Mr. John deposits* £ 2000 *per year in the bank for 10 years at 5% p.a. find how much will he receive at the end of 10 years if the money accumulates at compound interest.*

Solution

At the end of the 10^{th} deposit, the last deposit grows to £$2000(1+0.05)$

At the end of the 9^{th} deposit, the amount grows to £$2000(1+0.05)^2$

$\therefore$ the 1^{st} deposit grows to £$2000(1+0.05)^{10}$

$\therefore$ total amount accumulated $= 2000(1.05) + 2000(1.05)^2 + \cdots \ldots\ldots\ldots + 2000(1.05)^{10}$

$$= 2000(1.05)[1 + 1.05 + (1.05)^2 + \cdots \ldots\ldots\ldots + (1.05)^9]$$

$$= \frac{2000\,(1.05)[(1.05)^{10}-1]}{1.05-1} = \frac{2000\,(1.05)[(1.05)^{10}-1]}{0.05}$$

$$= £26\,413.57$$

10. *Jack operates an account with a certain bank which pays a compound interest rate of 13.5% p.a. he opened the account at the beginning of the year with Sh. 500 000 and deposits the same amount of money at the beginning of every year. Calculate how much will he receive at the end of 9 years. After how long will the money have accumulated to Sh. 3.32 million.*

Solution

At the beginning of the 1^{st} year, Sh. 500 000 is due, and the amount in 9 years is

$500\,000(1+0.135)^9 = 500000(1.135)^9$

At the beginning of the 2^{nd} year, another Sh. 500 000 is due and amounts in 8years to

$500\,000(1+0.135)^8 = 500000(1.135)^8$

$\therefore$ the last deposit in the 9^{th} year, amounts to $500\,000(1+0.135)^1 = 500\,000(1.135)$

$\therefore$ the total amount $= 500000(1.135) + 500000(1.135)^2 + \cdots \ldots\ldots. +500000(1.135)^9$

$$= 500000(1.135)[1 + 1.135 + (1.135)^2 + \cdots \ldots\ldots + (1.135)^8]$$

$$= \frac{500000\,(1.135)[(1.135)^9-1]}{1.135-1} = \frac{500000\,(1.135)[(1.135)^9-1]}{0.135}$$

$$= Sh.\,8\,936\,281$$

Let the money accumulate in $n-years$ to 3.32 million

The 1^{st} deposit grows to $500\,000(1+0.135)^n = 500\,000(1.135)^n$ in $n-years$.

The 2^{nd} deposit grows to $500\,000(1+0.135)^{n-1} = 500\,000(1.135)^n$ in $(n-1)-years$.

$\therefore$ the n^{th} deposit grows to $500\,000(1.135)$

$\therefore$ the total amount after $n-years$

$$= 500\,000(1.135) + 500\,000(1.135)^2 + \cdots \ldots\ldots + 500\,000(1.135)^n$$

$$= 500\,000(1.135)[1 + 1.135 + (1.135)^2 + \cdots \ldots\ldots. +(1.135)^{n-1}]$$

$$= \frac{500\ 000(1.135)[(1.135)^n-1]}{1.135-1} = \frac{500\ 000(1.135)[(1.135)^n-1]}{0.135}$$

$$\therefore 3.32\times10^6 = \frac{500\ 000(1.135)[(1.135)^n-1]}{0.135}$$

$0.789779735 = 1.135^n - 1 \ or\ 1.135^n = 1.789779735$

$\ln(1.135)^n = \ln 1.789779735$

$$\therefore n = \frac{\ln 1.789779735}{\ln 1.135} = 4.6\,years$$

11. *Five million shillings are invested each year at a rate of 20% interest. In how many years will it accumulate to have more than 50 million shillings?*

Solution

Since deposits are made each/every year, this is a case of *annuity due.*

For the 1^{st} year, $Sh.5\,000\,000$ is due, and amounts in n years to $Sh.5\,000\,000(1+0.2)^n$
For the 2^{nd} year, another $Sh.5\,000\,000$ is due, and amounts in $n-1$ years to $Sh.5\,000\,000(1+0.2)^{n-1}$
∴ the last deposit in the n^{th} year amounts to $Sh.5\,000\,000(1+0.2)$
∴ total accumulation after n years is:

$$= 5\,000\,000(1.2) + 5\,000\,000(1.2)^2 + \cdots \ldots\ldots\ldots . +5\,000\,000(1.2)^n$$
$$= 5\,000\,000(1.2)[1 + 1.2 + (1.2)^2 + \cdots \ldots\ldots . +(1.2)^{n-1}]$$
$$= \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{1.2-1} = \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{0.2}$$

Given the total accumulation is $Sh.50\,000\,000$

$\Rightarrow 50\,000\,000 = \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{0.2}$

$\Rightarrow \frac{5}{3} = 1.2^n - 1 \;\; or \;\; 1.2^n = \frac{8}{3}$

$\Rightarrow \ln 1.2^n = \ln\left(\frac{8}{3}\right) \;\; \therefore n = \frac{\ln\left(\frac{8}{3}\right)}{\ln 1.2} = 5.379666948$

∴ in 6 years the money would have accumulated more than 50 million.

Task

1. *Find the amount of annuity due of* £2000 *per year payable in the beginning of every year for 10 years at* 5% $p.a$ **ANSWER: £26 413.60**
2. *Five million shillings are invested each year at a rate of 15% interest. In how many years will it accumulate to more than Sh. 50 million?*
   ANSWER: 6 *years*
3. *A man puts Rs. 10 at the end of every year in the savings Bank at 2 ½ % compound interest. How much will his savings amount to in 15 years?*
   ANSWER: *Rs.* 179
4. *A company sets aside a sum of* £20 000 *annually to enable it pay off a debenture issue of* £230 000 *at the end of 10 years. Assuming that the sum accumulates at* 4% $p.a$ *compound interest, find the surplus after paying off a debenture.*

   HINT:
   £20 000 *will amount to (at the end of* 10 *years)* $20\,000(1.04)^{10}$
   The 2^{nd} *sum amounts to* $20\,000(1.04)^9$
   ∴ *total money at the end of the* 10^{th} *year wil be*:
   $20\,000(1.04)[1 + 1.04 + (1.04)^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots . +(1.04)^9]$
   $= \frac{20\,000(1.04)[(1.04)^{10} - 1]}{0.04} = £249\,080$
   Thus the surplus $= 249\,080 - 230\,000 = £19\,080$

5. *Phionah deposited* \$1000 *every 6 months in to an account earning 8% compounded semi annually. How much will the* \$1000 *amount to at the end of 10 years? How much of this value is interest?*

HINT:

1^{st} *deposit grows to* $1000\left(1+\frac{0.08}{2}\right)^{19}$ *in* 19 *months*

2^{nd} *deposit grows to* $1000\left(1+\frac{0.08}{2}\right)^{18}$ *in* 19 *months*

$\therefore$ *last deposit grows to* $1000\left(1+\frac{0.08}{2}\right)^{0}$

$\therefore Total\ value = 1000[1+1.04+(1.04)^2+\cdots\ldots\ldots+(1.04)^{19}]$
$= \$\,29\,778.4$

$\therefore Interest = 29\,778.4 - 20\times 1000 = \$\,9\,778.4$

6. *Beginning in January, a person plans to deposit* $\$\,100$ *at the end of each month in to an account earning 9% compounded monthly. Each year taxes must be paid on the interest earned during that year. Find the interest earned during each year for the first* 3 −*years.*

HINT:

$i=\frac{0.09}{12}=0.0075\ per\ month, P=\$\,100\ , n=1\ year=12\ months.$

$Value\ after\ 1^{st}\ year=\frac{100[(1+0.0075)^{12}-1]}{0.0075}=\$\,1\,250.76$

$Interest\ for\ 1^{st}\ year=1\,250.76-12\times 100=\$\,50.76$

$Value\ after\ 2^{nd}\ year=\frac{100[(1+0.0075)^{24}-1]}{0.0075}=\$\,2\,618.85$

$Interest\ after\ 2\ years=2\,618.85-24\times 100=\$\,218.85$

$\therefore Interest\ for\ 2^{nd}\ year=218.85-50.76=\$\,168.09$

$Value\ after\ 3^{rd}\ year=\frac{100[(1+0.0075)^{36}-1]}{0.0075}=\$\,4\,115.27$

$\therefore Interest\ after\ 3\ years=4\,115.27-36\times 100=\$\,512.27$

$Now\ interest\ for\ 3^{rd}\ year$

$= Interest\ for\ 3\ years\ minus\ interest\ for\ the\ 1^{st}\ two\ years$

$= 512.27-218.85=\$\,296.42$

7. *What is the annuity at the end of 5 years if* $\$\,100$ *per month is deposited in to an account earning 9% compounded monthly? How much of this value is interest?* ***Answer***: $\$\,\mathbf{1\,542.41}$

8. *A man pays a premium of* £100 *at the beginning of every year to an insurance company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? (Give your answer correct to* 3 *s.f)*

HINT:

At the beginning of 1^{st} *year,* £100 *is due and amounts to* £$100(1+0.05)^{15}$ *for remaining* 15 *years.*

At the beginning of 2^{nd} *year, another* £100 *is due and amounts to* £$100(1+0.05)^{14}$ *for the remaining* 14 *years.*

The last deposit grows to £$100(1+0.05)$

$\therefore Total = 100(1.05)+100(1.05)^2+\cdots\ldots\ldots\ldots.+100(1.05)^{15}$

$=\frac{100(1.05)[(1.05)^{15}-1]}{0.05}=2\,265.749177$

≈ 22700

9. *A man earned in a certain year* £2000 *from a certain source and his annual earnings from this time continued to increase at a rate of 5%. Find to the nearest* £ *the whole amount he received from this source in this year and the next seven years. Give your answer correct to* 3 *s.f.*

HINT:

$Amount\ in\ the\ 1^{st}\ year = 2000(1 + 0.05) = £2100$

$Now\ amount\ in\ 2^{nd}\ year\ grows\ to\ 2000(1 + 0.05)^7\ in\ the\ remaining\ 7\ years$
$or\ grows\ to\ 2000(1 + 0.05)^6\ in\ the\ remaining\ 6\ years, taking\ £2100\ as$
$the\ beginning\ amount.$

$Amount\ in\ 3^{rd}\ year\ grows\ to\ 2000(1 + 0.05)^6 in\ the\ remaining\ 6\ years$
$or\ grows\ to\ 2000(1 + 0.05)^5\ in\ the\ remaining\ 5\ years.$

$\therefore Amount\ in\ the\ last\ year\ grows\ to\ 2000(1 + 0.05)^0$

$\therefore Total\ amount\ accumulated\ in\ the\ next\ 7\ years\ is$:

$$= 2000 + 2000(1.05) + 2000(1.05)^2 + \cdots \ldots \ldots + 2000(1.05)^7$$
$$= 2000[1 + 1.05 + (1.05)^2 + \cdots \ldots . . +(1.05)^7]$$
$$= \frac{2000\,[(1.05)^8 - 1]}{0.05} = £19\,098.21775$$
$$\approx £19\,100$$

10. *A credit society gives out a compound interest of 4.5% p.a. Mugaga deposits Sh. 300 000 at the beginning of each year. How much money will he have at the end of 4 years if there are no withdraws during this period.* ***(Uneb 2006)*** ***Answer***: ***Sh.* 1 341 213**
11. *A man invests* £100 *at the end of 1971,* £200 *at the end of1972,* £300 *at the end of 1973,* £400 *at the end of 1974,* £500 *at the end of 1975. If all interest accumulated at* 5% *p.a, what do his investments amount to at the end of 1976?*

HINT:

$£100\ is\ invested\ for\ 5\ years, £200\ for\ 4\ years, £300\ for\ 3\ years, £400\ for\ 2\ years$
$£500\ for\ 1\ year$

$Total\ value\ at\ the\ end\ of\ 1976\ is;$

$A = 100(1.05)^5 + 100(1.05)^4 + 100(1.05)^3 + 100(1.05)^2 + 100(1.05) = £1\,684.2$

PERMUTATIONS AND COMBINATIONS
FACTORIALS:
Are used in the computations of permutations and combinations
Definition:
The continued product of the first $n - natural\ numbers\ i.e.$ the continued product of n-consecutive integers beginning with 1 and ending with n, is $ctorial\ n$, denoted by the symbol $n!$

$$\therefore \boldsymbol{n! = 1 \times 2 \times 3 \times \ldots\ldots\ldots\ldots \times n}$$
$$\boldsymbol{= n(n-1)(n-2)(n-3)\ldots\ldots\ldots \times 2 \times 1}$$

Examples

1. *How many different 3-digit number can be constructed from numerals 5,7 and 8 taking each once?*
   Solution
   3-choices arise for the 1^{st} numeral and for each choice; there are a further of 2-choices for the 2^{nd} numeral. The third numeral is the one that is left
   $\therefore 3 \times 2 \times 1 = 6$ choices for the 1^{st} *and* 2^{nd} numerals combined exist.
   $i.e. 3! = 3 \times 2 \times 1 = 6$

2. *How many 4-digit numbers can be constructed using numerals 1,2,3 and 4 once each?*
   Solution
   4-choices arise for the 1^{st} numeral; each choice leaves 3-ways to select2^{nd} numeral
   $\Rightarrow 4 \times 3 = 12\ ways$ of selecting the first 2-numerals exist.
   Each combination of the first two numerals leaves 2-ways to select the third numeral.
   $\Rightarrow Ther\ are\ 4 \times 3 \times 2 \times 1 = 24\ ways$ of selecting first 3-numerals. The last nmeral s the one left.
   $\therefore 4 \times 3 \times 2 \times 1 = 4! = 24$ are the number of ways.

3. *Evaluate*
   a) $6!$
   b) $\frac{8!}{3!}$
   c) $(7-2)!$
   d) $\frac{3!}{0!}$
   e) $\frac{5!}{(5-2)!2!}$

   Solution
   a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
   b) $\frac{8!}{3!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = 6720$
   c) $(7-2)! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   d) $\frac{3!}{0!} = \frac{3\times2\times1}{1} = 6 \qquad [\because 0! = 1]$
   e) $\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{(3\times2\times1).(2\times1)} = 10$

Note
1) From the definition of n!
   $n! = n(n-1)(n-2)(n-3)\ldots\ldots\ldots \times 2 \times 1$

$= n \times (n-1)!$

$\therefore n = \frac{n!}{(n-1)!}$

2) $2n! = 2 \times n!$; $(2n)! = (2n)(2n-1)(2n-2) \ldots\ldots \times 2 \times 1$

Example

1. *Simplify*
   a) $\frac{(2n+1)!}{(2n-1)!}$
   b) $\frac{n!}{(n+1)!}$
   c) $\frac{(n+1)!}{(n-1)!}$
   d) $\frac{(2n)!}{(2n+2)!}$

Solution

a) $\frac{(2n+1)!}{(2n-1)!} = \frac{(2n+1)(2n+1-1)(2n+1-2)\times\ldots\ldots\ldots\ldots\times 2\times 1}{(2n-1)!}$

$= \frac{(2n+1)(2n)(2n-1)!}{(2n-1)!} = 2n(2n+1)$

b) $\frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n+1-1)!} = \frac{n!}{(n+1)\times n!} = \frac{1}{n+1}$

c) $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n+1-1)(n+1-2)\ldots\ldots\times 2\times 1}{(n-1)!}$

$= \frac{(n+1)\times n\times(n-1)!}{(n-1)!} = n(n+1)$

d) $\frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+2-1)(2n+2-2)\ldots\ldots\times 2\times 1}$

e) $= \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{2(n+1)(2n+1)}$

2. *Write each of the following in factorial notation*
   a) $4 \times 3 \times 2 \times 1$
   b) $6 \times 5 \times 4$
   c) $\frac{(7\times 6)\times(3\times 2\times 1)}{2}$

Solution

a) $4 \times 3 \times 2 \times 1 = 4!$

b) $6 \times 5 \times 4 = \frac{(6\times 5\times 4)\times 3\times 2\times 1}{3\times 2\times 1} = \frac{6!}{3!}$

c) $\frac{(7\times 6)\times(3\times 2\times 1)}{2} = \frac{[(7\times 6)\times 5\times 4\times 3\times 2\times 1]\times(3\times 2\times 1)}{(2\times 1)\times(5\times 4\times 3\times 2\times 1)} = \frac{7!3!}{2!5!}$

Note

Some other terms have been introduced but without changing the meaning of the question so as to apply $n!$ definition.

PERMUTATIONS

The knowledge of permutations helps in finding *the number of ways a set of objects may be grouped/ arranged under certain conditions.*

Definition

Each arrangement/ *ordered* set of objects which can be obtained from a given set of objects by taking some or all at a time is called *permutation of the given set of objects.*

Notation

If the number of objects in the given set is n and the number of objects in each permutation is r ($i.e. n \geq r$), the total number of permutation is denoted as ${}^{n}\boldsymbol{p}_{r}$

Consider the example below

Qn. *Assuming in the department of math, there are four teachers A, B, C and D; a committee consisting of two members is to be formed for drafting the math syllabus, list the possible groups.*

ANSWER: AB, AC, AD, BC, BD, CD

Hence there is 6-groups altogether $i.e. 6\ combinations$

Qn. *Suppose one is to be selected as a president of the syllabus committee and the other as a secretary, list the possible groups.*

ANSWER:

Let the 1^{st} one denote the president and the 2^{nd} as the secretary.

The possibilities are;

$A_{president}\ B_{secretary}\ , BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC$

Hence there is now 12-possibilities altogether.

⇒ This problem is not only one of *grouping* but also *ordering* them.

Thus each of the 12-*arrangements* made out of 4-letters A, B, C, D by taking 2-letters at a time viz: $AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC$ is a *permutation of the 4-letters taken 2-at a time*. $i.e.$ ${}^{4}\boldsymbol{p}_{2} = \mathbf{12}$

NOTE

1. Permutations involve *ordering* and *arranging*. $i.e. AB$ and BA are different; AB means A is the president , B is the secretary, while BA means B is the president and A is the secretary.

   However for purpose of *grouping, AB* and *BA* mean one and same group.
2. Grouping is more in *combinations*

IMPORTANT RESULTS OF PERMUTATIONS

a) *Permutations of n-different objects*

   Permutations of n-different objects taken r-at a time $r < n$ is

$$\boxed{{}^{n}\boldsymbol{p}_{r} = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{r})!} = \boldsymbol{n}(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2}) \ldots\ldots\ldots\ldots (\boldsymbol{n}-\boldsymbol{r}+\mathbf{1})}$$

b) *Permutations in which objects are repeated*

   The number of permutations of n-different objects taken r-at a time in which each object is repeated r-times in any arrangement is $\boldsymbol{n}^{\boldsymbol{r}}$
c) *Permutations of objects not all different*

   The number of permutations of n- objects of which p-objects are alike of one kind, q-objects are alike of second kind, r-objects are alike of a third kind, and all the rest are all different is
d) *Circular permutations*

$$\boxed{\frac{\boldsymbol{n}!}{\boldsymbol{p}!\boldsymbol{q}!\boldsymbol{r}!}}$$

In circular permutations, fix the position of one object and then arrange the remaining $(n-1)$ out of n-objects in all the possible ways. This can be done in only $\boxed{(\boldsymbol{n-1})!\,\boldsymbol{ways}}$

e) *Restricted permutations*

i) The number of permutations of n-different objects taken r-at a time in which p-particular objects do not occur is

$$\boxed{{}^{n-p}\boldsymbol{p_r}}$$

ii) The number of permutations of n-different objects taken r-at a time in which p-particular objects are present is

$$\boxed{{}^{n-p}\boldsymbol{p_{r-p}} \times {}^{r}\boldsymbol{p_p}}$$

Examples

1. *Find the value of n if:*

a) ${}^{n}p_1 = 3$

b) ${}^{n}p_2 = 2$

c) ${}^{2n}p_2 = 6$

d) ${}^{3n}p_2 = 30$

Solution

a) ${}^{n}p_1 = 3 \Rightarrow \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = 3 \ \therefore n = 3$

b) ${}^{n}p_2 = 2 \Rightarrow \frac{n!}{(n-2)!} = 2 \ or \ \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = 2$

$\Rightarrow n^2 - n - 2 = 0 \ \therefore n = 2\,, -1 \ \ Required\, n = 2$

c) ${}^{2n}p_2 = 6 \Rightarrow {}^{2n}p_2 = \frac{(2n)!}{(2n-2)!} = \frac{(2n)(2n-1)(2n-2)!}{(2n-2)!} \ \therefore 2n(2n-1) = 6 \ or \ 4n^2 - 2n - 6 = 0$

solving$n = 1.5\,, -1$. Since n is a counting number, then $n = 2$

d) ${}^{3n}p_2 = 30 \Rightarrow {}^{3n}p_2 = \frac{(3n)!}{(3n-2)!} = \frac{(3n)(3n-1)(3n-2)!}{(3n-2)!}$

$\therefore 3n(3n-1) = 30 \ or \ 9n^2 - 3n - 30 = 0$. Solving $n = 2\,, -\frac{5}{3}$ $\therefore$

$Required\, n = 2$

2. *Find n if*

i) ${}^{n}p_5 : {}^{n}p_3 = 2 : 1$

ii) ${}^{2n+1}p_{n-1} : {}^{2n-1}p_n = 3 : 5$

iii) ${}^{56}p_{n+6} : {}^{54}p_{n+3} = 30800 : 1$

Solution

i) ${}^{n}p_5 : {}^{n}p_3 = 2 : 1 \ \Rightarrow {}^{n}p_5 \,/\, {}^{n}p_3 = \frac{2}{1} \ \ or \ \frac{\frac{n!}{(n-5)!}}{\frac{n!}{(n-3)!}} = \frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = 2$

$\Rightarrow \frac{(n-3)!}{(n-5)!} = 2 \ \ or \ \ \frac{(n-3)(n-4)(n-5)!}{(n-5)!} \ \therefore \ (n-3)(n-4) = 2$

$\Rightarrow n^2 - 7n + 10 = 0 \ \therefore n = 2 \ or \ 5$

$Since\ n > r, then\ required\, n = 5\ (Reject\ n = 2)$

ii) ${}^{2n+1}p_{n-1} : {}^{2n-1}p_n = 3 : 5 \ \Rightarrow {}^{2n+1}p_{n-1} \,/\, {}^{2n-1}p_n = 3\,/\,5$

$\therefore \frac{(2n+1)!}{[(2n+1)-(n-1)]!} \div \frac{(2n-1)!}{[(2n-1)-n]!} = \frac{3}{5}$

$\frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \Rightarrow \frac{(2n+1)!}{(2n-1)!} \times \frac{(n-1)!}{(n+2)!}$

$\frac{(2n+1)(2n+1-1)(2n+1-2)!}{(2n-1)!} \times \frac{(n-1)!}{(n+2)(n+2-1)(n+2-2)(n+2-3)!} = \frac{3}{5}$

$\frac{(2n+1)(2n)(2n-1)}{(2n-1)!} \times \frac{(n-1)!}{(n+2)(n+1).n.(n-1)!} = \frac{3}{5}$

$\Rightarrow \frac{2n(2n+1)}{n(n+1)(n+2)} = \frac{3}{5}$ or $\frac{2(2n+1)}{(n+1)(n+2)} = \frac{3}{5}$

$\therefore 10(2n+1) = 3(n^2+3n+2)$ or $3n^2 - 11n - 4 = 0$

$\Rightarrow (n-4)(3n+1) = 0 \ \therefore n = 1, -\frac{1}{3}$ $Required\, n = 1$

$and\ non\ integer\, n = -\frac{1}{3}\ rejected$

iii) ${}^{56}p_{n+6} : {}^{54}p_{n+3} = 30800 : 1$

$^{56}p_{n+6}\ /\ ^{54}p_{n+3} = 30800/1 \Rightarrow \frac{56!}{[56-(n+6)]!} \div \frac{54!}{[54-(n+3)]!} = \frac{30800}{1}$

$\Rightarrow \frac{56!}{[50-n]!} \div \frac{54!}{[51-n]!} = \frac{30800}{1} \ \Rightarrow \frac{56!}{54!} \times \frac{[51-n]!}{[50-n]!} = 30800$

$\Rightarrow \frac{56\times55\times54!}{54!} \times \frac{(51-n)(51-n-1)!}{(50-n)!} = 30800$

$\therefore 3080(51-n) = 30800$ or $51 - n = 10 \ \therefore n = 41$

Task

1) *If* ${}^{10}p_{n-1} : {}^{11}p_{n-2} = 30 : 11$, *find n*

   ANSWER: $\boldsymbol{n = 7}$

2) *Find n if*

   a) $5 \times {}^{56}p_{n+6} = 12 \times {}^{n-1}p_3$

   b) ${}^{n+1}p_5 : {}^{n-1}p_5 = 9 : 2$

   c) ${}^{2n-1}p_n : {}^{2n+1}p_{n-1} = 5 : 3$

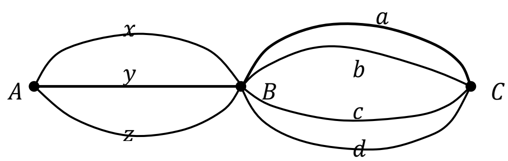
THE FUNDAMENTAL PRINCIPLE OF COUNTING

Consider the figure below

A, B *and* C are 3-stations.

x , y *and* z are 3-routes from A to B

a , b , c *and* d are 4-routes from B to C



There are 3-independent ways of going from A to B, and corresponding to each way of going from A to B, there are 4-ways of going from B to C. Hence there $3 \times 4 = 12$ *ways* of going from A to C.

i.e. xa , xb , xc , xd , ya, yb , yc , yd , za , zb , zc , zd

With this background the fundamental rules are;

1. **Multiplication principle**

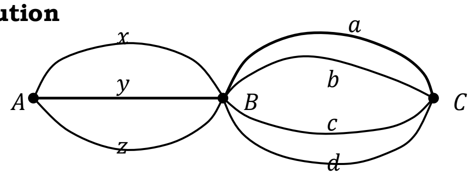
   If one operation can be performed independently in **m** different ways, and another operation can be performed independently in **n** different ways, the total number of ways in which both operations can be performed simultaneously is $(\boldsymbol{m} \times \boldsymbol{n})$ ***ways***

2. **Addition principle**
   If one operation can be performed in **m** ways, and another operation can be performed in **n** ways, and only one operation can be done at a time, then either of the two operations can be performed in $(\boldsymbol{m}+\boldsymbol{n})$ ***ways***

Examples

1. *There are four bus lines between A and B and three bus lines between B and C.*
   a) *In how many ways can a man travel by bus from A to C by way of B?*
   b) *In how many ways can a man travel round trip by bus from A to C by way of B, if he doe not want to use a bus line more than once?*

Solution



a) $4 \times 3 = 12\,ways$
b) Travelling from A to C, $no.of\ ways = 4 \times 3 = 12\,ways$. Since the man doe not want to use the bus line more than once, the number of bus lines from C to B are 2 and from B to A are 3.
$\therefore\ no.$ of ways in which he can travel from C to A by way of B is $2 \times 3 = 6\,ways$
$\therefore$ the total umber of ways in which a man can travel round trip by bus from A to C by way of B not using bus line more than once is $12 \times 6 = 72\,way$

General examples on permutations

1) *Find the number of words that can be formed by considering all possible permutations of the letters of the word "FATHER". How many of these words begin with A and end with R?*
Solution
Since all the letters are different,
$n = 6\,, r = 6\ (Taking\ all\ the\ letters\ for\ a\ word)$
$\Rightarrow {}^{n}p_r = {}^{6}p_6 = 6! = \frac{6!}{0!} = 720\,ways$
Since A and R are fixed, permutations are to be done on FTHE
$\therefore n = 4\,, r = 4\ \Rightarrow {}^{4}p_4 = 24\,ways$

2) *How many different words containing all the letters of the word TRIANGLE can be formed?*
How many of them
   a) Begin with T?
   b) Begin with E?
   c) Begin with T and end with E?

Solution
All the letters are different
$n = 8, r = 8\ \ \therefore {}^{8}p_8 = 8!$
   a) Since T is fixed for all words, permutations are to be made on R ,I ,A,N,G,L,E

$\therefore n = 7\,, r = 7 \Rightarrow {}^7p_7 = 7!$

b) Since E is now fixed, permutations are to be made on the remaining letters

$\therefore n = 7\,, r = 7 \Rightarrow {}^7p_7 = 7!$

c) Now T and E are fixed, hence permutations are to be made on the remaining 6-letters

$\therefore n = 6\,, r = 6 \Rightarrow {}^6p_6 = 6!$

3) i) *In how many ways can the letters of the word "MOBILE" be arranged so that the consonants always occupy the odd places?*

ii) *How many words can be formed out of the letters "ARTICLE" so that the vowels occupy the even places?*

Solution

i) There are 6-letters of which 3 (M, B, L) are consonants and 3 (I, O, E) are vowels.

The positions to be filled up with consonants are as below

1	2	3	4	5	6
(M)	–	(B)	–	(L)	–

The total no. of ways consonants can be filled $= {}^3p_3 = 3! = 6\,ways$

The total no. of ways 3-vowels can be arranged $= {}^3p_3 = 3! = 6\,ways$

By fundamental principle,

Total arrangements $= 6 \times 6 = 36\,ways$

ii) There are 7-letters of which 3 (I, O, E) are vowels

1	2	3	4	5	6	7
–	(A)	–	(I)	–	(E)	–

Arrangements of vowels$= {}^3p_3 = 3! = 6\,ways$

Remaining 4-letters can be arranged in 4 places in ${}^4p_4 = 4!\,ways$

By fundamental theorem, total no. of arrangements$= 3! \times 4! = 144\,ways$

4) *In how many ways can the letters of the word "STRANGE" be arranged so that the vowels may appear in the odd places?*

Solution

There are 7 letters with 2 (A, E) vowels.

1	2	3	4	5	6	7
(A)	–	(E)		–		–

The 2 vowels occupy only in the 2 out of 4 positions, in which this can be done in

$${}^4p_2 = \frac{4!}{2!} = 12\,ways$$

The rest of the letters occupy empty spaces. This can be done in

$${}^5p_5 = 5! = 120\,ways$$

$$\therefore total\ no. of\ arrangements\,are\ 120 \times 12 = 1440\,ways$$

5) *How many words can be formed from the letters of the word "FATEPUR" when*

i. The 3-letters PUR occur together?

ii. Vowels occur at even places?

Solution

i. There are 8 letters. Since the 3 (PUR) letters occur together, we may regard them as forming one letter, so there are now 6 letters for permutation.
⇒ The 6 letters can be arranged in ${}^6p_6 = 6!\ ways$
⇒ The 3 (PUR) letters can be arranged in ${}^3p_3 = 3!\ ways$
By fundamental principle, total arrangement $= 6! \times 3!\, ways$

ii.

1	2	3	4	5	6	7	8
$-$	(A)	$-$	(E)	$-$	(U)	$-$	$(-)$

3-vowels occupy 3 out of 4 positions in
${}^4p_3 = \frac{4!}{1!} = 24\, ways$
Remaining consonants occupy 5-empty spaces (1 , 3 , 5 , 7 , 8) in
${}^5p_5 = 5! = 120\, ways$
$\therefore total\ no. of\ arrangements = 120 \times 24 = 2880\, ways$

6) *How many different words containing all the letters of the word "TRIANGLE" can be formed? How many of them*
   i) *When consonants are never together?*
   ii) *When no two vowels are never together?*
   iii) *When consonants and vowels are both always together?*
   iv) *Vowels occupy odd places?*
   v) *Relative positions of vowels and consonants remain unaltered?*

Solution

TRIANGLE contain 8- different letters, which can be arranged in ${}^8p_8 = 8!\, ways$
Consonants are (T , R , N, G , L)
Vowels are $(I , A,\ E)$

i) When consonants are never together
$$= \begin{pmatrix} Total\ arrangements \\ of\ 8 - letters \end{pmatrix} - \begin{pmatrix} consonants\ when \\ altogether \end{pmatrix}$$
Permutations for consonants when together $= {}^5p_5 = 5!$
Permutations for the word with consonants together $.i.e. (TRNGL), I, A, E\ .i.e.$ 4-letters can be arranged in ${}^4p_4 = 4!\, ways$
⇒ The total arrangements when consonants are together $= 5! \times 4!$
∴ When consonants are never together $= 8! - 4! \times 5!\, ways$ can arise.

ii) When no two vowels are together = 8! − permutation for vowels when together. Different arrangement for 2 vowels out of 3
$$= {}^3p_2 = \frac{3!}{1!} = 3!\, ways$$
Arrangement for vowels (two) occurring together $= {}^2p_2 = 2!\, ways\ arise$
Arrangement for the remaining consonants $= {}^5p_5 = 5!\, ways\ arise$
$\therefore Total\ arrangement\ for\ vowels = 2! \times 3! \times 5!$
∴When no two vowels occur together $= (8! - 2! \times 3! \times 5!)\ ways\ arise$

iii) When consonants and vowels are both always together $.i.e. (TRNGL), (IAE)$ give two letters that can be arranged in
${}^2p_2 = 2!\, ways$
Consonants are arranged in ${}^5p_5 = 5!$
Vowels are arranged in ${}^3p_3 = 3!$

∴ Total arrangements= $2! \times 3! \times 5!$

iv) Vowels occupy odd places

1	2	3	4	5	6	7	8
(A)	–	(I)	–	(E)	–	( )	–

Vowels occupy 3 out of 4 positions in ${}^4p_3 = 4!\,ways$

Remaining consonants occupy 5 places in ${}^5p_5 = 5!\,ways$

∴ Total arrangements $4! \times 5!$

v) T R I A N G L E

⇒ For consonants, ${}^5p_5 = 5!\,ways$ occur

⇒ For vowels, ${}^3p_3 = 3!\,ways$ occur

∴ Relative positions of vowels and consonants remain unaltered in the word is $5! \times 3!\,ways$

7) *How many permutations can be made out of the letters of the word "MISSISSIPPI" taken altogether?*

Solution

MISSISSIPPI contains repeated letters $.i.e.$ "S" occurs 4-times, "I" occurs 4-times and "P" occurs 2-times.

Using permutation of objects not all different $.i.e.\ \frac{\boldsymbol{n}!}{\boldsymbol{p}!\boldsymbol{q}!\boldsymbol{r}!}$,⇒ number of permutations for 11-letters will be= $\frac{11!}{4!4!2!}$

8) *In how many ways can the letters of the word "CONSTITUTION" be arranged? How many of these will have the letter "N" both at the beginning and at the end?*

Solution

Word "CONSTITUTION" consists of 12-letters having O, N, I two times each, T- 3 times and
U, C, S once each.

Required number of different ways is $= \frac{12!}{2!2!2!!3!} = 9979200\,ways$

For having both N at the beginning and at the end, then N is fixed at the beginning and at the end, there by leaving 10-letters having the above characteristics.

∴ Required number of ways is $= \frac{10!}{2!2!3!} = 151200$

9) *In how many ways can the letters of the word "ARRANGE" be arranged? How many of these arrangements are there in which*

a) The two R's come together?
b) The two R's do not come together?
c) The two R's and the two A's come together?

Solution

ARRANGE has 7-letters out of which 2 are A, 2 are R, 1 is N, 1 is G and 1 is E

∴ Required no. of ways $= \frac{7!}{2!2!} = 1260\,ways$

a) Two R's come together give the word $A\ (RR)\ A\ N\ \ G\ E$ consisting of now 6-letters.

Now word has 2 A's, N, G, E are all one.

$\Rightarrow Required\ arrangements = \frac{6!}{2!} = 360\ ways$

$\left[For\ 2R's, arrangement = \frac{2!}{2!} = 1\right]$

b) Required no. of arrangements = $1260 - 360 = 900\ ways$

c) Treating both R and A as one individual letter the word becomes $(AA)(RR)NGE$ consisting of 5-letters that are different

$\therefore No. of\ ways = 5! - 120\ ways$

10) a) H*ow many different arrangements can be made by using all the letters in the word "MATHEMATICS"? How many of them begin with*

i) *C?* ii) *T?*

b) How many permutations can be made out of the letters of the word "INDEPENDENCE"? How many of them have vowels occurring together?

Solution

a) MATHEMATICS has 11-letters of which 'M' occurs twice, 'A' occurs twice and 'T' also occurs twice. Remaining letters $.i.e. 11 - 6 = 5\ letters\ H, E, I, C, S$ are all different from one another.

∴ No. of permutations of these 11- letters taken all at a time is

$\frac{11!}{2!2!2!} = 4989600\ ways$

i) Fixing C in first position, then the permutation shall be done on the remaining letters. $i.e. 10\ letters$ with the above stated characteristics

No. of ways $= \frac{10!}{2!2!2!} = 453\ 600\ ways$

ii) Fixing T in first position, permutation is to be done on the remaining letters. $i.e. T, M, A, T, H, E\ M, A, I, C, S$. $i.e.$ in these 10-letters, M and A both occur twice, T-no longer occurs twice for one of them occupied the first position.

∴ No. of ways $= \frac{10!}{2!2!} = 907\ 200\ ways$

b) INDEPENDENCE has 12-letters in which 'N' occurs thrice, 'D' occurs twice, 'E' occurs 4-times and the remaining 3-letters I, P, C are all different from each other.

∴ No. of permutations of 12-letters taking all at a time= $\frac{8!}{3!2!4!}$

INDEPENDENCE, consists of 5-vowels $i.e. I, E, E, E, E$ $(4 - letters\ are\ similar)$

Taking the 5-vowels as one letter, then $(IEEEE)NDPNDNC$ is the word of 8-letters consisting of 3 N's and 2 D's

$\therefore Arrangement = \frac{8!}{3!2!}$

Permutation of the tied/bracketed vowels= $\frac{5!}{4!}$

∴ Total no. of arrangements $= \frac{8!}{3!2!} \times \frac{5!}{4!}$

11) a) How *many numbers greater than 1 million can be formed with digits 4, 5, 5, 0, 5, 4, 3?*

Solution

There are 7 digits in all, of which there are 2-fours, 3-fives and the rest are different.

∴ Total no. of arrangements $= \frac{7!}{2!3!} = 420$

Of these numbers, some begin with 0 and are less than 1-million, so they must be rejected.

Fixing 0 at the beginning, possible no. of permutations are= $\frac{6!}{2!3!} = 60$
∴ Required number = 420 − 60 = 360

b) *(i) Find all the six digit numbers formed from the digits 0, 1, 2, 3, 4, 5.*
(ii) How many numbers of 4-digits can be formed from out of the digits 3, 4, 5, 6, 7, 8, 0 if no digit is to be repeated?
Solution
i) The digits can be arranged in ${}^6p_6 = 6!\, ways$
Out of these 6! Ways, there are numbers that begin with "0"
∴ Number of such numbers=, ${}^5p_5 = 5!$
6-digit number = 6! − 5! = 600
ii) From the 7-digits, 4-digit number is to be formed.
Number of ways for this, ${}^7p_4 = \frac{7!}{3!}$
Out of these numbers, there are numbers starting with 0-digit hence reducing to 3- digit number.
Number of ways for this = ${}^6p_3 = \frac{6!}{3!}$
Required numbers = $\frac{7!}{3!} - \frac{6!}{3!} = 720$

c) *How many odd numbers of 5-digits can be formed with digits 3, 2, 7, 4, 0 when no digit is repeated?*
Solution
Odd numbers have 3 or 7 digit in the last place.
Odd numbers starting with 0 and have 3 or 7 in the last place exist.
Let us find all 5-digit odd numbers with 3 in last place $= {}^4p_4 = 4! = 24\, ways$
Now all 5-digit odd numbers with 7 in last place $= {}^4p_4 = 4! = 24\, ways$
Now finding all odd numbers with 0 in first place but ending with 3 $.i.e. 4 - digit\ number$
$= {}^3p_3 = 3! = 6\, ways$
Also finding numbers with 0 in first place and 7 in the last place. $i.e. 4 - digit\ number$
$= {}^3p_3 = 3! = 6\, ways$
Now total number of odd numbers with 0 in first place $= 6 + 6 = 12\, numbers$
Total number of odd numbers = 24 + 24 = 48
∴ Number of odd numbers with 5-digits = 48 − 12 = 36

d) (i) *How many numbers of 4-digits can be formed with digits 3, 5, 7, 8, 9, o digit being repeated? How many of these will be greater 8000?*
ii) *A number of 4-different digits is to formed using digits 1, 2, 3, 4, 5, 6, 7in all possible ways. Find how many of such numbers can be formed. Hence how many such numbers are greater than 3400?*
Solution
i) 4-digit numbers $= {}^5p_4 = 5!$
For 4-digit number to be greater than 8000, 8 or 9 must occupy the first place.
Permutations of the remaining digits $.i.e$ 3out of 4 after the first being occupied by 8 or 9

$={}^4p_3 = 4!\ ways$

Since numbers are of 2 types $.i.e\ 8\ or\ 9$ starting digits, then

Total number of numbers= $4! \times 2 = 48$

Alternatively

Number greater than 8000= $5! - (Numbers\ less\ than\ 8000)$

Finding $4 - digit\ numbers < 8000$

Numbers $<$ 8000 begin with 3, 5 or 7

Since first place is being occupied in 3-ways, then

$4 - digit\ numbers < 8000 = 3 \times$ Permutation of the 3-digits after the first one being occupied by 3 or 5 or 7

$\therefore Numbers < 8000 = 3 \times {}^4p_3 = 3 \times 4!$

Required $numbers > 8000 = 5! - 3 \times 4! = 48$

ii) 4-digit numbers formed out of 7-digits in ${}^7p_4 = \frac{7!}{3!}\ ways$

For a number to be greater than 3400,

Case I

The first digit on LHS must be 4, 5, 6, 7 and the 3-digis occupied in ${}^6p_3 = \frac{6!}{3!}$

The first place in 4-ways

Case II

The first digit on LHS is 3, the second digit must be 5, 6, 7 and the other two places are occupied by any of the two remaining 5-digits

Case III

The first digit on LHS is occupied by 3, second digit by 4 and the other two places are occupied by two of the remaining digits

For case I: Total arrangement= $4 \times {}^6p_3$

For case II: Total arrangement = $3 \times {}^5p_2$

For case III: Total arrangement $={}^5p_2$

Total $= 4 \times {}^6p_3 + 3 \times {}^5p_2 + {}^5p_2 = 560\ ways$

e) *How many numbers greater than 7000 can be formed with digits 3, 5, 7, 8, 9 no digit being repeated?*

Solution

Cases in the question

Case I: All numbers with 5-digits are $>$ 7000

This is done in ${}^5p_5 = 5!\ ways$

Case II: There will be 4-digit numbers which are> 7000. These start with 8, 9 or 7 but never with 3 or 5

A. 4-digit number $>$ 7000, starting with 8 or 9 are:

$$2 \times {}^4p_3 \begin{bmatrix} \because the\ remaining\ 3 - digits \\ are\ permutated\ out\ of\ 4 \end{bmatrix}$$

B. If the number is to start with 7, the 3-digits must be filled up from the 4-remaninig digits in ${}^4p_3 = 4!\ ways$

$\therefore Total\ arrangement = {}^4p_3 + 2 \times {}^4p_3 + 5! = 192$

Task

1. **Uneb 2004**

   How many arrangements can be made from the letters of the name "MISSISSIPPI"

   a. *When all letters are taken at a time?*
   b. *If the two letters P begin every word?*

   ANSWER: ***a.*** $\frac{11!}{4!4!2!}$ ***b.*** $\frac{9!}{4!4!}\left[\begin{matrix} HINT: MISSISSI(PP)I\ has\ 10\ letters\ of\ which\ 9 \\ are\ to\ be\ permutated \end{matrix}\right]$

2. *How many different words can be formed with the letter of the word "CAPTAIN"? In how many of these C and T are never together?*

   ANSWER: $\frac{7!}{2!}$**,** *taking C and T as one unit*$\Rightarrow \frac{6!}{2!}$, (CT) *occur in* ${}^2p_2 = 2!\ \therefore total =$ $\frac{6!}{3!} \times 2!$

3. *In how many ways the letters of the word "PETROL" can be arranged? How many of these do not begin with P? How many begin with P but do not end with L? Also find the number of words which can be formed if O and L are never together?*

   ANSWER: ***i)*** *720* ***ii)*** $720 - 120$ ***iii)*** $120 - 24$ ***iv)*** $720 - 2 \times 120$

4. *How many permutations can be made out of letters all taken together*

   a) *EXAMINATION*
   b) *PERMUTATION*
   c) *ACCOUNTANT*
   d) *MATAAMA*

   ANSWER: ***a)*** *4 989 600* ***b)*** $\frac{11!}{2!}$ ***c)*** $\frac{10!}{2!2!2!}$ ***d)*** $\frac{7!}{2!4!}$

Circular permutation

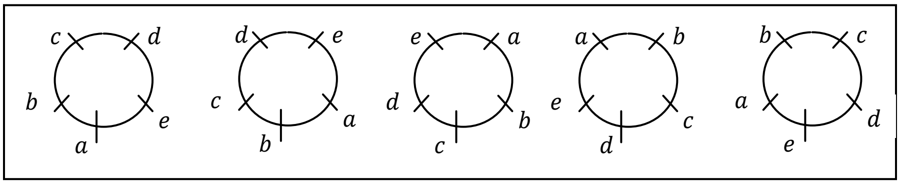
1. *In how many ways can 5 persons be seated on a round table conference?*

Solution

Let the 5 persons be denoted by letters a, b, c, d and e

NOTE

If these persons are to be seated in a *row* the five linear arrangements are; $abcde, bcdea, cdeab, deabc, eabcd$, are all different from one another. But if are to be seated at a round table in the same sense as in the figure below,



All the arrangements are the same, for starting with different letters and reading them in clockwise direction.$i.e.$ $abcde, bcdea, cdeab, deabc, eabcd$, which are the same as the linear arrangements.

$\therefore$ A single circular arrangement of the n letters is equivalent to n different linear arrangements.

$$\text{i.e. } \mathbf{n \times a\ circular\ arrangement} = \mathbf{n!} \times (\mathbf{linear\ arrangement\ of\ n - letters})$$

$$\Rightarrow \mathbf{No.\ of\ circular\ arrangements} = (\mathbf{n-1})!$$

Let the circular arrangement $= x$

$\Rightarrow 5 \times x = 5!$ $(linear\ arrangements\ of\ 5 - letters)$

$\therefore x = \frac{1}{5} \times 5! = 4!$

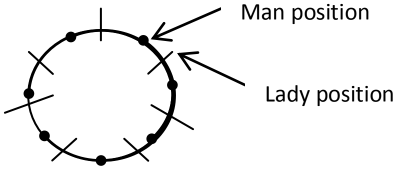
$\therefore No.\ of\ circular\ arrangements\ of\ 5 - persons = (5-1)! = 4!$

2. a) *In how many ways can 7 gentlemen and 7 ladies sit at a round table so that no two ladies are together?*

b) In how many ways can 5 gentlemen and 3 ladies sit at a round table so that no two ladies are together?

Solution

a)

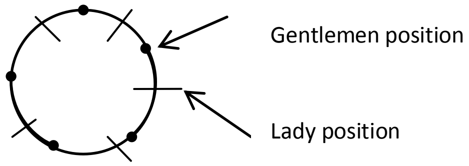


No. of ways in which 7 gentlemen can sit on round table$= (7-1)! = 6!$

Since no two ladies can be sitting together, ladies can be put in between the men .$i.e.$ there are exactly 7 places, 7 ladies can use. This can be done in 7! Ways.

$\therefore Total\ no.of\ ways = 6! \times 7! = 3\ 628\ 800$

b)



5 gentlemen sit on a round table in $(5-1)! = 4!\ ways$

There are 5 positions 3 ladies can occupy in ${}^5p_3 = \frac{5!}{2!}\ ways$

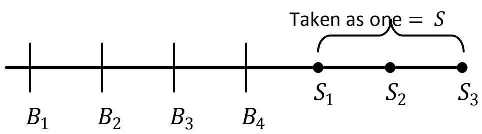
$\therefore Total\ nu.of\ ways\ required = 4! \times \frac{5!}{2!} = 1440\ ways$

Task

1. *In how many ways can 7 people be arranged at a round table so that two particular persons be together?*
2. *A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if*
   a) *All sisters sit together?*
   b) *No two sisters sit together?*

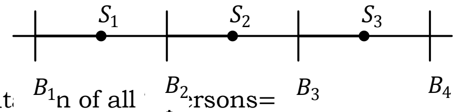
HINT:

a) $Total\ no.of\ arrangements\ in\ which\ all\ sisters\ sit\ together = 5! \times 3! = 720$



$\therefore$ Permutation of $B_1B_2B_3B_4S = 5!$
Now permutation of $S_1S_2S_3 = 3!$
Total permutation= $5! \times 3!$

ii)



Permut n of all rsons=
Permutations of sisters seated together
$= 5! \times 3!$
$\therefore$ Total= $7! - 5! \times 3! = 4320$

3. *There are 6 books of Economics, 3 of Math and 2 of Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?*
HINT: 6E's, 3M's and 2A's are different$\Rightarrow 6! \times 3! \times 2!$

4. *A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of 3 books and a single copy of 8 books. In how many ways can all the books be arranged on the shelf?*
HINT: total number of books of same kind= $1 \times 5 + 4 \times 2 + 6 \times 3 + 8 =$
$39\ books$ of which same types are 5, 4, 4, 6, 6, and 6
$\therefore$ Total number of arrangements= $\frac{39!}{5!4!4!6!6!6!} = \frac{39!}{5!(4!)^2(6!)^3}$

COMBINATIONS
Definition:
The different groups/collections/selections that can be formed out of a given set of objects by taking some or all of them at a time (without regard to the order of arrangements) are called
Combinations
NOTATION
The number of combinations of n-different objects taken r-at a time is denoted as

$$\boldsymbol{{}^nC_r = \frac{{}^np_r}{r!} = \frac{n!}{r!(n-r)}}$$, called *combinatorial coefficient*

IMPORTANT DEFINATIONS IN COMBINATIONS
a) **Complementary combinations**
i) $\boxed{\boldsymbol{{}^nC_r = {}^nC_{n-r}}}$ $.i.e.$The number of combinations of n different objects taken r at a time is equal to the number of combinations of n different objects taken $(n-r)$ at a time.
Proof
$$\boldsymbol{{}^n}C_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-n+r]} = \frac{n!}{(n-r)![n-(n-r)]} = \boldsymbol{{}^n}C_{n-r}$$

ii) $\boxed{\boldsymbol{{}^nC_n = {}^nC_0 = 1}}$

Proof

${}^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$

Also ${}^nC_0 = \frac{n!}{n!(n-0)!} = \frac{n!}{n!0!} = 1$

iii) $\boxed{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1} \quad \textbf{OR} \quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$

i.e. The number of combination of n objects taken r at a time plus the number of combinations of n taken (r+1) or (r-1) in the above formulae respectively at a time is equal to the number of combinations (n+1) objects taken (r+1) or respectively at a time.

Proof

${}^nC_n + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

${}^nC_n + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-r-1]!}$

$= \frac{n!}{(n-r-1)!r!}\left[\frac{1}{n-r} + \frac{1}{r+1}\right]$ *taking out common factor*

$= \frac{n!}{(n-r-1)!r!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right]$ *adding two fractions* $= \frac{n!}{(n-r-1)!r!}\cdot\frac{(n+1)}{(n-r)(r+1)}$

$= \frac{(n+1)!}{(n-r)!(r+1)!}\left[\begin{array}{c} \because\ (n-r-1)!\,(n-r) = (n-r)! \\ r!\,(r+1) = (r+1)! \\ n!\,(n+1) = (n+1)! \end{array}\right]$

$= \frac{(n+1)!}{[(n+1)-(r+1)]!(r+1)!}$ $[\because (n+1)-(r+1) = n-r]$

$= {}^{n+1}C_{r+1}$

b) **Restricted combinations**

i. The number of combinations on n objects taken r at a time in which p particular objects are always included is

$$\boxed{{}^{n-p}C_{r-p}}$$

ii. The number of combinations of n objects taken r at a time in which p particular objects never occur is $\boxed{{}^{n-p}C_r}$

c) **Total number of combinations of n-objects all different**

The total number of combinations of n-different objects taken some or all at a time is

$$\boxed{{}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n - 1}$$

d) **Group division**

The number of ways in which $(m+n)$objects m and n being un equal, can be divided in to two groups containing m and n objects respectively is

$$\boxed{\frac{(m+n)!}{m!n!}}$$

Example

1. *Evaluate*

   i) $^6c_3$ ii) $^7c_2$ iii) $^4c_4$ iv) $^5c_1$

   Solution

   i) $^6c_3 = \frac{6!}{3!3!} = 20$ iii) $^4c_4 = \frac{4!}{4!0!} = 1$

   ii) $^7c_2 = \frac{7!}{5!2!} = 21$ iv) $^5c_1 = \frac{5!}{4!1!} = 5$

2. *Find n if:*

   a) $^nc_2 = 3$ b) $^{2n}c_3 : {}^nc_2 = 44:3$ c) $^{28}c_{2n} : {}^{24}c_{2n-4} = 225:11$

   Solution

   a) $^nc_2 = \frac{n!}{(n-2)!2!} = 3 \Rightarrow \frac{n(n-1)(n-2)!}{(n-2)!2!} = 3$

   $\Rightarrow \frac{n(n-1)}{2!} = 3 \; or \; n^2 - n - 6 = 0 \;\therefore n = 3, -2$

   Taking $n = 3$; $n = -2$ *is rejected*

   b) $^{2n}c_3 : {}^nc_2 = \frac{(2n)!}{(2n-3)!3!} \div \frac{n!}{(n-2)!2!} = \frac{44}{3} \Rightarrow \frac{(2n)!}{(2n-3)!3!} \times \frac{(n-2)!2!}{n!} = \frac{44}{3}$

   $\Rightarrow \frac{2n(2n-1)(2n-2)}{3!} \times \frac{(n-2)!2!}{n(n-1)(n-2)!} = \frac{44}{3}$

   $\Rightarrow \frac{2n(2n-1)(2n-2)(2n-3)}{6} \times \frac{2}{n(n-1)} = \frac{44}{3} \; or \; \frac{4(2n-1)}{3} = \frac{44}{3}$

   $\therefore 2n - 1 = 11 \; or \; n = 6$

3. *Find n if* $^np_4 = 30.{}^nc_5$

   Solution

   $\frac{n!}{(n-4)!} = 30 \times \frac{n!}{(n-5)!5!} \Rightarrow \frac{1}{(n-4)(n-5)!} = \frac{30}{(n-5)!5!}$

   $\Rightarrow \frac{1}{n-4} = \frac{1}{5!} = \frac{1}{4} \;\therefore n - 4 = 4 \; or \; n = 8$

4. *There are 12 men and 9 women. In how man ways can they stand in a row so that no two women are together?*

   Solution

   Let the 12 men (denoted by M's) be put in a row as shown below.

   × M × M × M × M × M × M × M × M × M × M × M × M ×

   In order that no two women may be put together the women can only be put in places marked as ×, and the men are 12, the number of such places is 13.

   ∴ In order to place the 9 women we must choose any 9 out of 13 places. *i.e.*

   $^{13}c_9$ or $^{13}c_4 = \frac{13!}{9!4!} = 715 \; ways$

5. *In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen?*

   Solution

   3 ladies can be selected from 8 in $^8c_3$ ways.

   4 gentlemen can be selected from 7 in $^7c_4$ ways

   ∴ The number of ways in which a committee can be selected is

   $^8c_3 \times {}^7c_4 = 1\,960 \; ways$

6. *A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed?* (***uneb* 2011**)

Solution
4 men can be selected from 10 men in ${}^{10}c_4$
3 women are selected from 8 women in ${}^{8}c_3$
Total number of ways of selecting committee $={}^{10}c_4 \times {}^{8}c_3 = 11\,760\ ways$

7. *In how many ways can a committee of 6 men and 2 women be formed out of 10 men and 5 women?*
Solution
6 men selected from 10 men in ${}^{10}c_6$ ways
2 women selected from 5 women in ${}^{5}c_2$
Total number of ways of selecting a committee $={}^{10}c_6 \times {}^{5}c_2 = 2\,100$

8. *A group of nine has to be selected from 10 boys and 8 girls. It can consist of either 5 boys and 4 girls or 4 boys and 5 girls. How many different groups can be chosen?*($\boldsymbol{Uneb}\ \mathbf{2006}$)
Solution
Case I
5 boys selected from 10 boys in ${}^{10}c_5$
4 girls selected from 8 girls in ${}^{8}c_4$
Total number of selection $= {}^{10}c_5 \times {}^{8}c_4$
Case II
4 boys selected from 10 boys in ${}^{10}c_4$
5 girls selected from 8 girls in ${}^{8}c_5$
Total number of selection $= {}^{10}c_4 \times {}^{8}c_5$
∴ Overall total number of selection $={}^{10}c_5 \times {}^{8}c_4 + {}^{10}c_4 \times {}^{8}c_5 = 29\,400$

9. *In a multinational company, 3 branches in a particular country are managed by 4, 5 and 8 persons. In how many ways can 20 persons be allotted to the different branches?*

Solution
Let the branches be A, B and C
For branch A
Let the 1st 4 persons be selected out 20 persons.
Number of ways $={}^{20}c_4$
Having selected 4 persons for branch A, 5 persons can be selected out of the remaining 16 for **branch B** in ${}^{16}c_5 ways$
After this 8 persons for **branch C** out of the remaining 11 in ${}^{11}c_8\ ways$
By fundamental theorem, selection of persons can be made in
${}^{20}c_4 \times {}^{16}c_5 \times {}^{11}c_8\ ways$

10. *At an election 3 wards of a town are canvassed by 4, 5 and 8 respectively. If 20 men volunteer, in how many ways can they be allotted to the different wards?*
Solution
Let 3 wards be A, B and C
Let A be canvassed by the 1st 4 out of 20 in ${}^{20}c_4\ ways$
B will be canvassed by 5 out of remaining 16 in ${}^{16}c_5\ ways$

C will be canvassed by 8 out of the remaining 11 in ${}^{11}c_8$ *ways*

$\therefore$ Total number of ways $= {}^{20}c_4 \times {}^{16}c_5 \times {}^{11}c_8$

11. *From 6 boys and 4 girls, 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls?*

Solution

Type of selection	**Boys (6)**	**Girls (4)**
A	3	2

$\Rightarrow$ A can be done in ${}^{6}c_3 \times {}^{4}c_2 = 120\, ways$

12. *From 5 gentlemen and 6 ladies, a committee of 5 is to be formed. In how many ways can this be done if*
    a) The committee is to include at least a lady
    b) There is no restriction about its formation
    c) Not more than 3 gentlemen?

Solution

a)

Lady (6)	**Gentlemen (5)**	**No. ways of formation**
1	4	${}^{6}c_1 \times {}^{5}c_4$
2	3	${}^{6}c_2 \times {}^{5}c_3$
3	2	${}^{6}c_3 \times {}^{5}c_2$
4	1	${}^{6}c_4 \times {}^{5}c_1$
5	0	${}^{6}c_5 \times {}^{5}c_0$

$\therefore$ Number of ways in which a committee includes at least a lady is

$$= {}^{6}c_1 \times {}^{5}c_4 + {}^{6}c_2 \times {}^{5}c_3 + {}^{6}c_3 \times {}^{5}c_2 + {}^{6}c_4 \times {}^{5}c_1 + {}^{6}c_5 \times {}^{5}c_0$$

$$= 30 + 50 + 200 + 75 + 6 = 461\, ways$$

b) When there is no restriction about formation 5 are selected out of 11 in

$${}^{11}c_5 = 231\, ways$$

c)

Gentlemen (5)	**Lady (6)**	**Type of committee**	**No. of ways**
0	5	A	${}^{5}c_0 \times {}^{6}c_5$
1	4	B	${}^{5}c_1 \times {}^{6}c_4$
2	3	C	${}^{5}c_2 \times {}^{6}c_3$
3	2	D	${}^{5}c_3 \times {}^{6}c_2$

$\therefore$ Number of ways in which a committee consisting of not more than 3 gentlemen is formed in

$${}^{5}c_0 \times {}^{6}c_5 + {}^{5}c_1 \times {}^{6}c_4 + {}^{5}c_2 \times {}^{6}c_3 + {}^{5}c_3 \times {}^{6}c_2 = 6 + 75 + 200 + 150 = 431\, ways$$

13. *The question paper on math and statistics contains 10 questions divided in to two groups of 5 questions each. In how many ways can an examinee select 6 questions taking at least 2 questions from each group?*

Solution

Group (A) [5 questions]	**Group (B) [5 questions]**	**No. of ways**
2	4	${}^{5}c_2 \times {}^{5}c_4$
3	3	${}^{5}c_3 \times {}^{5}c_3$
4	2	${}^{5}c_4 \times {}^{5}c_2$
5	1	
6	0	

Not considered since at least two questions from each group are to be considered.

∴ Total number of selection $= {}^{5}c_2 \times {}^{5}c_4 + {}^{5}c_3 \times {}^{5}c_3 + {}^{5}c_4 \times {}^{5}c_2$
$= 50 + 100 + 50 = 200\, ways$

14. *A council consists of 11 members of which 5 are permanent and the remaining are temporary. A solution is declared to be passed by the council if it receives at least 7 votes including all the permanent members. Find the number of ways in which the resolution can be passed. What will be the number of ways if;*
    a) *For passing a resolution, only a simple majority was selected*
    b) *Any 7 or more votes could secure the passing of the resolution?*

Solution

Permanent vote (5)	**Temporary vote (6)**	**No. of selection**
5	2	${}^{5}c_5 \times {}^{6}c_2 = 15$
5	3	${}^{5}c_5 \times {}^{6}c_3 = 20$
5	4	${}^{5}c_5 \times {}^{6}c_4 = 15$
5	5	${}^{5}c_5 \times {}^{6}c_5 = 6$
5	6	${}^{5}c_5 \times {}^{6}c_6 = 1$

∴ Total number of ways = 15 + 20 + 15 + 6 + 1 = 57

a) For simple majority either 6 or 7 or 8 or 9 or 10 or 11 votes out of 11 are required. This is done in
${}^{11}c_6 + {}^{11}c_7 + {}^{11}c_8 + {}^{11}c_9 + {}^{11}c_{10} + {}^{11}c_{11} = 1024\, ways$

b) Number of ways $= {}^{11}c_7 + {}^{11}c_8 + {}^{11}c_9 + {}^{11}c_{10} + {}^{11}c_{11} = 562\, ways$

15. *A council consists of 10 members, 6 belonging to the party A and 4 to the party B. in how many ways can a committee of 5 be selected so that the members of the party A are in the majority?*

Solution

A (6)	**B (4)**	**No. of ways**
5	0	${}^{6}c_5 \times {}^{4}c_0 = 6$
4	1	${}^{6}c_4 \times {}^{4}c_1 = 60$
3	2	${}^{6}c_3 \times {}^{4}c_2 = 120$
2	3	
1	4	

, cannot be taken since party (A) must be the majority.

∴ Total number of selection = 6 + 60 + 120 = 186

16. *Among 20 members of a cricket club there are 2 wicket keepers and 5 bowlers. In how many ways can 11 be chosen so as to include only one of the wicket keepers and at least 3 bowlers?*

Solution

Type	**Wicket players** $[2\, max]$	**Bowler [5max]**	**No. of players other than Wicket of Bowlers [13 max]**	**Number of ways**
A	1	3	7	${}^{2}c_1 \times {}^{5}c_3 \times {}^{13}c_7$
B	1	4	6	${}^{2}c_1 \times {}^{5}c_4 \times {}^{13}c_6$
C	1	5	5 (Exhausted)	${}^{2}c_1 \times {}^{5}c_5 \times {}^{13}c_5$

$\therefore$ Total number of ways $= {}^{2}c_1 \times {}^{5}c_3 \times {}^{13}c_7 + {}^{2}c_1 \times {}^{5}c_4 \times {}^{13}c_6 + {}^{2}c_1 \times {}^{5}c_5 \times {}^{13}c_5 = 54\,054\, ways$

16. *A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket keepers. In how many ways can a team be formed so that the team consists of at least 3 bowlers and at least one wicket keeper?*

Solution

Wicket keeper (2 max)	bowlers (4 max)	Neither wicket nor keeper (10 max)
1	3	7
2	4	5
1	4	6
2	3	6

Further two possibilities

$\therefore$ Total selection $= {}^{2}c_1 \times {}^{4}c_3 \times {}^{10}c_7 + {}^{2}c_1 \times {}^{4}c_4 \times {}^{10}c_6 + {}^{2}c_2 \times {}^{4}c_3 \times {}^{10}c_6 + {}^{2}c_2 \times {}^{4}c_4 \times {}^{10}c_5 = 2\,472\, ways$

3. a) *A bag contains a mixture of 8 Rupee, 6 Fifty paise and 4 Twenty paise coins. In how many ways selection of 3 coins can be made so that*
   i) *all the three are Rupee coins*
   ii) *one is of each denomination*
   iii) *none is a Rupee coin*

Solution

i) For all to be Rupee coins, the 3 coins are selected from 8 Rupee in

$${}^{8}c_3 \times {}^{6}c_0 \times {}^{4}c_0 = 56\, ways$$

ii) One of each denomination is to be chosen; 1 Rupee can be chosen out of 8 in ${}^{8}c_1$, 1 Fifty paise is taken out of 6 in ${}^{6}c_1$ and 1 Twenty paise in ${}^{4}c_1$ ways.

$$\therefore \text{Total number of choices} = {}^{8}c_1 \times {}^{6}c_1 \times {}^{4}c_1 = 192\, ways$$

iii) None is a Rupee coin if the 3 coins are to be selected from Fifty paise and Twenty paise $.i.e.$ a total of $6 + 4 = 10$

$$\therefore \text{Total number of ways} = {}^{10}c_3 = 120\, ways$$

b) *A box contains 7 red, 6 white and 4 blue balls. How many selection of 3 balls can be made so that*
   i) *All three are red balls*
   ii) *None is a red ball*
   iii) *There is one ball of each color?*

Solution

i) All 3 balls are red and are selected from only 7 red balls in ${}^{7}c_3 = 35\, ways$

ii) No red ball, the 3 balls are selected from the other colors $.i.e.$ out of $6 + 4 = 10$ balls

$$\therefore {}^{10}c_3 = 120\, ways\ of\ selection$$

iii) One ball of each color is selected in ${}^{7}c_1 \times {}^{6}c_1 \times {}^{4}c_2 = 168\, ways$

c) *A committee of 4 persons is to be appointed from 3 officers of the production department, 3 officers of the sales department, 2 officers of the purchase department and 1 cost accountant. In how many ways should a committee be formed so that*

i) *There must be one from each category*
ii) *It should have at least one from purchase department?*

Solution

i) Number of ways of forming a committee from each category $= {}^3c_1 \times {}^3c_1 \times {}^2c_1 \times {}^1c_1 = 18\, ways$

ii) A committee with at least 1 from purchase department
$=$ No. of ways of (1 − from purchase department +3 from other depts.) +(2 − from purchase dept.+ 2 from other depts.)
$= {}^2c_1 \times {}^7c_3 + {}^2c_2 \times {}^7c_2 = 92\, ways$

d) *An urn contains 8 white and 3 red balls. In how many ways should 2 balls be drawn so that*
i) *Both are white*
ii) *Both are red*
iii) *One is of each color?*

Solution

i) 2 white balls are drawn from 8 in ${}^8c_2 = 28\, ways$
ii) 2 red balls are drawn from 3 in ${}^3c_2 = 3\, ways$
iii) No. of ways in which 1 white and 1 red can be drawn out of 8 white and 3red $= {}^8c_1 \times {}^3c_1 = 24\, ways$

e) *A subcommittee of 16 members is to be formed out of a group of 7 men and 4 ladies. In how many ways can the subcommittee be formed in order to consist of*
i) *Exactly two ladies*
ii) *At least 3 ladies*

Solution

i) 2 ladies out of 4 and 4 men out of 7 can be selected in ${}^4c_2 \times {}^7c_4 = 210\, ways$
ii) The subcommittee will consist of at least 2 ladies if there are 2, 3 or 4 ladies in it in each case the number of men would be 4, 3 and 2 respectively.
No. of ways $= {}^4c_2 \times {}^7c_4 + {}^4c_3 \times {}^7c_3 + {}^4c_4 \times {}^7c_2 = 371\, ways$

f) *A bag contains 6 white and 9 black balls. Two drawings of 4 balls are made such that*
i) *The balls are replaced before the second trial*
ii) *The balls are not replaced before the second trial*
In how many ways should the first drawing give 4 white and the second 4 black balls in each case? What if no restriction of which is the first or second drawn, find the number of ways this can be done in each case.

Solution

i) No. of ways in which 4 white balls are drawn $= {}^6c_4$ ways
No. of ways in which 4 black balls are drawn $= {}^9c_4$ ways

ii) At the first trial number of ways of drawing 4 white $= {}^6c_4$ ways. When 4 white balls have been removed the bag contains 2 white and 9 black balls
$\therefore$ No. of ways $= {}^9c_4$ ways
If no restriction; (i) ${}^{15}c_4 \times {}^{15}c_4$ (ii) ${}^{15}c_4 \times {}^{11}c_4$

Task

1. A student has to answer 8 out of 10 questions in an examination.
   a) How many choices have he?
   b) How many if he must answer the first three questions
   c) How many if he must answer at least 4 of the first five questions?

Solution

a) 8 out of 10 can be selected in ${}^{10}c_8 = 45\, ways$

b) Since the first three questions are compulsory only 5 out of the remaining 7 questions must be selected. This can be done in ${}^{7}c_5 = 21\, ways$

c) **Case I:** 4 from the first 5 questions and 4 from the remaining 5 questions

Case II: 5 from the first 5 questions and 3 from the 3 remaining questions

$\therefore$ No. of ways $= {}^{5}c_4 \times {}^{5}c_4 + {}^{5}c_5 \times {}^{5}c_3 = 35$

2. How many combinations can be formed of 8 counters marked 1. 2, 3, 4, 5, 6, 7 , 8 taking them 4 at a time, there being at least one odd and one even counter in each combinations?

Solution

${}^{4}c_3 \times {}^{4}c_1 + {}^{4}c_2 \times {}^{4}c_2 + {}^{4}c_1 \times {}^{4}c_3 = 68$

3. Find the number of combinations of the words "ACCOMMODATION", "COLLEGE" taken 4 at a time. **ANSWER**: For ACCOMMODATION: **167** , COLLEGE, $\boldsymbol{{}^{5}c_4 + {}^{2}c_1 \times {}^{4}c_2 + {}^{2}c_2 = 18}$
4. i) ${}^{n}c_{10} = {}^{n}c_{12}$ ii) ${}^{n}c_7 = {}^{n}c_3$ iii) ${}^{n}c_5 + {}^{n}c_3 = 6$ **ANSWER**: (i) 22 (ii) 10
5. Find the value of r if
   a) ${}^{n}p_r = 30240$ and ${}^{n}c_r = 252$ **ANSWER**: r = 5
   b) ${}^{n}c_r = {}^{n}c_{2r-5}$

A committee of 4 men and 3 women is to be formed from 1o men and 8 women. In how many ways can the committee be formed? (***Uneb* 2011**)

ANSWER: $\boldsymbol{{}^{10}c_4 \times {}^{8}c_3 = 11\,760}$

POLYNOMIAL FUNCTIONS AND EQUATIONS
Polynomial functions of one variable

A Polynomial in the symbol x over the real numbers $\mathbb{R}$, is an algebraic expression that can be written:
$\boldsymbol{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + a_1 x + a_0}$**,** where n is a non-negative integer, the *coefficient* $\boldsymbol{a_i}$in$\mathbb{R}$, and $\mathbb{R}\boldsymbol{a_n} \neq \boldsymbol{0}$
The number **n** is said to be the *degree* of a polynomial and $\boldsymbol{a_n}$ is called the *leading coefficient.*
A *polynomial function* $\boldsymbol{f(x)}$ is an algebraic function that can be written as:
$\boldsymbol{f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + a_1 x + a_0}$, where x is a variable whose domain is the set of real number $\mathbb{R}$

Examples of polynomial functions

i) $f(x) = x^5 + 2x^3 + 1$, degree = 5
ii) $f(x) = 2x^3 + 5x + 1$, degree = 3

Examples of non-polynomial functions

a) $f(x) = \sqrt{x} + 2x^3 + 1$; $x^{\frac{1}{2}}$ non permissible
b) $r(x) = \frac{3}{2}ix^2 + 2$; $\frac{3}{2}i \notin \mathbb{R}$

Division of a polynomial by another polynomial

Two methods may be applied

- Long division
- Synthetic division

The polynomial $\boldsymbol{f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots + a_1 x + a_0}$ can be divided by a <u>DIVISOR</u> to either get a <u>QUOTIENT</u> only or *quotient* + <u>REMAINDER</u>. The polynomial being divided is called a <u>DIVIDEND</u>.

LONG DIVISION

Steps followed are:

1) Arrange the dividend in descending powers of the variable
2) Insert with 0 coefficients any missing terms of less than degree n
3) Divide the 1^{st} term of the divisor in to the 1^{st} term of the dividend
4) Multiply the divisor by the result from step (3) above, line up like terms
5) Subtract as arithmetic and bring down now the remaining terms
6) Repeat the process until the degree of the remainder is less than that of the divisor.

NOTE:

Division of a polynomial by a divisor will be possible if the degree of polynomial is greater than that of divisor.

Examples

Divide

a) $f(x) = 5 + 4x^3 - 3x$ by $2x - 3$
b) $f(x) = x^6 - 2x^4 + 2$ by $x^2 - x - 2$
c) $f(x) = x^5 + 2x^3 + 2$ by $x^3 + x + 1$

Solution

a)

$$
\begin{array}{r|l}
\multicolumn{1}{r}{\textit{Divisor}} & 2x^2+3x+3 \quad \textit{Quotient} \\ \hline
2x-3 & 4x^3+0x^2-3x+5 \quad \textit{Dividend} \\
 & \underline{4x^3-6x^2\qquad\qquad\quad} \\
 & \qquad 6x^2-3x+5 \\
 & \quad \underline{\;\; 6x^2-9x\qquad\quad} \\
 & \qquad\qquad\quad 6x+5 \\
 & \qquad\qquad \underline{\;\; 6x-9\;} \\
 & \qquad\qquad\qquad 14 \quad \textit{Remainder}
\end{array}
$$

$\therefore \frac{f(x)}{2x-3} = 2x^2+3x+3+\frac{14}{2x-3}$ $\; or\; f(x) = (2x^2+3x+3)(2x-3)+14$

In general, $\boxed{Dividend = (Divisor)(Quotient) + Remainder}$

NOTE:

i) Divisor 2x – 3 is *linear*

ii) Dividend $4x^3-3x+5$ is of degree n = 3

iii) Quotient $2x^2+3x+3$ is of degree n – 1 = 2

iv) Remainder 14 is a *constant*

b)

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & x^4+x^3+x^2+3x+6 \\ \hline
x^2-x-2 & x^6+0x^5-2x^4+0x^3+x^2+0x-2 \\
 & \underline{x^6-x^5-2x^4\qquad\qquad\qquad\qquad\quad} \\
 & \qquad\quad x^5+0x^4+0x^3+x^2+0x \\
 & \quad\;\; \underline{x^5-x^4-2x^3\qquad\qquad\qquad\;} \\
 & \qquad\qquad\quad x^4+2x^3+x^2+0x-2 \\
 & \qquad\quad \underline{\;\; x^4-x^3-2x^2\qquad\qquad\;} \\
 & \qquad\qquad\qquad\quad 3x^3+3x^2+0x-2 \\
 & \qquad\qquad\quad \underline{\;\; 3x^3-3x^2-6x\qquad\;} \\
 & \qquad\qquad\qquad\qquad\quad 6x^2+6x-2 \\
 & \qquad\qquad\qquad\quad \underline{\;\; 6x^2+6x-12\;\;} \\
 & \qquad\qquad\qquad\qquad\quad 12x+10
\end{array}
$$

$\therefore \frac{f(x)}{x^2-x-2} = x^4+x^3+x^2+3x+\frac{12x+10}{x^2-x-2}$

NOTE:

i) Divisor x^2-x-2 is *a Quadratic*

ii) Dividend x^6-2x^4+2 by x^2-x-2 is of degree n = 6

iii) Quotient $x^4+x^3+x^2+3x+6$ is of degree n – 2 = 4

iv) Remainder $12x+10$ is *linear*

c)

$$\begin{array}{r|l} & x^4+x^3+x^2+3x+6 \\ \hline x^3+0x^2-x-2 & x^5+0x^4+2x^3+0x^2+0x+2 \\ & \underline{x^5+0x^4+x^3+x^2} \\ & \quad x^3-x^2+0x+2 \\ & \quad \underline{x^3+0x^2+x+1} \\ & \qquad -x^2-x+1 \end{array}$$

$\therefore \frac{f(x)}{x^3+x+1} = x^2+1+\frac{-x^2-x+1}{x^3+x+1}$

NOTE:

i) Divisor x^3+x+1 is *a Cubic*

ii) Dividend $x^5+2x^3+2$is of degree n $=5$

iii) Quotient x^2+1 is of degree n $-3=2$

iv) Remainder $-x^2-x+1$ is *Quadratic*

Summary of the results:

Degree of Polynomial	Divisor	Degree of Quotient	Remainder
$\boldsymbol{n}$	$(\boldsymbol{ax+b})$ *Linear*	$n-1$	$\boldsymbol{R}$ *Constant*
	$(\boldsymbol{ax+b})^2$ *or* $(\boldsymbol{ax+b})(\boldsymbol{px+q})$ *or* $\boldsymbol{ax^2+bx+c}$ *Quadratic*	$n-2$	$\boldsymbol{Ax+B}$ *Linear*
	$(\boldsymbol{ax+b})^3$ *or*$(\boldsymbol{ax+b})(\boldsymbol{px+q})(\boldsymbol{mx+n})$ *or* $\boldsymbol{ax^3+bx^2+cx+d}$ *Cubic*	$n-3$	$\boldsymbol{Ax^2+Bx+C}$

The above table is important in finding the remainder basing on the type of the divisor. It is so much applied in remand theorem.

SYNTHETIC DIVISION

Key steps in synthetic division process

To divide $f(x) = Polynomial$ by$\boldsymbol{x-a}$

1. Arrange the coefficients of $f(x)$ in descending powers of x. write zero as the coefficient for each missing power
2. After writing the divisor in the form$(x-a)$, use $\boldsymbol{a}$ to generate the 2^{nd} and 3^{rd} row s as follows: bring down the 1^{st} coefficient of the dividend and multiplying it by$\boldsymbol{a}$**;** then add the product to the 2^{nd} coefficient of the dividend. Multiplying this sum by $\boldsymbol{a}$ and add the product to the 3^{rd} coefficient of the dividend.
   Repeat the procedure until a product is added to the constant term of the$f(x)$.
3. The last nu number to the right in the 3^{rd} row of numbers is the remainder. The other numbers in the 3^{rd} row are the coefficients of the quotient which is of degree$< f(x)$.

Examples

Use synthetic division to find the quotient and remainder resulting from dividing;

a) $f(x) = 4x^5-30x^3-50x-2$ by $x+3$

b) $f(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$

Solution

a) $Divisor = x + 3 = x - (-3) \therefore a = -3$

$Dividend = 4x^5 + 0x^4 - 30x^3 + 0x^2 - 50x - 2$

1^{st} *row*:	4	0	− 30	0	− 50	− 2
2^{nd} *row*:		−12	36	− 18	54	− 12
−3	4	− 12	6	− 18	4	− 14

$\therefore$ The quotient is: $4x^4 - 12x^3 + 6x^2 - 18x + 4$

The remainder is: -14

$\therefore \frac{f(x)}{x+3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 - \frac{14}{x+3}$

b) $Divisor = x + 2 = x - (-2) \therefore a = -2$

$Dividend = 2x^4 + 3x^3 + 0x^2 - x - 5$

	2	3	0	− 1	−	Dividend Coefficient
		−4	2	− 4	10	
−2	2	− 1	2	−		Remainder

Quotient Coefficient

$\therefore$ The quotient is: $2x^3 - x^2 + 2x - 5$

The remainder is: 5

$\therefore \frac{f(x)}{x+2} = 2x^3 - x^2 + 2x - 5 - \frac{5}{x+2}$

THE REMAINDER AND FACTOR THEOREM

Remainder Theorem

> If $\boldsymbol{f(x)}$ **is** a polynomial of degree n , $n > 0$ and if $\boldsymbol{c}$ is a number, then the remainder in division of $\boldsymbol{f(x)}$ by $(\boldsymbol{x - c})$ is $\boldsymbol{f(c)}$. In symbols,
>
> $\boldsymbol{f(x) = (x - c).Q(x) + f(c)}$, where $\boldsymbol{Q(x)}$ is a quotient of degree $\boldsymbol{n - 1}$

Proof:

Dividing $\boldsymbol{f(x)}$by $(\boldsymbol{x - c})$, you obtain a quotient and a remainder **r** (a real number)

$\Rightarrow \frac{f(x)}{(x-c)} = Q(x) + \frac{r}{(x-c)}$

$\therefore f(x) = Q(x).(x - c) + r$

This is an identity, true for all values of x.

If $x = c$, then $f(xc) = Q(c).(c - c) + r$

$\therefore \boldsymbol{r = f(c)}$

NOTE:

The theorem is stated in terms of $(\boldsymbol{x - c})$**,** the *difference* between x and the number c.

The factor Theorem

A number c is a zero of polynomial function f of degree $n, n > 0$, if and only if $x - c$ is a factor of $f(x)$.

Proof:
From the remainder theorem $f(x) = Q(x).(x - c) + r$, if $f(c) = 0$ (Remainder$f(c) = 0$), then $f(x) = (x - c).Q(x)$.
Consequently $(x - c)$ is a factor of $f(x)$.
Now from $f(x) = (x - c).Q(x)$, for $x = c$ then $f(c) = Q(c).(c - c) = 0$. Thus for a polynomial function $f(x)$, if $x - c$ is its factor, then $f(c) = 0$. $i.e.$ remainder is zero.

To test whether or not a linear polynomial $mx + b$, $m \neq 0$ is a factor of $f(x)$ you write;
$mx + b = m\left(x + \frac{b}{m}\right) = m\left[x - \left(-\frac{b}{m}\right)\right]$
, and see whether $f\left(-\frac{b}{m}\right) = 0$
By factor theorem,$mx + b$ is a factor of $f(x)$ if and only if$f\left(-\frac{b}{m}\right) = 0$

Examples

1. *Find the remainder when*$x^5 + x - 9$ *is divided by;*
   a) $x + 2$
   b) $x - 3$

Solution
Let $f(x) = x^5 + x - 9$
a) $x + 2 = x - (-2)$
$\therefore f(-2) = (-2)^5 + (-2) - 9 = -43$
b) $x - 3$
$\therefore f(3) = (3)^5 + (3) - 9 = 239$

2. *Find the remainder when* $4x^3 - 5x + 4$ *when divided by;*
   a) $2x - 1$
   b) $2x + 3$

Solution
Let $f(x) = 4x^3 - 5x + 4$
a) $2x - 1 = 2\left(x - \frac{1}{2}\right)$
$\therefore f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right) + 4 = 2$
$\therefore f\left(-\frac{3}{2}\right) = 4\left(-\frac{3}{2}\right)^3 - 5\left(-\frac{3}{2}\right) + 4 = -2$

3. *Find the value of constant a if;*
   i) $x^3 - 3x^2 + ax + 5$ has a remainder 17 when divided by $x - 3$
   ii) $x^5 + 4x^4 - 6x^2 + ax + 2$ has a remainder 6 when divided by $x + 2$
   iii) $4x^3 - ax + 5$ has a remainder 2 ½ when divided by $2x - 1$
   iv) $3x^3 + ax^2 - 6x + 3$ has a remainder 5 when divided by $3x + 1$

Solution
i) Let $f(x) = x^3 - 3x^2 + ax + 5$, Remainder= 17 when divided by $x - 3$
$\therefore$ Remainder= $f(3) = 17$
$\Rightarrow (3)^3 - 3(3)^2 + a(3) + 5 = 17$

$\therefore 3a + 5 = 17 \ \therefore a = 4$

ii) Let $f(x) = x^5 + 4x^4 - 6x^2 + ax + 2$, Remainder= 6 when divided by $x + 2$;
Divisor $= x + 2 = x - (-2)$
Remainder= $f(-2) = 6$
$\Rightarrow (-2)^5 + 4(-2)^4 - 6(-2)^2 + a(-2) + 2 = 6$
$\therefore -2a + 10 = 6 \ \therefore a = 2$

iii) Let $f(x) = 4x^3 - ax + 5$, Remainder= $2\frac{1}{2}$ when divided by $2x - 1$
Divisor $= 2x - 1 = 2\left(x - \frac{1}{2}\right)$
Remainder= $f\left(\frac{1}{2}\right) = 2\frac{1}{2}$
$\Rightarrow 4\left(\frac{1}{2}\right)^3 - a\left(\frac{1}{2}\right) + 5 = 2\frac{1}{2}$
$\therefore \frac{a}{2} + \frac{11}{2} = \frac{5}{2} \ \therefore a = -6$

iv) Let $f(x) = 3x^3 + ax^2 - 6x + 3$, Remainder= 5 when divided by $3x + 1$
Divisor $= 3x + 1 = 3\left(x + \frac{1}{3}\right) = 3\left[x - \left(\frac{1}{3}\right)\right]$
Remainder= $f\left(-\frac{1}{3}\right) = 5$
$\Rightarrow 3\left(-\frac{1}{3}\right)^3 + a\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 3 = 5$
$\therefore \frac{a}{9} + \frac{46}{9} = 5 \ \therefore a = -1$

4. *Show that;*
   i) $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$
   ii) $2x^3 + x^2 - 8x - 4$is divisible by $2x + 1$
   iii) $x + 2$ and $x + 1$ are factors of $2x^3 + 5x^2 + x - 2$

Solution

i) Let $f(x) = 12x^3 + 16x^2 - 5x - 3$
$f(x)$is divisible by $2x - 1$ if Remainder $= 0$
Remainder= $f\left(\frac{1}{2}\right) = 0 \ \left[\because 2x - 1 = 2\left(x - \frac{1}{2}\right)\right]$
$\Rightarrow f\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^3 + 16\left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) - 3 = 0$
$\therefore 2x - 1$is a factor of $12x^3 + 16x^2 - 5x - 3$ or $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$

ii) Let $f(x) = 2x^3 + x^2 - 8x - 4$
$2x + 1 = 2\left[x - \left(-\frac{1}{2}\right)\right]$is a factor of $f(x)$ if Remainder $= 0$
Remainder $= f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) - 4 = 0$
$\therefore 2x + 1$is a factor of$2x^3 + x^2 - 8x - 4$

iii) Let $f(x) = 2x^3 + 5x^2 + x - 2$
$x + 2 = x - (-2)$is a factor of $f(x)$ if Remainder $= f(-2) = 0$
$\Rightarrow f(-2) = 2(-2)^3 + 5(-2)^2 + (-2) - 2 = 0$. Hence $x + 2$ is a factor.
Also $x + 1 = x - (-1)$ is a factor of $f(x)$ if Remainder $= f(-1) = 0$
$\Rightarrow f(-1) = 2(-1)^3 + 5(-1)^2 + (-1) - 2 = 0$. Hence $x + 1$ is a factor.

NOTE:
For all the above examples the factors and divisors used are all *linear*. Now let us look at examples of non-linear divisors.

Examples

1. *Find the remainder when;*
   a) $x^4 - 5x^3 + 6x^2 - 7$ is divided by $(x-1)(x-3)$
   b) $x^4 + x^2 - 7$ is divided by $x^2 - 4$
   c) $x^5 - 7x^3 + 4x - 2$ is divided by $(x+1)(x-1)(x-3)$
   d) $x^5 + x^2 - 4x - 3$ is divided by $(x^2-4)(x+1)$

Solution

a) Let $f(x) = x^4 - 5x^3 + 6x^2 - 7$
Divisor $= (x-1)(x-3)$ $.i.e. Quadratic\ divisor.$
Let Remainder$= Ax + B$ $.i.e. Linear\ remainder.$
$\Rightarrow f(x) = (x-1)(x-3).Q(x) + Ax + B$
For $x = 1$
$\Rightarrow f(1) = (1-1)(1-3).Q(1) + A(1) + B = A + B\ \therefore f(1) = A + B$
Also from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(1) = (1)^4 - 5(1)^3 + 6(1)^2 - 7 = -5$
$\therefore f(1) = -5$
$\therefore A + B = -5$ ………………………………………………… (1)
Also for $x = 3$
$\Rightarrow f(3) = (3-1)(3-3).Q(3) + A(3) + B = 3A + B\ \therefore f(3) = 3A + B$
Now from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(3) = (3)^4 - 5(3)^3 + 6(3)^2 - 7 = -7$
$\therefore f(3) = -7$
$\therefore 3A + B = -7$ ……………………………………………… (2)
Solving (1) and (2) simultaneously, eqn. (1)– eqn. (2)
$\Rightarrow -2A = 2\ \therefore A = -1$
From eqn. (1), $B = -5 - A = -5 + 1 = -4$
$\therefore$ Remainder $= -x - 4$

b) Let $f(x) = x^4 + x^2 - 7$
Divisor $= (x^2 - 4) = (x+2)(x-2)$ $.i.e. Quadratic\ divisor.$
Let Remainder$= Ax + B$ $.i.e. Linear\ remainder.$
$\Rightarrow f(x) = (x+2)(x-2).Q(x) + Ax + B$ or $x^4 + x^2 - 7 = (x+2)(x-2).Q(x) + Ax + B$
For $x = 2$
$f(2) = 2A + B = (2)^4 + (2)^2 - 7 = 13$
$\therefore 2A + B = 13$ …………………………………………………… (1)
For $x = -2$
$f(-2) = -2A + B = (-2)^4 + (-2)^2 - 7 = 13$
$\therefore -2A + B = 13$ ……………………………………………… (2)
Solving (1) and (2) simultaneously, eqn. (1)+ eqn. (2)
$\Rightarrow 2B = 26\ \therefore B = 13$
From eqn. (1), $B = \frac{1}{2}(13 - B) = \frac{1}{2}(13 - 13) = 0$
$\therefore$ Remainder $= 13$

c) Let $f(x) = x^5 - 7x^3 + 4x - 2$
Divisor $= (x+1)(x-1)(x-3)$ $.i.e. Cubic\ divisor.$
Let Remainder$= Ax^2 + Bx + C$ $.i.e. Quadratic\ remainder.$
$\Rightarrow f(x) = (x+1)(x-1)(x-3).Q(x) + Ax^2 + Bx + C = x^5 - 7x^3 + 4x - 2$
For $x = 1$
$f(1) = A + B + C = (1)^5 - 7(1)^3 + 4(1) - 2 = -4$
$\therefore A + B + C = -4$ …………………………………………………… (1)

For $x = -1$
$f(-1) = A - B + C = (-1)^5 - 7(-1)^3 + 4(-1) - 2 = 0$
$\therefore A - B + C = 0$ .... (2)
For $x = 3$
$f(3) = 9A + 3B + C = (3)^5 - 7(3)^3 + 4(3) - 2 = 64$
$\therefore 9A + 3B + C = 64$ .... (3)
Solving the equations simultaneously
$Eqn.(1) - eqn.(2)$ to eliminate C

$$\begin{array}{rl} & A + B + C = -4 \\ - & A - B + C = 0 \\ \hline & 2B \quad = -4 \quad \therefore B = -2 \end{array}$$

$Eqn.(1) - eqn.(3)$ to eliminate C

$$\begin{array}{l} A - B + C \quad = 0 \\ 9A + 3B + C = 64 \\ \hline -8A - 4B \quad = -64 \end{array}$$

From $B = -2, -8A = -64 + 4B = -64 - 8 = -72$
$\therefore A = 9$
From $eqn.(1),\ A + B + C = -4$
$\Rightarrow 9 - 2 + C = -4 \therefore C = -11$
$\therefore Remainder = 9x^2 - 2x - 11$

d) Let $f(x) = x^5 + x^2 - 4x - 3$
Divisor $= (x + 1)(x^2 - 4) = (x + 1)(x - 2)(x + 2)$. $i.e. Cubic\ divisor.$
Let Remainder$= Ax^2 + Bx + C$. $i.e. Quadratic\ remainder.$
$\Rightarrow f(x) = (x + 2)(x - 2)(x + 1).Q(x) + Ax^2 + Bx + C = x^5 + x^2 - 4x - 3$
For $x = 2$
$f(2) = 4A + 2B + C = (2)^5 + (2)^2 - 4(2) - 3 = 25$
$\therefore 4A + 2B + C = 25$ .... (1)
For $x = -2$
$f(-2) = 4A - 2B + C = (-2)^5 + (-2)^2 - 4(-2) - 3 = -23$
$\therefore 4A - 2B + C = -23$ .... (2)
For $x = -2$
$f(-1) = 4A - 2B + C = (-1)^5 + (-1)^2 - 4(-1) - 3 = 1$
$\therefore A - B + C = 1$ .... (3)
Solving the equations simultaneously
$Eqn.(1) - eqn.(2)$ to eliminate C

$$\begin{array}{rl} & 4A + 2B + C = 25 \\ - & 4A - 2B + C \quad = -23 \\ \hline & 4B \qquad = 48 \end{array}$$

$\therefore B = 12$
$Eqn.(2) - eqn.(3)$ to eliminate C

$$\begin{array}{rl} & 4A + 2B + C = 25 \\ - & A - \ B + C \ = 1 \\ \hline & 3A - \ B \qquad = -24 \end{array}$$

From $B = 12$, $3A = B - 24 = 12 - 24 = -12$
$\therefore A = -4$
From $eqn.(3)$, $A - B + C = 1$
$\Rightarrow -4 - 12 + C = 1 \therefore C = 17$
$\therefore Remainder = -4x^2 + 12x + 17$

NOTE:
If the divisor is not factorisable to linear factors, you can use long division to find the remainder.

2. *Find the quotient when;*
   a) $2x^3 - 3x^2 + 2x - 4$ is divided by $x - 2$
   b) $x^4 - x^3 + x - 5$ is divided by $x + 1$

Solution

a) Let $f(x) = 2x^3 - 3x^2 + 2x - 4$
$\Rightarrow f(x) = (x-2).Q(x) + R\,; R = f(2) = constant$
Now $f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$
$\therefore f(x) = (x-2).Q(x) + 4$
$\Rightarrow \frac{f(x)-4}{x-2} = Q(x)$. Here $Q(x)$ has no remainder.
$\therefore Q(x) = \frac{2x^3-3x^2+2x-4-4}{x-2} = \frac{2x^3-3x^2+2x-8}{x-2}$
Using long division,

$$\begin{array}{r|l} & 2x^2 + x + 4 \\ \hline x-2 & 2x^3 - 3x^2 + 2x - 8 \\ & \underline{2x^3 - 4x^2} \\ - & \quad x^2 + 2x - 8 \\ & \quad \underline{x^2 - 2x} \\ - & \qquad 4x - 8 \\ & \qquad \underline{4x - 8} \end{array}$$

$\therefore Q(x) = 2x^2 + x + 4$

Alternatively
Let $f(x) = 2x^3 - 3x^2 + 2x - 4, Degree{:}\, n = 3$
Divisor $= x - 2$
$\therefore f(x) = (x-2).Q(x) + Remainder$
But $Remainder = f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$
$\therefore f(x) = (x-2).Q(x) + 4$
Since the quotient has degree $n - 1 = 3 - 1 = 2$, then the quotient must be a quadratic. $.i.e. Ax^2 + Bx + C$
$\Rightarrow f(x) = (x-2).(Ax^2 + Bx + C) + 4$
$\therefore 2x^3 - 3x^2 + 2x - 4 \equiv (x-2).(Ax^2 + Bx + C) + 4$
$\equiv Ax^3 + Bx^2 + Cx - 2Ax^2 - 2Bx - C + 4$
$\equiv Ax^3 + x^2(B - 2A) + x(C - 2B) + (4 - 2C)$
Comparing coefficients;
For x^3 : $2 = A$
For x^2 : $-3 = B - 2A; B = -3 + 2A = -3 + 4 = 1$
For x : $2 = C - 2B \Rightarrow C = 2 + 2B = 2 + 2 = 4$
For $constants$: $; -4 = 4 - 2C \Rightarrow 2C = 8 \therefore C = 4$
$\therefore Q(x) = 2x^2 + x + 4$

b) Let $f(x) = x^4 - x^3 + x - 5; Divisor = x + 1$

$\Rightarrow f(x) = (x+1).Q(x) + R\ ; R = f(-1) = constant$

Now $f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$

$\therefore f(x) = (x+1).Q(x) - 4$

$\Rightarrow \frac{f(x)+4}{x+1} = Q(x)$. Here $Q(x)$ has no remainder.

$\therefore Q(x) = \frac{x^4-x^3+x-5+4}{x+1} = \frac{x^4-x^3+x-1}{x+1} = \frac{x^4-x^3+0x^2+x-1}{x+1}$

By long division.

$$x^3 - 2x^2 + 2x - 1$$
$$x+1 \,\overline{)\, x^4 - x^3 + 0x^2 + x - 1}$$
$$\underline{x^4 + x^3}$$
$$-2x^3 + 0x^2 + x - 1$$
$$-\quad \underline{-2x^3 - 2x^2}$$
$$2x^2 + x - 1$$
$$-\quad \underline{2x^2 + 2x}$$
$$-x - 1$$
$$-\quad \underline{-x - 1}$$

$\therefore Q(x) = x^3 - 2x^2 + 2x - 1$

Alternatively

Let $f(x) = x^4 - x^3 + x - 5, Degree: n = 4$

Divisor $= x + 1$

$\therefore f(x) = (x+1).Q(x) + Remainder$

But $Remainder = f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$

$\therefore f(x) = (x+1).Q(x) - 4$

Since the quotient has degree $n - 1 = 4 - 1 = 3$, then the quotient must be a cubic. $.i.e. Ax^3 + Bx^2 + Cx + D$

$\Rightarrow f(x) = (x+1).(Ax^3 + Bx^2 + Cx + D) - 4$

$\therefore x^4 - x^3 + x - 5 \equiv (x+1).(Ax^2 + Bx + C) - 4$

$\equiv Ax^4 + Ax^3 + Bx^3 + Bx^2 + Cx^2 + Cx + Dx + D - 4$

$\equiv Ax^4 + x^3(A+B) + x^2(B+C) + x(C+D) + D - 4$

Comparing coefficients;

For x^4 : $1 = A$

For x^3 : $-1 = A + B \Rightarrow B = -1 - A = -1 - 1 = -2$

For x^2 : $0 = B + C \Rightarrow C = -B = -(-2) = 2$

For x : $1 = C + D \Rightarrow D = 1 - C = 1 - 2 = -1$

For $constants$: $; -5 = D - 4 \Rightarrow D = -1$

$\therefore Q(x) = x^3 - 2x^2 + 2x - 1$

NOTE:

To determine the quotient using method II, given the polynomial function and divisor, first establish the degree of the quotient basing on that of the divisor, then determine the coefficients as presented as above.

3. *Determine the quotient when;*
   a) $x^4 + x^2 - 7$ is divided by $x^2 - 4$
   b) $x^5 - 7x^3 + 4x - 2$ is divided by $(x^2 - 1)(x - 3)$

Solution

a) Let $f(x) = x^4 + x^2 - 7, Degree: n = 4$

Divisor $= x^2 - 4$
$\therefore f(x) = (x^2 - 4).Q(x) + Remainder$
Let the remainder$= Ax + B$
$\Rightarrow f(x) = (x^2 - 4).Q(x) + Ax + B$
$\therefore f(x) = (x + 2)(x - 2).Q(x) + Ax + B$
$Or\ x^4 + x^2 - 7 \equiv (x + 2)(x - 2).Q(x) + Ax + B$
First obtaining the remainder,
For $x = 2$
$\Rightarrow (2)^4 + (2)^2 - 7 \equiv (2 + 2)(2 - 2).Q(2) + A(2) + B$
$\therefore 2A + B = 13$ ………………………………………………… (1)
For $x = -2$
$\Rightarrow (-2)^4 + (-2)^2 - 7 \equiv (-2 + 2)(2 - 2).Q(-2) + A(-2) + B$
$\therefore -2A + B = 13$ ……………………………………………… (1)
Solving eqn. (1) and eqn. (2) simultaneously,
$eqn.(1) + eqn.(2)\ to\ eliminate\ A$
$2B = 26\ \therefore B = 13$
From eqn. (1), $A = \frac{1}{2}(13 - B) = \frac{1}{2}(0) = 0$
Remainder $= 13$
Now $f(x) = (x + 2)(x - 2).Q(x) + 13$
Since the degree of $f(x)$ is $n = 4$ and the divisor is of degree $n - 2 = 2$, then the quotient is a quadratic $.i.e. mx^2 + nx + p$
Let the quotient $Q(x) = mx^2 + nx + p$
$\Rightarrow f(x) = (x + 2)(x - 2).(mx^2 + nx + p) + 13$
$\therefore x^4 + 0x^3 + x^2 - 7 \equiv (x + 2)(x - 2).(mx^2 + nx + p) + 13$
$\equiv (x^2 - 4)(mx^2 + nx + p) + 13$
$\equiv mx^4 + nx^3 + px^2 - 4mx^2 - 4nx - 4p + 13$
$\equiv mx^4 + nx^3 + x^2(p - 4m) - 4nx + (-4p + 13)$
Comparing coefficients;
For $x^4 : 1 = m$
For $x^3 :\ 0 = n$
For $x^2 :\ 1 = p - 4m \Rightarrow p = 1 + 4m = 1 + 4 = 5$
For $x\ :\ 0 = -4n\ \therefore n = 0$
For $constants$: ; $-7 = 13 - 4p \therefore p = 5$
$\therefore Q(x) = x^2 + 5$

b) Let $f(x) = x^5 - 7x^3 + 4x - 2, Degree: n = 5$
Divisor $= (x^2 - 1)(x - 3)$
$\therefore f(x) = (x^2 - 1)(x - 3).Q(x) + Remainder$
Let the remainder$= px^2 + qx + r$
$\therefore x^5 - 7x^3 + 4x - 2 \equiv (x^2 - 1)(x - 3).Q(x) + px^2 + qx + r$
$Or\ x^5 - 7x^3 + 4x - 2 \equiv (x + 1)(x - 1)(x - 3).Q(x) + px^2 + qx + r$
First obtaining the remainder,
For $x = 1$
$(1)^5 - 7(1)^3 + 4(1) - 2 \equiv (1 + 1)(1 - 1)(1 - 3).Q(1) + p(1)^2 + q(1) + r$
$\therefore p + q + r = -4$ ……………………………………………………… (1)
For $x = -1$
$(-1)^5 - 7(-1)^3 + 4(-1) - 2 \equiv (-1 + 1)(-1 - 1)(-1 - 3).Q(-1) + p(-1)^2 + q(-1) + r$

$\therefore p - q + r = 0$ …………………………………………………………… (2)
For $x = 3$
$(3)^5 - 7(3)^3 + 4(3) - 2 \equiv (3+1)(3-1)(3-3).Q(3) + p(3)^2 + q(3) + r$
$\therefore 9p + 3q + r = 64$ ……………………………………………………… (3)
Solving eqn. (1), eqn. (2) and eqn. (3) simultaneously,
$eqn.(1) - eqn.(2)\ to\ eliminate\ r$
$\Rightarrow 2q = -4 \therefore q = -2$
$eqn.(2) - eqn.(3)\ to\ eliminate\ r$
$\Rightarrow -8p - 4q = -64\ , q = -2$
$\therefore -8p + 8 = -64\ , p = 9$
From eqn. (2), $p - q + r = 0 \Rightarrow r = q - p = -2 - 9 = -11$
$\therefore Remainder = 9x^2 - 2x - 11$
Now $\therefore f(x) = (x^2 - 1)(x - 3).Q(x) + 9x^2 - 2x - 11$
Since the degree of $f(x)$ is $n = 5$ and the divisor is of degree $n - 3 = 2$, then the quotient is a quadratic $.i.e. Ax^2 + Bx + C$
Let the quotient $Q(x) = Ax^2 + Bx + C$
$\Rightarrow f(x) = (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\therefore f(x) \equiv (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\equiv (x^3 - 3x^2 - x + 3)(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\equiv Ax^5 + Bx^4 + Cx^3 - 3x^4 - 3Bx^3 - 3Cx^2 - Ax^3 - Bx^2 - Cx + 3Ax^2 + 3Bx + 3C + 9x^2 - 2x - 11$
$\therefore f(x) \equiv Ax^5 + x^4(B - 3) + x^3(C - 3B - A) + x^2(-3C - B + 3A + 9) + x(-C + 3B - 2) + 3C - 11$
Given $f(x) = x^5 + 0x^4 - 7x^3 + 0x^2 + 4x - 2$
Comparing coefficients,
For $x^5 : 1 = A$
For $x^4 : 0 = B - 3 \Rightarrow B = 3$
For $x^3 :\ -7 = C - 3B - A \Rightarrow C = -7 + 9 + 1 = 3$
For $x^2 :\ 0 = -3C - B + 3A + 9 \Rightarrow C = \frac{-3+3+9}{3} = 3\ As\ before$
For $x\ \ : 4 = -C + 3B - 2\ \Rightarrow C = 9 - 2 - 4 = 3\ As\ before$
For $constants$: $; -2 = 3C - 11 \Rightarrow C = 3\ As\ before$
$\therefore Q(x) = x^2 + 3x + 3$

NOTE:
The above examples on determination of remainder and quotient when a polynomial is divided by a divisor using the above methods can still be done using long division.

4. *Find the constants* $p\ , q\ , r$ *if;*
   a) $x^3 - px + q$ is divided by $x^2 - 3x + 2$ leaves a remainder $4x - 1$
   b) $px^4 + qx^3 + 3x^2 - 2x + 3$ has a remainder $x + 1$ when divided by $x^2 - 3x + 2$
   c) $(x - 1)$ and $(x + 1)$ are factors of the expression $x^3 + px^2 + qx + r$ a nd it leaves a remainder of 12 when divided by $x - 2$

Solution

a) Let $f(x) = x^3 - px + q$
Divisor $= x^2 - 3x + 2 = (x - 1)(x - 2)$

Remainder $= 4x - 1$

$\Rightarrow\ f(x) \equiv Q(x).(x-1)(x-2) + 4x - 1$

$\therefore x^3 - px + q \equiv\ Q(x).(x-1)(x-2) + 4x - 1$

For $x = 1$, $(1)^3 - p(1) + q \equiv\ Q(1).(1-1)(1-2) + 4(1) - 1$

$\Rightarrow 1 - p + q = 3\ \therefore q - p = 2$ ………………………………………………… (1)

For $x = 2$, $(2)^3 - p(2) + q \equiv\ Q(2).(2-1)(2-2) + 4(2) - 1$

$\Rightarrow 8 - 2p + q = 7\ \therefore q - 2p = -1$ …………………………………………… (2)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$, eliminate q

$\Rightarrow p = 3$

From $eqn.(1), q - p = 2 \Rightarrow q = 2 + 3 = 5$

$\therefore p = 2, q = 5$

Alternatively

By long division

$$
\begin{array}{r|l}
 & x + 3 \\
\hline
x^2 - 3x + 2 & x^3 + 0x^2 - px + q \\
 & \underline{x^3 - 3x^2 + 2x} \\
 & \quad -\ 3x^2 + (-p-2)x + q \\
 & \quad\ \ \underline{3x^2 - 9x + 6} \\
 & \qquad\quad (-p+7)x + q - 6
\end{array}
$$

Comparing the remainder,

$(-p + 7)x + q - 6 \equiv 4x - 1$

$\Rightarrow -p + 7 = 4\ \therefore p = 3$

$\Rightarrow q - 6 = -1\ \therefore q = 5$

b) Let $f(x) = px^4 + qx^3 + 3x^2 - 2x + 3$

Divisor $= x^2 - 3x + 2 = (x-1)(x-2)$

Remainder $= x + 1$

$\Rightarrow\ f(x) \equiv Q(x).(x-1)(x-2) + x + 1$

$\therefore px^4 + qx^3 + 3x^2 - 2x + 3 \equiv Q(x).(x-1)(x-2) + x + 1$

For $x = 1$, $p(1)^4 + q(1)^3 + 3(1)^2 - 2(1) + 3 \equiv Q(1).(1-1)(1-2) + 1 + 1$

$\Rightarrow p + q + 3 - 2 + 3 = 2\ \therefore p + q = -2$ ………………………………… (1)

For $x = 2$, $p(2)^4 + q(2)^3 + 3(2)^2 - 2(2) + 3 \equiv Q(2).(2-1)(2-2) + 2 + 1$

$\Rightarrow 16p + 8q + 12 - 4 + 3 = 3$

$\therefore 16p + 8q = -8\ or\ 2p + q = -1$ …………………………………………… (2)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$, eliminate q

$\Rightarrow -p = -1\ \therefore p = 1$

From $eqn.(1)$, $p + q = -2 \Rightarrow q = -2 - 1 = -3$

$\therefore p = 1, q = -3$

c) Let $f(x) = x^3 + px^2 + qx + r$

Divisor $= (x - 1)$, Remainder $= 0$

Divisor $= (x + 1)$, Remainder $= 0$

Divisor $= (x - 2)$, Remainder $= 12$

$\Rightarrow\ f(x) \equiv Q(x).(x-1)(x+1)$

$\therefore x^3 + px^2 + qx + r \equiv Q(x).(x-1)(x+1)$

For $x = 1$, $(1)^3 + p(1)^2 + q(1) + r \equiv Q(1).(1-1)(1+1)$

$\Rightarrow p + q + r = -1$ ………………………………………………… (1)

For $x = -1$, $(-1)^3 + p(-1)^2 + q(-1) + r \equiv Q(-1).(-1-1)(-1+1)$

$\Rightarrow p - q + r = 1$ ………………………………………………… (2)

$\Rightarrow\ f(x) \equiv H(x).(x-2) + 12$

$\therefore x^3 + px^2 + qx + r \equiv H(x).(x-2) + 12$
For $x = 2$, $(2)^3 + p(2)^2 + q(2) + r \equiv H(2).(2-2) + 12$
$\Rightarrow 4p + 2q + r = 4$ .... (3)
Eliminating r from eqn. (1) and eqn. (2), $eqn.(1) - eqn.(2)$
$2q = -2 \therefore q = -1$
Eliminating r from eqn. (2) and eqn. (3), $eqn.(2) - eqn.(3)$
$-3p - 3q = -3$ *or* $p + q = 1$
But $q = -1$ $\therefore p = 1 - q = 1 + 1 = 2$
From (1) $r = -1 - p - q = -1 - 2 + 1 = -2$
$\therefore p = 2, q = -1, r = -2$

5. *Given the polynomial* $f(x) = Q(x).g(x) + R(x)$ *where* $Q(x)$ *is the quotient,* $g(x) = (x-\alpha)(x-\beta)$ *and* $R(x)$ *is the remainder, show that* $R(x) = \frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$ *when* $f(x)$ *is divided by* $g(x)$.
Hence find the remainder when $f(x)$ *is divided by* $x^2 - 9$ *given that* $f(x)$ *divided by* $x - 3$ *and when divided by* $x + 3$*, the remainders are respectively* 2 *and* -3.

Solution
$f(x) = Q(x).g(x) + R(x)$
Divisor $= g(x) = (x-\alpha)(x-\beta)$
Remainder $= R(x) = Ax + B$
$\Rightarrow f(x) = Q(x).(x-\alpha)(x-\beta) + Ax + B$
Now for $x = \alpha$
$f(\alpha) = A\alpha + B$ .... (1)
Now for $x = \beta$
$f(\beta) = A\beta + B$ .... (2)
Since $f(x)$ is known, $f(\alpha)$ and $f(\beta)$ will be known. Now finding the unknowns A and B,
Eliminating B from eqn. (1) and eqn. (2), $eqn.(1) - eqn.(2)$
$\Rightarrow f(\alpha) - f(\beta) = A(\alpha - \beta)$
$\therefore A = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ .... (3)
From eqn. (1), $f(\alpha) = A\alpha + B \Rightarrow B = f(\alpha) - A\alpha$
$\Rightarrow B = f(\alpha) - \alpha\left[\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right] = \frac{\alpha f(\alpha)-\beta f(\alpha)-\alpha f(\alpha)+\alpha f(\beta)}{\alpha-\beta}$
$\therefore B = \frac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$ .... (4)
$\therefore R(x) = Ax + B$ becomes
$R(x) = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}x + \frac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta} = \frac{xf(\alpha)-xf(\beta)+\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$
$\therefore R(x) = \frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$, as required.
From $f(x) = Q(x).(x-\alpha)(x-\beta) + Ax + B$
Divisor $= g(x) = (x-\alpha)(x-\beta) \equiv x^2 - 9 = (x-3)(x+3)$
By comparison, $\alpha = 3, \beta = -3$
Given: $f(x) = G(x).(x-3) + 2 \Rightarrow f(3) = 2 \equiv f(\alpha)$
$f(x) = H(x).(x+3) - 3 \Rightarrow f(-3) = -3 \equiv f(\beta)$
, where $G(x)$ and $H(x)$ are the new quotients due to different divisors.
Remainder$= Ax + B$ is given by $\frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$

$\Rightarrow R(x) = \frac{(x-3)(2)+(3-x)(-3)}{3-(-3)} = \frac{5x-3}{6}$

$\therefore$ Remainder when $f(x)$ is divided by $x^2 - 9$ is $\frac{5x-3}{6}$

6. *A polynomial $P(x)$ leaves a remainder 6 when divided by $x^2 - 4$ and a remainder $x + a$ when divided by $x^2 - x - 2$. Find;*
   a) *The value of the constant a*
   b) *The remainder when $P(x)$ is divided by;*
      i) $x^2 + 3x + 2$
      ii) $x^3 + x^2 - 4x - 4$

Solution

Case I:

$P(x) = Q(x).(x^2 - 4) + 6$

$\quad = Q(x).(x+2)(x-2) + 6$

For $x = 2$, $P(2) = 6$

For $x = -2$, $P(-2) = 6$

Case II:

$P(x) = H(x).(x^2 - x - 2) + x + a$

$\quad = H(x).(x-2)(x+1) + x + a$

For $x = 2$, $P(2) = 2 + a$

For $x = -1$, $P(-1) = -1 + a$

a) But $P(2) = 6 = 2 + a \therefore a = 4$

b) (i) $P(x) = G(x).(x^2 + 3x + 2) + Ax + B$

[$\because$ *Remainder is $Ax + B$ (Linear) since Divisor is Quadratic*]

$P(x) = G(x).(x+1)(x+2) + Ax + B$

For $x = -1$, $P(-1) = -A + B$

For $x = -2$, $P(-2) = -2A + B$

But $P(-1) = -1 + a = -A + B$; $-1 + a = -1 + 4 = 3$

$\Rightarrow -A + B = 3$ ………………………………………………… (1)

Also $P(-2) = -2A + B = 6$……………………………… (2)

Eliminating B from eqn. (1) and eqn. (2)

$eqn.(1) - eqn.(2)$

$\therefore A = -3$

From eqn. (1) $-A + B = 3$

$\Rightarrow B = A + 3 = -3 + 3 = 0 \therefore B = 0$

Remainder when $P(x)$ is divided by $x^2 + 3x + 2$ is $-3x$

(ii) $P(x) = F(x).(x^3 + x^2 - 4x - 4) + mx^2 + nx + l$

$\quad = F(x).(x+1)(x+2)(x-2) + mx^2 + nx + l$

For $x = 2$, $P(2) = 4m + 2n + l$

For $x = -2$, $P(-2) = 4m - 2n + l$

For $x = -1$, $P(-1) = m - n + l$

But $P(2) = 6, P(-2) = 6, P(-1) = 3$

$\Rightarrow 4m + 2n + l = 6$ ………………………………………… (i)

$\Rightarrow 4m - 2n + l = 6$ ………………………………………… (ii)

$\Rightarrow m - n + l = 3$ ……………………………………………… (iii)

Eliminating l from eqn. (i) and eqn. (ii)

$eqn.(i) - eqn.(ii)$

$4n = 0 \therefore n = 0$

Eliminating l from eqn. (ii) and eqn. (iii)

$eqn.(ii) - eqn.(iii)$
$3m - n = 3\ , n = 0$
$\therefore 3m = 3 \Rightarrow m = 1$
From eqn. (iii) $m - n + l = 3 \Rightarrow l = 3 - m + n$
$\Rightarrow l = 3 - 1 + 0 = 2$
$\therefore$ Remainder is $x^2 + 2$

Task

1. *Find the value of the constant k so that* $f(x) = x^4 + kx^3 - 2kx^2 + 3x - 5$ *has* $(x-1)$ *as a factor.* $\boldsymbol{Answer}: \boldsymbol{k = -1}$
2. *Find the value of k so that* $f(x) = x^3 - x^2 + kx - 12$ *has a factor* $(x-3)$ $\boldsymbol{Answer}: \boldsymbol{k = -2}$
3. *Find the value of a if;*
   i) $x^3 - 3x^2 + ax + 5$ *has a remainder 17 when divided* $x - 3$.
   $\boldsymbol{Answer}: \boldsymbol{4}$
   ii) $x^3 + x^2 - 2ax + a^2$ *has remainder 8 when divided by* $x - 2$
   $\boldsymbol{Answer}: \boldsymbol{2}$
   iii) $x - a$ *will divide* $f(x) = x^2 + 4x + 2$ *with a remainder of* -1
   $\boldsymbol{Answer}: \boldsymbol{a = -3\ , -1}$
4. *Find the values of a and b if;*
   i) When $x^5 + 4x^2 + ax + b$ *is divided by* $x^2 - 1$ *the remainder is* $2x + 3$
   $\boldsymbol{Answer}: \boldsymbol{a = 1\ , b = -1}$
   ii) When $ax^4 + bx^3 - x^2 + 2x + 3$ *has a remainder* $3x + 5$ *when divided by* $x^2 - x - 2$
   $\boldsymbol{Answer}: \boldsymbol{a = 1\ , b = -1}$
   iii) $x - 3$ *and* $x + 7$ *are factors of the quadratic* $ax^2 + 12x + b$
   $\boldsymbol{Answer}: \boldsymbol{a = 3\ , b = -63}$
5. a) *If* $f(x) = x^2 + px + q$ *is exactly divisible by* $x - a$ *and* $x - b$, *show that* $p = -a - b$ *and*
   $q = ab$
   b) *Show that;*
   i) $x - a$ *is a factor of* $x^n - a^n$ *when* $n = even$
   ii) $x + a$ *is a factor of* $x^n + a^n$ *when* $n = odd$

$$\begin{bmatrix} \boldsymbol{HINT}: \\ f(-a) = (-a)^n - a^n. If\ n\ is\ odd, (-a)^n = -a^n\ and\ f(-a) = 0. Thus\ x + a\ is\ a\ facor\ of \\ x^n + a^n. Q(x) = x^{n-1} - ax^{n-2} + ax^{n-3} - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a^{n-1} \end{bmatrix}$$

   c) *Show that the remainder when the polynomial* $f(x)$ *is divided by* $x - a$ *is* $f(a)$. *Show further that if* $f(x)$ *is divided by* $(x-a)(x-b)$ *where* $a \neq b$, *then the remainder is;* $\left[\frac{f(a)-f(b)}{a-b}\right]x + \left[\frac{af(b)-bf(a)}{a-b}\right]$

   d) *A polynomial* $f(x)$ *is divided by* $x^2 - a^2$ *where* $a \neq 0$ *and the remainder is* $px + q$ *prove that* $p = \frac{1}{2a}[f(a) - f(-a)]\ , q = \frac{1}{2a}[f(a) + f(-a)]$
6. *When a polynomial* $p(x)$ *is divided by* $2x + 1$, *the quotient is* $x^2 - x + 4$ *and the remainder is 3. Find* $p(x)$ $[\boldsymbol{HINT}: p(x) = (2x+1).(x^2 - x + 4) + 3]$
7. *When a polynomial* $p(x)$ *is divided by* $3x - 4$, *the quotient is* $x^3 + 2x + 2$ *and the remainder is* -1. *Find* $p(x)$. $[\boldsymbol{HINT}: p(x) = (3x-4).(x^3 + 2x + 2) - 1]$

8. a) *When the quadratic expression* $ap^2 + bp + c$ *is divided by* $p - 1$, $p - 2$ *and* $p + 1$, *the remainders are* 1 , 2 *and* 25 *respectively. Determine the factors of the expression.*
$$\left[\begin{array}{l} \boldsymbol{HINT}: Solve\ eqns.\ a + b + c = 1, 4a - 12b + c = 2\ and\ a - b + c = 25. Then \\ \qquad f(p) = 4p^2 - 12p + 9. Now\ factoring\ f(p) = (2p - 3)(2p - 3) \end{array}\right]$$
b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2).Q(x) + Ax + B$ where $Q(x)$a polynomial in is x and A and B are constants. Determine the values of constants A and B and the expression $Q(x)$. $\boldsymbol{Answer}: \boldsymbol{Q(x) = 2x + 3}$, $\boldsymbol{A = -3}$, $\boldsymbol{B = 3}$ $(\boldsymbol{Uneb\ 1993})$

9. *if* $4x^3 + kx^2 + px + 2$ *is divisible by* $x^2 + \lambda^2$, *prove that* $kp = 8$.
$$\left[\begin{array}{l} \boldsymbol{HINT}: Let\ f(x) = 4x^3 + kx^2 + px + 2 \equiv (x^2 + \lambda^2)(Ax + B), \\ since\ the\ quotient\ must\ be\ of\ degree\ 1\ when\ the\ divisor\ is\ of\ degree\ 2\ for\ f(x) \\ is\ of\ degree\ 3 \end{array}\right]$$
10. *A polynomial expression* $P(x)$ *when divided by* $x - 1$ *leaves a remainder of* 3 *and when divided by* $x - 2$ *leaves a remainder of* 1. *Show that when divided by* $(x - 1)(x - 2)$ *it leaves a remainder of* $-2x + 5$.
$$\left[\begin{array}{l} \boldsymbol{HINT}: P(x) = (x - 1).Q(x) + 3\ \ or\ P(1) = 3 \\ \qquad P(x) = (x - 2).H(x) + 1\ \ or\ P(2) = 1 \\ Now\ P(x) = (x - 1)(x - 2).G(x) + Ax + B \\ \qquad P(1) = A + B = 3;\ P(2) = 2A + B = 1 \\ \qquad Solving\ the\ eqns.\ A = -2, B = 5 \\ \qquad\quad \therefore Remainder = -2x + 5 \end{array}\right]$$
11. *Find the quotient and the remainder when* $f(x)$ *is divided by a linear factor*
   i) $f(x) = 2x^3 - 3x^2 + 2x - 8$, $x - 2$
   ii) $f(x) = x^7 - 31x^2$, $x - 2$
   ANSWER: $(i)\ \boldsymbol{2x^2 + x + 4}\ ; \boldsymbol{0}\ \ (ii)\ \boldsymbol{x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + x + 2}\ ; \boldsymbol{2}$
12. *Find the constant p such that* $x^2 + 2$ *is a factor of* $x^4 - 6x^2 + p$. *Hence factorize it.* **ANSWER**: $\boldsymbol{p = -16}$, $\boldsymbol{(x^2 + 2)(x - 2\sqrt{2})(x + 2\sqrt{2})}$

REPEATED ROOTS
(Equal Roots)
From factor theorem which states that "*If for a polynomial function* $f(x)$ *if* $f(a) = 0$ *then* $(x - a)$ *is a factor of* $f(x)$"
If $f(x)$ has a repeated factor $(x - a)$. $i.e.$ $f(x) = (x - a)^2.g(x)$, then;
$$f'(x) = \frac{d}{dx}[(x - a)^2.g(x)] = g(x).2(x - a) + (x - a)^2.g'(x)$$
$$= (x - a)[(x - a).g'(x) + 2g(x)]$$
$i.e.$ If $f(x)$ has a repeated factor $(x - a)$, then $f'(x)$ has a repeated factor $(x - a)$. This is true if and only if $f(a) = 0$, $f'(a) = 0$.
Examples
1. *Without using long division find the remainder when* $x^3 - 5x^2 + 6x - 2$ *is divided by* $(x - 2)^2$.

Solution
Let $f(x) = x^3 - 5x^2 + 6x - 2$
$\Rightarrow f(x) = (x - 2)^2.Q(x) + Ax + B$ ………………………………………………… (*)
For $x = 2$
$f(2) = 2A + B = (2)^3 - 5(2)^2 + 6(2) - 2$

$\therefore 2A + B = -2$ .... (1)
No other condition linking A and B can be obtained from eqn. (*) without involving $Q(x)$.
But from $f(x) = x^3 - 5x^2 + 6x - 2$, $f'(x) = 3x^2 - 10x + 6$
$f'(x) = \frac{d}{dx}[(x-2)^2.Q(x) + Ax + B] = Q(x).2(x-2) + (x-2)^2.Q'(x) + A$
$\Rightarrow 3x^2 - 10x + 6 = Q(x).2(x-2) + (x-2)^2.Q'(x) + A$
$= (x-2)\left[2Q(x) + (x-2).Q'(x)\right] + A$
For $x = 2$
$f'(2) = 3(2)^2 - 10(2) + 6 = A \;\therefore A = -2$
From eqn. (1) $2A + B = -2 \Rightarrow B = -2 + 4 = 2$
$\therefore Remainder = -2x + 2$

2. *Find the constants m and n such that when $x^4 - mx^2 + n$ is divided by $(x+1)^2$ the remainder is $5x - 2$.*
Solution
Let $P(x) = x^4 - mx^2 + n = (x+1)^2.Q(x) + 5x - 2$
For $x = -1$
$P(-1) = (-1)^4 - m(-1)^2 + n = 5(-1) - 2$
$\therefore n - m = -8$ .... (1)
Since the divisor is a repeated factor, then
$P'(x) = 4x^3 - 2mx = Q(x).2(x+1) + (x+1)^2.Q'(x) + 5$
$= (x+1)\left[2Q(x) + (x+1).Q'(x)\right] + 5$
For $x = -1$
$P'(-1) = 4(-1)^3 - 2m(-1) = 5$
$\Rightarrow 2m - 4 = 5 \;\therefore m = \frac{9}{2}$
From eqn. (1), $n = m - 8 = \frac{9}{2} - 8 = -\frac{7}{2}$
$m = \frac{9}{2}, n = -\frac{7}{2}$

3. *If the equation $3x^4 + 2x^3 - 6x^2 - 6x + p = 0$ has two equal roots, find the possible values of p.*
Solution
Let $f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$
The equation $f(x) = 0$ has two equal roots if $f(x)$ has two equal factors.$i.e.$ $Repeated\,factors$.
Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$.
$f'(x) \equiv 12x^3 + 6x^2 - 12x - 6$
$\equiv 6(x-1)(x+1)(2x+1)$
For $(x-1)$ to be a repeated factor of $f(x)$, then $f(1) = 0$
$\Rightarrow f(1) = 3(1)^4 + 2(1)^3 - 6(1)^2 - 6(1) + p = 0 \;\therefore p = 7$
Similarly $(x+1)$ is a repeated factor of $f(x)$, if $f(-1) = 0$
$\Rightarrow f(-1) = 3(-1)^4 + 2(-1)^3 - 6(-1)^2 - 6(-1) + p = 0 \;\therefore p = -1$
For $(2x+1)$ to be a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$
$\Rightarrow f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0$
$\therefore p = -\frac{23}{16}$

4. *Determine whether $f(x) \equiv 3x^4 - 8x^3 - 6x^2 + 24x - 13$ has any repeated factors, and if so find them.*

Solution

$f'(x) \equiv 12x^3 - 24x^2 - 12x + 24$

$\equiv 12(x^3 - 2x^2 - x + 2)$

$\equiv 12\big((x-1)(x^2 - x - 2)\big) \equiv 12(x-1)(x+1)(x-2)$

Now $f'(x) = 0$ if $x = 1, -1, 2$

Checking the value of $f(x)$ for these values of x

$\Rightarrow f(1) = 0, f(-1) \neq 0, f(2) \neq 0$

So $(xx + 1)$ and $(xx - 2)$ are not factors of $f(x)$. Hence $(xx - 1)$ is the only repeated factor of $f(x)$

5. a) *Find the values of p and q which make $x^4 + 6x^3 + 13x^2 + px + q$ a perfect square.*

b) *The polynomial $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ is a perfect square of degree two. Show that $c + 8 = 2b$, $16d = c^2$. Given that $P(x)$ leaves a remainder of 4 when divided by $x + 1$, determine the possible values of the constants b, c and d.*

Solution

a) Since $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ is of degree four$\left(4^0\right)$, then its perfect square must be a quadratic.

Let $x^4 + 6x^3 + 13x^2 + px + q \equiv (Ax^2 + Bx + C)^2 \equiv [(Ax^2 + Bx) + C]^2$

$\equiv (Ax^2 + Bx)^2 + 2C(Ax^2 + Bx) + C^2$

$\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ACx^2 + 2BCx + C^2$

$\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AC)x^2 + (2BC)x + C^2$

Comparing coefficients

For x^4: $1 = A^2 \therefore A = \pm 1$

For x^3: $6 = 2AB \therefore B = \frac{3}{A} = \pm 3$

For x^2: $13 = B^2 + 2AC \therefore C = \frac{13 - B^2}{2A} = \frac{13 - 9}{2(\pm 1)} = \pm 2$

For x: $p = 2BC \therefore p = 2(\pm 3)(\pm 2) = \pm 12$

For $Constants$: $q = C^2 \therefore q = (\pm 2)^2 = 4$

$\therefore p = 12, q = 4$

b) Let $P(x) = x^4 + 4x^3 + bx^2 + cx + d \equiv (Ax^2 + Bx + D)^2$

$\equiv [(Ax^2 + Bx) + D]^2$

$\equiv (Ax^2 + Bx)^2 + 2D(Ax^2 + Bx) + D^2$

$\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ADx^2 + 2BDx + D^2$

$\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AD)x^2 + (2BD)x + D^2$

Comparing coefficients

For x^4: $1 = \mathrm{A}^2 \therefore \mathrm{A} = \pm 1$

For x^3: $4 = 2\mathrm{AB} \therefore \mathrm{B} = \frac{2}{\mathrm{A}} = \pm 2$

For x^2: $\mathrm{b} = \mathrm{B}^2 + 2\mathrm{AD} \therefore \mathrm{b} = 4 \pm 2\mathrm{D}$

For x: $\mathrm{c} = 2\mathrm{BD} \therefore \mathrm{c} = \pm 4\mathrm{D}$

For Constants: $\mathrm{d} = \mathrm{D}^2 \therefore \mathrm{q} = (\pm 2)^2 = 4$

From $\mathrm{b} = 4 + 2\mathrm{D}$ and $c = 4D$ (Taking the $+ve$ sign for both), eliminating D from the equations,

$2b = 8 + 4D$, but $c = 4D$

$\Rightarrow 2b = 8 + c \therefore c + 8 = 2b$

From $d = D^2$ and $c = 4D$ or $D = \frac{c}{4}$, eliminating D from the equations,

$\Rightarrow d = \left(\frac{c}{4}\right)^2 \ \therefore 16d = c^2$
Given that $P(x) = Q(x).(x+1) + 4$
For $x = -1$, $P(-1) = 4$
From $P(x) = x^4 + 4x^3 + bx^2 + cx + d$
$\Rightarrow P(-1) = (-1)^4 + 4(-1)^3 + b(-1)^2 + c(-1) + d$
$\Rightarrow 4 = 1 - 4 + b - c + d$
$\therefore b - c + d = 7$ ……………………………………………… (1)
From $16d = c^2$ *or* $d = \frac{c^2}{16}$
$\therefore b - c + \frac{c^2}{16} = 7$ ……………………………………………… (2)
From $c + 8 = 2b$ *or* $\frac{c+8}{2} = b$
Eqn. (2) becomes,
$\frac{c+8}{2} - c + \frac{c^2}{16} = 7$ ……………………($\times 16$)
$\Rightarrow 64 + 8c - 16c + c^2 = 112$ *or* $c^2 - 8c - 48 = 0$
Solving the equation, $c = 12, -4$
$\therefore b = \frac{c+8}{2} = \frac{12+8}{2}$ *or* $\frac{-4+8}{2}$
$\therefore b = 10, c = 12; b = 2, c = -4$
Now from $d = \frac{c^2}{16}$
For $c = -4$, $d = \frac{16}{16} = 1$
For $c = 12$, $d = \frac{144}{16} = 9$
$\therefore b = 10, c = 12, d = 9; b = 2, c = -4, d = 9$

Task

1) *Find the remainder when* $x^3 - 5x^2 + 7$ *is divided by* $(x-1)^2$. **ANSWER**: $\mathbf{10 - 7x}$
2) *If the equation* $2x^3 - 9x^2 + 12x + p = 0$ *has two equal roots, find the possible values of p.* **ANSWER**: $\mathbf{-5, -4}$
3) *Find the value of a for which the function* $2x^3 - ax^2 - 12x - 7$ *has a repeated factor.* **ANSWER**: **3**
4) *If* $(x+1)^2$ *is a factor of* $2x^4 + 7x^3 + 6x^2 + Ax + B$. *Find the values of A and B.* **ANSWER**: $\boldsymbol{A = 7, B = 0}$
5) *Given that* $x^2 + c^2$ *is a factor of* $5x^4 + px^3 + qx^2 + rx + 4$, *show that* $5r^2 + 4p^2 - 4pr = 0$
6) *Show that if* $x^3 + 3px^2 + qx + r$ *is a perfect cube for all real values of x then* $q^3 = 27r^2$
7) *Show that* $x^2 + 3$ *is a factor of* $x^3 - x^2 + 3x - 3$
8) *Find the constant m for which* $x^2 + 1$ *is a factor of* $mx^4 + x^2 - 1$. **ANSWER**: **2**
9) *Determine whether the given functions have any repeated factors, and if any, find them.*
    a) $x^4 - 16$ **ANSWER**: $\boldsymbol{No}$
    b) $x^4 - 18x^2 + 81$ **ANSWER**: $\boldsymbol{(x-3)(x+3)}$
    c) $2x^3 - 3x^2 + 1$ **ANSWER**: $\boldsymbol{x - 1}$
    d) $x(x^4 - 4)$ **ANSWER**: $\boldsymbol{No}$
10) *Find the constants p and q such that* $(x-1)$ *is a common factor of* $x^4 - 2px^2 + 2$ *and* $x^4 + x^2 - q$. **ANSWER**: $\boldsymbol{p = \frac{3}{2}, q = 2}$

SOLVING POLYNOMIAL EQUATIONS BY FACTORING

Important Theorems used:

a) **The Factor Theorem**

"For a polynomial $P(x)$, $x-a$ is a factor if and only if $P(a)=0$"

b) **The Rational Theorem**

"Let $P(x)$ be a polynomial of degree n with integral coefficients a_i and non zero constant term, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_1x+a_0$ where $a_i\in I$.

If one of the roots of the equation $P(x)=0$ is $x=\frac{p}{q}$, $\frac{p}{q}\neq 0$, $q>0$ and $\frac{p}{q}$ is expressed in lowest terms, then p must be a factor of a_0 and q must be a factor of a_n

c) **The Fundamental Theorem of Algebra**

If the complex system consisting of all real and imaginary numbers, if $P(x)$ is a polynomial of degree n, $(n>0)$ with complex coefficients, then the equation $P(x)=0$ has exactly n –roots (provided a double root is counted as two roots, a triple is counted as $3-$ roots

Example

- $2ix^3+\sqrt{5}x^2+(3+2i)x+7-0$ has $3-$ roots
- $3x^4-11x^3+19x^2+25x-36=0$ has $4-$ roots

Other Theorems

- " If $P(x)$ is a polynomial of odd degree with real coefficients, then the equations $P(x)=0$ has at least on real root"
- The relationship between the roots of a polynomial equation and coefficients of the polynomial is;

  For the equation $\boldsymbol{a_nx^n}+\boldsymbol{a_{n-1}x^{n-1}}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\boldsymbol{a_1x}+\boldsymbol{a_0}=\mathbf{0}$ with $a_n\neq 0$, then

  Sum of roots $=-\frac{\boldsymbol{a_{n-1}}}{\boldsymbol{a_n}}$

  Product of roots $=\begin{cases}\frac{\boldsymbol{a_0}}{\boldsymbol{a_n}} & if\ n\ is\ even\\ -\frac{\boldsymbol{a_0}}{\boldsymbol{a_n}} & if\ n\ is\ odd\end{cases}$

Example

1. a) If one of the roots of the equation $P(x)=3x^4+13x^3+15x^2-4=0$ is in the form $x=\frac{p}{q}$, state the possible rational roots of the equation $P(x)=0$.

   b) Determine whether any of the possible rational roots really are roots. Then find all the other roots, real/imaginary.

Solution

a) $x=\frac{p}{q}$ is a possible rational root if p divides -4 and q divides 3

$\Rightarrow$ p could be equal to: ± 1 , ± 2 $or\ \pm 4$ and

q could be equal to: ± 1 , ± 3

$\therefore \frac{p}{q}=\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

b) Using Factor Theorem

From $P(x)=3x^4+13x^3+15x^2-4$

Let $x=1$, $P(1)\neq 0$ $\therefore x-1$ is not a factor

$x = -1\,, P(-1) \neq 0\ \therefore x + 1$ is not a factor
$x = 2\,, P(2) \neq 0\ \therefore x - 2$ is not a factor
$x = -2\,, P(-2) = 0\ \therefore x + 2$ is a factor

Now by long division.

$$\begin{array}{r|l} & 3x^3 + 7x^2 + x - 2 \\ \hline x+2 & 3x^4 + 13x^3 + 15x^2 + 0x - 4 \\ & \underline{3x^4 + 6x^3} \\ - & \quad 7x^3 + 15x^2 + 0x - 4 \\ & \quad \underline{7x^3 + 14x^2} \\ - & \qquad x^2 + 0x - 4 \\ & \qquad \underline{x^2 + 2x} \\ - & \qquad\quad -2x - 1 \\ & \qquad\quad \underline{-2x - 1} \end{array}$$

$\therefore P(x) = (x + 2)(3x^3 + 7x^2 + x - 2) = 0$

Now finding roots of the depressed equation $3x^3 + 7x^2 + x - 2 = 0$
If $x = \frac{p}{q}$ is the root of the equation, p divides -2 and q divides 3
p could be equal to: $\pm 1\,, \pm 2$ and
q could be equal to: $\pm 1\,, \pm 3$
$\therefore \frac{p}{q} = \pm 1\,, \pm 2\ , \pm\frac{1}{3}\,, \pm\frac{2}{3}$
Let $Q(x) = 3x^3 + 7x^2 + x - 2 = 0$
For $x = 1\,, Q(1) \neq 0\ \therefore x - 1$ is not a factor
$x = -1\,, Q(-1) \neq 0\ \therefore x + 1$ is not a factor
$x = 2\,, Q(2) \neq 0\ \therefore x - 2$ is not a factor
$x = -2\,, Q(-2) = 0\ \therefore x + 2$ is a factor

By long division,

$$\begin{array}{r|l} & 3x^2 + x - 1 \\ \hline x+2 & 3x^3 + 7x^2 + x - 2 \\ & \underline{3x^3 + 6x^2} \\ - & \quad x^2 + x - 2 \\ & \quad \underline{x^2 + 2x} \\ & \qquad -x - 2 \\ & \qquad \underline{-x - 2} \end{array}$$

$\therefore Q(x) = (x + 2)(3x^2 + x - 1) = 0$

$\therefore P(x) = (x + 2)^2(3x^2 + x - 1) = 0$

Now solving $3x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{(1)^2 + 4(3)(1)}}{2(3)} = \frac{-1 \pm \sqrt{13}}{6}$

$\therefore$Roots are: $x = \frac{-1 \pm \sqrt{13}}{6}\,, x = -2\,, x = -2$

2. *Find all the zeros and if possible, all the other real zeros of;*
   a. $f(x) = x^3 - x^2 - 14x + 24$
   b. $f(x) = x^3 - 8x^2 + 5x + 14$
   c. $f(x) = 3x^3 - x^2 - 6x + 2$

Solution

For rational zeros, $f(x) = 0$ is an equation to be solved.

a. $f(x) = x^3 - x^2 - 14x + 24 = 0$
 Let $x = \frac{p}{q}$, p divides 24, q divides 1
 p could be equal to: $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm12, \pm24$ and
 q could be equal to: ±1
 $\therefore$Possible values of $x = \frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm12, \pm24$
 By factor theorem,
 For $x = 1, f(1) \neq 0 \ \therefore x - 1$ is not a factor
 $x = -1, f(-1) \neq 0 \ \therefore x + 1$ is not a factor
 $x = -2, f(-2) \neq 0 \ \therefore x + 2$ is not a factor
 $x = 2, f(2) = 0 \ \therefore x - 2$ is a factor
 By long division,

$$\begin{array}{r|l}
 & x^2 + x - 12 \\
\hline
x - 2 & x^3 - x^2 - 14x + 24 \\
 & \underline{x^3 - 2x^2} \\
 & \qquad x^2 - 14x + 24 \\
 & - \ \underline{x^2 - 2x} \\
 & \qquad\qquad -12x + 24 \\
 & \qquad\qquad \underline{-12x - 24}
\end{array}$$

$\therefore f(x) = \ (x-2)(x^2 + x - 12) = 0$
Solving $x^2 + x - 12 = 0$
$\Rightarrow (x + 4)(x - 3) = 0$
$\therefore x = -4, 3$
$\therefore$The rational zeros are:
$\therefore x = 2, -4, 3$

b. $f(x) = x^3 - 8x^2 + 5x + 14 = 0$
 Let $x = \frac{p}{q}$, p divides 14, q divides 1
 Possible values of $p = \pm1, \pm2, \pm7, \pm14$
 Possible values of $q = \pm1$
 Possible values of $x = \pm1, \pm2, \pm7, \pm14$
 By factor theorem,
 For $x = 1, f(1) \neq 0 \ \therefore x - 1$ is not a factor
 $x = -1, f(-1) \neq 0 \ \therefore x + 1$ is a factor
 $x = 2, f(2) = 0 \ \therefore x - 2$ is a factor
 $x = 7, f(2) = 0 \ \therefore x - 7$ is a factor
 $\therefore f(x) = (x + 1)(x - 2)(x - 7) = 0$
 The zeros of $f(x)$ are: $-1, 2, 7$

c. $f(x) = 3x^3 - x^2 - 6x + 2 = 0$
 Let $x = \frac{p}{q}$, p divides 2, q divides 3
 Possible values of $p = \pm1, \pm2$
 Possible values of $q = \pm1, \pm3$
 Possible values of $x = \pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}$
 By factor theorem,
 For $x = 1, f(1) \neq 0 \ \therefore x - 1$ is not a factor
 $x = -1, f(-1) \neq 0 \ \therefore x + 1$ is not a factor
 $x = 2, f(2) \neq 0 \ \therefore x - 2$ is not a factor
 $x = -\frac{1}{3}, f\left(-\frac{1}{3}\right) \neq 0 \ \therefore x + \frac{1}{3}$ is not a factor
 $x = \frac{1}{3}, f\left(\frac{1}{3}\right) \neq 0 \ \therefore x - \frac{1}{3}$ is a factor

By long division

$$\begin{array}{r|l}
 & 3x^2 - 6 \\
\hline
x - \frac{1}{3} & 3x^3 - x^2 - 6x + 2 \\
 & \underline{3x^3 - x^2} \\
 & \quad -6x + 2 \\
 & - \ \underline{-6x + 2}
\end{array}$$

$\therefore f(x) = (3x^2 - 6)\left(x - \frac{1}{3}\right) = 0$

$\therefore f(x) = (3x^2 - 6)\left(x - \frac{1}{3}\right) = 0 \Rightarrow 3x^2 - 6 = 0 \ \therefore x = \pm\sqrt{2}$

The zeros of $f(x)$ are: $\pm\sqrt{2}, \frac{1}{3}$

3. *Factorize the polynomial* $P(x)$, *hence find the zeros of* $P(x)$.

   a) $P(x) = x^3 + 2x^2 - 9x - 18$

   b) $P(x) = x^4 - 11x^2 + 28$

   c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Solution

a) $P(x) = x^3 + 2x^2 - 9x - 18$

Let $x = \frac{p}{q}$, p divides-18, q divides 1

Possible values of $p = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

Possible values of $q = \pm1$

Possible values of $x = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

By factor theorem,

For $x = 1, P(1) \neq 0 \ \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \ \therefore x + 1$ is not a factor

$x = -2, P(-2) = 0 \ \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \ \therefore x - 3$ is a factor

$x = -3, P(-3) \neq 0 \ \therefore x + 3$ is a factor

$\therefore P(x) = (x + 3)(x - 3)(x + 2)$

The zeros of $P(x)$ are: $\pm3, -2$

b) $P(x) = x^4 - 11x^2 + 28$

Let $x = \frac{p}{q}$, p divides 28, q divides 1

Possible values of $p = \pm1, \pm2, \pm3, \pm4, \pm7, \pm14, \pm28$

Possible values of $q = \pm1$

Possible values of $x = \pm1, \pm2, \pm3, \pm4, \pm7, \pm14, \pm28$

By factor theorem,

For $x = 1, P(1) \neq 0 \ \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \ \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \ \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \ \therefore x + 2$ is a factor

$\therefore P(x) = (x + 2)(x - 2).M(x) = (x^2 - 4).M(x)$

Since $x + 2, x - 2$ are factors of $P(x)$, then $(x + 2)(x - 2) = x^2 - 4$ must also be a factor.

By long division to obtain $M(x)$

Divisor $= x^2 - 4$

$$\begin{array}{r|l} & x^2 - 7 \\ \hline x^2 - 4 & x^4 + 0x^3 - 11x^2 + 0x + 28 \\ & \underline{x^4 + 0x^3 - 4x^2} \\ & \qquad -7x^2 + 0x + 28 \\ & \qquad \underline{7x^2 + 0x + 28} \end{array}$$

$\therefore M(x) = x^2 - 7$

$\therefore P(x) = (x+2)(x-2)(x^2-7) = (x+2)(x-2)\left(x \pm \sqrt{7}\right)$

The zeros of $P(x)$ are: $\pm 2, \pm\sqrt{7}$

c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Let $x = \frac{p}{q}$, p divides 12, q divides 2

Possible values of $p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Possible values of $q = \pm 1, \pm 2$

Possible values of $x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$

By factor theorem,

For $x = 1, P(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \therefore x - 3$ is a factor

$\therefore P(x) = (x+2)(x-2)(x-3).M(x) = (x^2-4)(x-3).M(x)$

Since $x+2, x-2, x-3$ are factors of $P(x)$, then $(x+2)(x-2)(x-3)$ must also be a factor. By long division to obtain $M(x)$

Divisor $= (x^2-4)(x-3) = x^3 - 3x^2 - 4x + 12$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x^3 - 3x^2 - 4x + 12 & 2x^4 - 5x^3 - 11x^2 + 20x + 12 \\ & \underline{2x^4 - 6x^3 - 8x^2 + 24x} \\ & \qquad x^3 - 7x^2 + 0x + 28 \\ & - \quad \underline{x^3 - 7x^2 + 0x + 28} \end{array}$$

$\therefore P(x) = (x+2)(x-2)(x-3)(2x+1)$

The zeros of $P(x)$ are: $\pm 2, -\frac{1}{2}, -3$

Task

1) *Factorize $f(x)$, hence state the zeros of $f(x)$*

a) $f(x) = x^3 + 4x^2 + x - 6$

ANSWER: $\boldsymbol{(x+2)(x+3)(x-1)}$; $\boldsymbol{Zeros}$: $\boldsymbol{-2, -3, 1}$

b) $f(x) = 6x^3 - 5x^2 - 2x + 1$

ANSWER: $\boldsymbol{(3x-1)(2x+1)(x-1)}$; $\boldsymbol{Zeros}$: $\boldsymbol{-\frac{1}{2}, \frac{1}{3}, 1}$

c) $f(x) = 4x^3 - 13x - 6$

ANSWER: $\boldsymbol{(x-2)(2x+1)(2x+3)}$; $\boldsymbol{Zeros}$: $\boldsymbol{-\frac{1}{2}, -\frac{3}{2}, 2}$

2) *Solve the equation giving real and imaginary roots if any;*

a) $2x^3 - 7x + 2 = 0$ **ANSWER**: $\boldsymbol{\frac{2\pm\sqrt{2}}{2}, -2}$

b) $3x^3 - 4x^2 - 5x + 2 = 0$ **ANSWER**: $\boldsymbol{-1, 2, \frac{1}{3}}$

c) $x^4 - 10x^2 + 9 = 0$ **ANSWER**: $\boldsymbol{\pm 1, \pm 3}$

THE BINOMIAL THEOREM

Pascal's triangle

This is used to give the coefficients of the emerging terms during the expansion.
The coefficients are obtained as below.
Consider an array of triangular combinatorial coefficients.

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	$^0C_0$					
	1	$^1C_0$	$^1C_1$				
	2	$^2C_0$	$^2C_1$	$^2C_2$			
	3	$^3C_0$	$^3C_1$	$^3C_2$	$^3C_3$		
	4	$^4C_0$	$^4C_1$	$^4C_2$	$^4C_3$	$^4C_4$	

Superscript on the left of each of the coefficients denotes *row number* and the *subscript* on the right denote *column number*.

Note:

1. Superscript *must* be greater than or equal to subscript. *i.e.* for $^nC_r, n \geq r$
2. The above array is generated using $\boxed{^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}}$
   Simplifying the table

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	1					
	1	1	$^1C_1$				
	2	1	2	1			
	3	1	3	3	1		
	4	1	4	6	4	1	

From the above table, from the coefficients of each expansion, the coefficient of the next expansion can be obtained using $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

$^1C_0 \quad {}^1C_1$

$\quad {}^1C_0 \quad {}^1C_1$] **add**

$^1C_0 \quad \left[{}^1C_0 + {}^1C_1 = {}^2C_1\right] \quad {}^1C_1$

$\quad {}^1C_0 \quad {}^2C_1 \quad {}^1C_1$] **add**

$^1C_0 \quad \left[{}^2C_1 + {}^1C_0 = {}^3C_1\right] \quad \left[{}^1C_1 + {}^2C_1 = {}^3C_2\right] \quad {}^1C_1$

$\quad {}^1C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^1C_1$] **add**

$^1C_0 \quad \left[{}^3C_1 + {}^1C_0 = {}^4C_1\right] \quad \left[{}^3C_2 + {}^3C_1 = {}^4C_2\right] \quad \left[{}^1C_1 + {}^3C_2 = {}^4C_3\right] \quad {}^1C_1$

Observation

1 1
 1 1 **add**

$(a+b)^2$ coeff: **1 2 1**
 1 2 1 **add**

$(a+b)^3$ coeff: **1 3 3 1**
 1 3 3 1 **add**

$(a+b)^4$ coeff: **1 4 6 4 1**
 1 4 6 4 1 **add**

$(a+b)^5$ coeff: **1 5 10 10 5 1**

Pascal's triangle is

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Binomials

Definition

An algebraic expression containing two numbers/terms is a *binomial.*

Examples

$(x+y), (a+b), (3x-2y), (4a-3b)\ etc$

Pascal's triangle shall be applied to binomials raised to a real number. $i.e. (a+b)^n$, where n is a particular number.

$(a+b)^n$ when being expanded using Pascal's triangle has the following features:

i) There are $(n+1)\ terms$ in the expansion.
ii) The sum of the powers of 'a' and 'b' in each term equal to 'n' in the expansion.
iii) The first term of expansion is a^n and the last term is b^n.
iv) The power of 'a' decrease by 1 and subsequent terms and that of b start with 1 in the second term, and increasing by 1 in every succeeding term.

Examples

1) *Expand in descending powers of x in each of the following.*

i. $(x+6)^6$ iii. $(2x+1)^3$

ii. $\left(2x-\frac{1}{4}\right)^4$ iv. $\left(x-\frac{1}{x}\right)^5$

Solution

i. $(x+6)^6 = ?$

There are 7 $terms. i.e.$

$x^6 \quad x^5(6)^1 \quad x^4(6)^2 \quad x^3(6)^3 \quad x^2(6)^4 \quad x^1(6)^5 \quad (6)^6$

Coefficients from Pascal's triangle

1 6 15 20 15 6 1

$\Rightarrow (x+6)^6 = \mathbf{1}.x^6 + 6.x^5(6)^1 + 15.x^4(6)^2 + 20.x^3(6)^3 + \mathbf{15}.\,x^2(6)^4 + \mathbf{6}.x^1(6)^5 + \mathbf{1}.(6)^6$

$= x^6 + 36x^5 + 540x^4 + 4320x^3 + 19440x^2 + 46656x + 46656$

ii. $\left(2x - \frac{1}{4}\right)^4 = ?$

There are 5 $terms.\,i.e.$

$(2x)^4 \quad (2x)^3\left(-\frac{1}{4}\right)^1 \quad (2x)^2\left(-\frac{1}{4}\right)^2 \quad (2x)^1\left(-\frac{1}{4}\right)^3 \quad \left(-\frac{1}{4}\right)^4$

Coefficients from Pascal's triangle

1 4 6 4 1

$\Rightarrow \left(2x - \frac{1}{4}\right)^4 = \mathbf{1}.(2x)^4 + \mathbf{4}.(2x)^3\left(-\frac{1}{4}\right)^1 + \mathbf{6}.(2x)^2\left(-\frac{1}{4}\right)^2 + \mathbf{4}.(2x)^1\left(-\frac{1}{4}\right)^3 + \mathbf{1}.\left(-\frac{1}{4}\right)^4$

$= 16x^4 - 8x^3 + \frac{3}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{256}$

iii. $(2x+1)^3 = ?$

There are 4 $terms.\,i.e.$

$(2x)^3 \quad (2x)^2(1)^1 \quad (2x)^1(1)^2 \quad (1)^3$

Coefficients from Pascal's triangle

1 3 3 1

$\Rightarrow (2x+1)^3 = \mathbf{1}.(2x)^3 + \mathbf{3}.(2x)^2(1)^1 + \mathbf{3}.(2x)^1(1)^2 + \mathbf{1}.(1)^3$

$= 8x^3 + 12x^2 + 6x + 1$

iv. $\left(x - \frac{1}{x}\right)^5 = ?$

There are 6 $terms.\,i.e.$

$x^5 \quad x^4\left(-\frac{1}{x}\right)^1 \quad x^3\left(-\frac{1}{x}\right)^2 \quad x^2\left(-\frac{1}{x}\right)^3 \quad x\left(-\frac{1}{x}\right)^4 \quad \left(-\frac{1}{x}\right)^5$

Coefficients from Pascal's triangle

1 5 10 10 5 1

$\Rightarrow \left(x - \frac{1}{x}\right)^5 = \mathbf{1}.x^5 + \mathbf{5}.x^4\left(-\frac{1}{x}\right)^1 + \mathbf{10}.x^3\left(-\frac{1}{x}\right)^2 + \mathbf{10}.x^2\left(-\frac{1}{x}\right)^3 + \mathbf{5}.x\left(-\frac{1}{x}\right)^4 + \mathbf{1}.\left(-\frac{1}{x}\right)^5$

$= x^5 - 5x^2 + \frac{10}{x} - \frac{10}{x^4} + \frac{\mathbf{5}}{x^7} - \frac{1}{x^{10}}$

Finding approximations by Pascal's triangle expansions

2) *Expand* $\left(3 + \frac{1}{8}x\right)^4$ *in ascending powers of x. by putting* $x = 0.2$ *in the first three terms of the expansion, find the approximate value of* $(3.025)^4$ *to* $3d.p.$

Solution

$\left(3 + \frac{1}{8}x\right)^4$ has $5 - terms.\,i.e.$

$(3)^4 \quad (3)^3\left(\frac{x}{8}\right) \quad (3)^2\left(\frac{x}{8}\right)^2 \quad (3)\left(\frac{x}{8}\right)^3 \quad \left(\frac{x}{8}\right)^4$

Coefficients: **1 4 6 4 1**

$\therefore \left(3 + \frac{1}{8}x\right)^4 = \mathbf{1}.(3)^4 + \mathbf{4}.(3)^3\left(\frac{x}{8}\right) + \mathbf{6}.(3)^2\left(\frac{x}{8}\right)^2 + \mathbf{4}.(3)\left(\frac{x}{8}\right)^3 + \mathbf{1}.\left(\frac{x}{8}\right)^4$

$= 81 + \frac{108}{8}x + \frac{54}{64}x^2 + \frac{12}{512}x^3 + \frac{x^4}{4096}$

$= 81 + \frac{27}{2}x + \frac{27}{32}x^2 + \frac{3}{128}x^3 + \frac{x^4}{4096}$

Taking the first 3 –terms

$\left(3 + \frac{1}{8}x\right)^4 = 81 + \frac{27}{2}x + \frac{27}{32}x^2$

For $x = 0.2$

$\Rightarrow \left(3 + \frac{1}{8} \times 0.2\right)^4 = 81 + \frac{27}{2} \times 0.2 + \frac{27}{32} \times (0.2)^2$

$\therefore (3.025)^4 \approx 83.734$

3) *Expand* $\left(1+\frac{1}{x}\right)^3$ *as far as* $\frac{1}{x^3}$. *Hence find the value of* $(1.025)^3$ *giving your answer to* $4\ s.f.$

Solution

$\left(1+\frac{1}{x}\right)^3$ has $4-terms.i.e.$

$(1)^3 \quad (1)^2\left(\frac{1}{x}\right)^1 \quad (1)^1\left(\frac{1}{x}\right)^2 \quad \left(\frac{1}{x}\right)^3$

Coefficients from Pascal's triangle: **1 3 3 1**

$\therefore \left(1+\frac{1}{x}\right)^3 = \mathbf{1}.(1)^3+\mathbf{3}.(1)^2\left(\frac{1}{x}\right)^1+\mathbf{3}.(1)^1\left(\frac{1}{x}\right)^2+\mathbf{1}.\left(\frac{1}{x}\right)^3$

$= 1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}$

Now finding a suitable value of x for substitution

$\Rightarrow \left(1+\frac{1}{x}\right)^3 = (1.025)^3 \quad \therefore 1+\frac{1}{x} = 1.025 \ \ or\ x = 40$

$\therefore \left(1+\frac{1}{40}\right)^3 = (1.025)^3 = 1+\frac{3}{40}+\frac{3}{40^2}+\frac{1}{40^3} = 1.0779$

$\therefore (1.025)^3 \approx 1.077$

4) *Write down the expansion of* $\left(1+\frac{x}{4}\right)^4$ *af far as* x^3. *Hence find the value of* $(1.025)^4$ *correct to* $3d.ps.$

Solution

$\left(1+\frac{x}{4}\right)^4$ has $5-terms.i.e.$

$(1)^4 \quad (1)^3\left(\frac{x}{4}\right)^1 \quad (1)^2\left(\frac{x}{4}\right)^2 \quad (1)^1\left(\frac{x}{4}\right)^3 \quad \left(\frac{x}{4}\right)^4$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$\therefore \left(1+\frac{1}{x}\right)^3 = \mathbf{1}.(1)^4+\mathbf{4}.(1)^3\left(\frac{x}{4}\right)^1+\mathbf{6}.(1)^2\left(\frac{x}{4}\right)^2+\mathbf{4}.(1)^1\left(\frac{x}{4}\right)^3+\mathbf{1}.\left(\frac{x}{4}\right)^4$

$\approx 1+x+\frac{3}{8}x^2+\frac{1}{16}x^3\ ;\ \left(\frac{x}{4}\right)^4\ is\ ignored$

Now finding a suitable value of x for substitution

$\Rightarrow \left(1+\frac{x}{4}\right)^4 = (1.025)^4 \quad \therefore 1+\frac{x}{4} = 1.025 \ \ or\ x = 0.1$

$\therefore \left(1+\frac{0.1}{4}\right)^3 = (1.025)^4 = 1+0.1+\frac{3}{8}(0.1)^2+\frac{1}{16}(0.1)^3 = 1.104$

$\therefore (1.025)^4 \approx 1.104$

5) *Using Pascal's triangle, find the exact value of*
   a) $(2.95)^3$
   b) $(2.998)^4$

Solution

a) $(2.95)^3 = (3-0.05)^3 = (2+0.95)^3 \quad [Use\ any\ of\ binomials\ of\ your\ choice]$

Using $(2.95)^3 = (3-0.05)^3$, has $4-terms.i.e.$

$(3)^3 \quad (3)^2(-0.05)^1 \quad (3)^1(-0.05)^2 \quad (-0.05)^3$

Coefficients from Pascal's triangle: **1 3 3 1**

$\Rightarrow (2.95)^3 = (3-0.05)^3 = \mathbf{1}.(3)^3+\mathbf{3}.(3)^2(-0.05)^1+\mathbf{3}.\ \ (3)^1(-0.05)^2+\mathbf{1}.(-0.05)^3$

$= 27-1.35+0.0225-0.000125$

$= 25.672375$

b) $(2.998)^4 = (3-0.002)^4 = (2+0.998)^4$

Using $(2.998)^4 = (3-0.002)^4$, has $5-terms.i.e.$

$(3)^4 \quad (3)^3(-0.002)^1 \quad (3)^2(-0.002)^2 \quad (3)^1(-0.002)^3 \quad (-0.002)^4$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$\Rightarrow (2.998)^4 = (3 - 0.05)^3$

$= \mathbf{1}.(3)^4 + \mathbf{4}.(3)^3(-0.002)^1 + \mathbf{6}.(3)^2(-0.002)^2 + \mathbf{4}.(3)^1(-0.002)^3 + \mathbf{1}.(-0.002)^4$

$= 81 - 0.216 + 0.000216$

$= 80.784$

Task

1) *Expand* $(2 - x)^6$ *in ascending powers of* x *.Taking the first 3-terms, find the value of* $(1.998)^6$ *as accurately as possible.* ***Answer***: $\mathbf{63.617}$
2) *Write down the first three terms of* $(3 - x)^6$. *Hence find the value of* $(2.998)^6$ *as accurately as possible.* ***Answer***: $\mathbf{80.784}$

Note:

1) Pascal's triangle is applicable to $(a + b)^n$ where n is relatively small. $i.e. n \leq 6$ and is a natural number.
2) If n is large or negative of fractional, Binomial theorem can be used.

THE BINOMIAL THEOREM

This states that if n is a particular number then

$$(a + b)^n = {}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b^1 + {}^{n}C_2 a^{n-2} b^2 + \ldots\ldots\ldots\ldots + {}^{n}C_n a^{n-n} b^n$$
$$= a^n + {}^{n}C_1 a^{n-1} b^1 + {}^{n}C_2 a^{n-2} b^2 + \ldots\ldots\ldots\ldots\ldots\ldots + b^n$$

However if few terms of an expansion are required the theorem is used in the form

$$(a + b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots + b^n$$

Note:

${}^{n}C_1 = \frac{n!}{(n-1)!} = n$, ${}^{n}C_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!}$, *and* ${}^{n}C_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$ $e.t.c$

Putting all these results in the first stated Binomial theorem yields the second stated theorem above.

Example

Expand by Binomial theorem.

a) $\left(x + \frac{1}{x}\right)^5$ *as far as term in* $\frac{1}{x^5}$
b) $\left(3 + \frac{x}{5}\right)^4$ *as far as term in* x^3
c) $\left(4 + \frac{x}{8}\right)^8$ *as far as term in* x^4

Solution

a) $\left(x + \frac{1}{x}\right)^5 = ?$

Using $(a + b)^n = a^n + {}^{n}C_1 a^{n-1} b^1 + {}^{n}C_2 a^{n-2} b^2 + \ldots\ldots\ldots\ldots\ldots\ldots + b^n$

$\Rightarrow \left(x + \frac{1}{x}\right)^5 = x^5 + {}^{5}C_1 x^{5-1}\left(\frac{1}{x}\right)^1 + {}^{5}C_2 x^{5-2}\left(\frac{1}{x}\right)^2 + {}^{5}C_3 x^{5-3}\left(\frac{1}{x}\right)^3 + {}^{5}C_4 x^{5-4}\left(\frac{1}{x}\right)^4 + {}^{5}C_5 x^{5-5}\left(\frac{1}{x}\right)^5$

$= x^5 + \frac{5!}{4!1!}.\frac{x^4}{x} + \frac{5!}{3!2!}.\frac{x^3}{x^2} + \frac{5!}{2!3!}.\frac{x^2}{x^3} + \frac{5!}{1!4!}.\frac{x}{x^4}$

$\therefore \left(x+\frac{1}{x}\right)^5 = x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$

b) $\left(3+\frac{x}{5}\right)^4 = ?$

$\left(3+\frac{x}{5}\right)^4 = (3)^4 + {}^4C_1(3)^{4-1}\left(\frac{x}{5}\right)^1 + {}^4C_2(3)^{4-2}\left(\frac{x}{5}\right)^2 + {}^4C_3(3)^{4-3}\left(\frac{x}{5}\right)^3 + {}^4C_4\left(\frac{x}{5}\right)^4$

$= 81 + \frac{4!}{3!1!}(3)^3\left(\frac{x}{5}\right) + \frac{4!}{2!2!}(3)^2\left(\frac{x}{5}\right)^2 + \frac{4!}{1!3!}(3)\left(\frac{x}{5}\right)^3 + \frac{x^4}{625}$

$\therefore \left(3+\frac{x}{5}\right)^4 = 81 + \frac{108}{5}x + \frac{54}{25}x^2 + \frac{12}{125}x^3$

c) $\left(4+\frac{x}{8}\right)^8 = (4)^8 + {}^8C_1(4)^7\left(\frac{x}{8}\right)^1 + {}^8C_2(4)^6\left(\frac{x}{8}\right)^2 + {}^8C_3(4)^5\left(\frac{x}{8}\right)^3 + {}^8C_4(4)^4\left(\frac{x}{8}\right)^4$

$\therefore \left(4+\frac{x}{8}\right)^8 = 65536 + 16384x + 1792x^2 + 112x^3 + \frac{35}{8}x^4$

The general term of Binomial expansion

Recall: $(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots\ldots + {}^nC_na^{n-n}b^n$

Each term looks like ${}^nC_ra^{n-r}b^r$ where $0 \le r \le n$.

Thus the expression $\boldsymbol{{}^nC_ra^{n-r}b^r}$ is typical of each term in the expansion of $(a+b)^n$. Hence it is called **general term.**

Any specific term can be derived from the general term.

Example:

If $r = 2$, *the general term becomes;*

${}^nC_2a^{n-2}b^2 = \frac{n!}{(n-2)!2!}a^{n-2}b^2$, which is the 3^{rd} *term* of the expansion of $(a+b)^n$.

$\therefore$ The 3^{rd} *term* is obtained by letting $r = 2$.

General deduction

$\boldsymbol{{}^nC_ra^{n-r}b^r}$ represents $(r+1)^{th}$ *term* in the expansion of $(a+b)^n$ *for* $0 \le r \le n$

Example

1. *State the required term in the expansions written in ascending powers of x.*
   a) 10^{th} *term in* $(1+x)^{15}$
   b) 8^{th} *term in* $\left(2-\frac{x}{3}\right)^{12}$
   c) 4^{th} *term in* $\left(2-\frac{3}{x}\right)^8$

Solution

Using $(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{n}}$ expansion with general term $\boldsymbol{{}^nC_ra^{n-r}b^r}$ for $(\boldsymbol{r}+\boldsymbol{1})^{\boldsymbol{th}}$ ***term***

a) $(1+x)^{15}$ gives: $a = 1, b = x, n = 15, r+1 = 10 \Rightarrow r = 9$

$\therefore 10^{th}\ term = {}^{15}C_9(1)^{15-9}x^9 = \frac{15!}{(15-9)!9!}x^9$

$= \frac{151}{6!9!}x^9 = 5005x^9$ $[obtained\ from\ calculator]$

b) $\left(2-\frac{x}{3}\right)^{12}$ gives: $a = 2, b = -\frac{x}{3}, n = 12, r+1 = 8 \Rightarrow r = 7$

$\therefore 8^{th}\ term = {}^{12}C_7(2)^{12-7}\left(-\frac{x}{3}\right)^7 = \frac{12!}{5!7!}(2)^5.\frac{x^7}{(-3)^7}$

$= \frac{792\times32}{-2187}x^7 = -\frac{2816}{243}x^7$

c) $\left(2-\frac{3}{x}\right)^8$ gives: $a = 2, b = -\frac{3}{x}, n = 8, r+1 = 4 \Rightarrow r = 3$

$\therefore 4^{th}\ term = {}^{8}C_3(2)^{8-3}\left(-\frac{3}{x}\right)^3 = \frac{8!}{5!3!}(2)^5.\frac{(-3)^3}{x^3} = -\frac{48384}{x^3}$

Note:

Sometimes a coefficient of term can be required. The general term can be used to state the required coefficient.

The required coefficient of term is derived by finding a suitable *r.*

2. *Find the coefficient of* x^{10} *in the expansion of* $(2x-3)^{14}$.

Solution

From the general term $\boldsymbol{U_{r+1}} = {}^{\boldsymbol{n}}\boldsymbol{C_r a^{n-r} b^r}$ of expansion of $(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{n}}$

For $(2x-3)^{14}$, $a = 2x\,, b = -3\,, n = 14\,, r+1 = ?\Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{14}C_r(2x)^{14-r}(-3)^r = {}^{14}C_r.(2)^{14-r}.(x)^{14-r}.(-3)^r$

Since term in x^{10} is required, then equating $x^{10} = (x)^{14-r}$ gives the value of r which is the key determinant of required coefficient.

$\Rightarrow 10 = 14 - r \ \therefore r = 4$

Thus the required coefficient is in the $5^{th}\ term$ of the expansion of $(2x-3)^{14}$

$\therefore U_5 = {}^{14}C_4.(2)^{14-4}.(x)^{14-4}.(-3)^4 = \frac{14!}{10!4!}.2^{10}.(-3)^4.x^{10}$

Required coefficient $= 1001 \times 2^{10} \times (-3)^4 = 83026944$

3. *Find the coefficient of* x^{-4} *in the expansion of* $\left(2-\frac{3}{x}\right)^8$.

Solution

For $\left(2-\frac{3}{x}\right)^8$, $a = 2\,, b = -\frac{3}{x}\,, n = 8\,, r+1 = ?\Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{8}C_r(2)^{8-r}\left(-\frac{3}{x}\right)^r = {}^{8}C_r.(2)^{8-r}.(x)^{-r}.(-3)^r$

Finding a suitable value of *r;*

Since term in x^{-4} is required, then equating $x^{-4} = x^{-r}$ gives the value of r $.i.e. r = 4$

Thus the required coefficient is in the $5^{th}\ term$ of the expansion of $\left(2-\frac{3}{x}\right)^8$

$\therefore U_5 = {}^{8}C_4.(2)^{8-4}.(x)^{-4}.(-3)^4 = \frac{8!}{4!4!}.2^4.(-3)^4.x^{-4}$

Required coefficient $= \frac{8!}{4!4!}.2^4.(-3)^4 = 90\,720$

4. *Find the coefficient of* a^6x^{14} *in* $(a+x)^{20}$.

Solution

For $(a+x)^{20}$, $A = a\,, B = x, N = 20\,, R+1 = ?\Rightarrow R = ?$

$\Rightarrow U_{R+1} = {}^{20}C_R a^{20-R}x^R$

Finding a suitable value of *R;*

Since term in a^6x^{14} is required, then equating either $x^{14} = x^R$ giving the value of $R = 14$ or $a^6 = a^{20-R} \Rightarrow R = 14\ as\ before$

Thus the required coefficient is in the $15^{th}\ term$ of the expansion of $(a+x)^{20}$

$\therefore U_{15} = {}^{20}C_{14}a^{20-14}x^{14} = {}^{20}C_{14}.\ a^6x^{14}$

$\therefore coefficient = {}^{20}C_{14} = 38\,760$

5. *Write down the coefficient of the term in* x^4y^5 *in the expansion of* $(2x+3y)^9$

Solution

For $(2x+3y)^9$, $a = 2x\,, b = 3y\,, n = 9\,, r+1 = ?\Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{9}C_r(2x)^{9-r}(3y)^r = {}^{9}C_r.(2)^{9-r}.(x)^{9-r}.(3)^r(y)^r$

Finding a suitable value of *r;*

Since term in x^4y^5 is required, then equating either $x^4 = x^{9-r}$ giving the value of $r = 5$ or $y^5 = y^r \Rightarrow r = 5$ *as before*
Thus the required coefficient is in the 6^{th} *term* of the expansion of $(2x+3y)^9$
$\therefore U_6 = {}^9C_5.(2)^{9-5}.(x)^{9-5}.(3)^5(y)^5 = {}^9C_5.(2)^{9-5}.(3)^5.x^4y^5$
$\therefore coefficient = {}^9C_5.(2)^{9-5}.(3)^5 = 489\,988$

6. *Expand* $\left(1+\frac{10}{x}\right)^{10}$ *in terms of descending powers of* x *as far as term in* $\frac{1}{x^3}$.
Find: a) *The* 8^{th} *term*
b) *The coefficient of* x^{-8}

Solution

The expansion is left as an exercise.

a) Using $U_{r+1} = {}^nC_r a^{n-r}b^r$ for $(r+1)^{th}$ *term*
For $\left(1+\frac{10}{x}\right)^{10}$, $a = 1$, $b = \frac{10}{x}$, $n = 10$
Given: $r + 1 = 8 \therefore r = 7$
$\therefore U_8 = {}^{10}C_7(1)^{10-7}\left(\frac{10}{x}\right)^7 = {}^{10}C_7.\frac{10^7}{x^7} = \frac{10!}{7!3!}.\frac{10^7}{x^7}$
$\therefore 8^{th}\ term = \frac{1.2\times10^9}{x^7}$

b) From $U_{r+1} = {}^nC_r a^{n-r}b^r$, $a = 1$, $b = \frac{10}{x}$, $n = 10$, $r + 1 = ? \Rightarrow r = ?$
$U_{r+1} = {}^{10}C_r(1)^{10-r}\left(\frac{10}{x}\right)^r = {}^{10}C_r.10^r.x^{-r}$ $[\because (1)^{10-r} = 1]$
Finding a suitable value of *r;*
Equating $x^{-r} = x^{-8} \therefore r = 8$
$\Rightarrow U_9 = {}^{10}C_8.10^8.x^{-8}$
$\therefore coefficient = {}^{10}C_8.10^8 = \frac{10!}{2!8!}\times10^8 = 4.5\times10^8$

7. *Write down the middle terms of the expansions*
a) $(a+b)^{14}$
b) $(a-b)^{16}$
c) $(1+x)^9$

Solution

a) $(a+b)^{14}$, has $15-$ *terms*. Required term is the $\left(\frac{15+1}{2}\right)^{th}$ $value = 8^{th}$ *term*
$\left[\because For\ odd\ terms, the\ n^{th}\ term\ is\ located\ using\ \left(\frac{n+1}{2}\right)^{th}\ value\right]$
Using $U_{r+1} = {}^nC_r a^{n-r}b^r$ for $(r+1)^{th}$ *term*
$\Rightarrow 8 = r + 1 \therefore r = 7$
$\therefore Middle\ term = 8^{th}\ term = {}^{14}C_7a^{14-7}b^7$
$= {}^{14}C_7a^7b^7 = 3432a^7b^7$

b) $(a-b)^{16}$, has $17-$ $terms(odd)$. Required term is the $\left(\frac{17+1}{2}\right)^{th}$ $value = 9^{th}$ *term*
The $9^{th}\ term = r + 1 \therefore r = 8$
$\Rightarrow U_9 = {}^{16}C_8a^{16-8}(-b)^8 = {}^{16}C_8a^8b^8 = 12\,870a^8b^8$

c) $(1+x)^9$, has $10-$ $terms(even)$. Hence there are $2-$ *middle terms*.
$\left[\because For\ even\ terms, the\ \frac{n^{th}}{2}\ and\ \left(\frac{n^{th}}{2}+1\right)\ are\ the\ middle\ terms\right]$
Required terms are $\left(\frac{10}{2}\right)^{th}$ *and* $\left(\frac{10}{2}+1\right)^{th}$ *terms*
For $5^{th}\ term, r + 1 = 5 \therefore r = 4$
$\Rightarrow 5^{th}\ term = {}^9C_4(1)^{9-4}(x)^4 = {}^9C_4.x^4 = 126x^4$

For 6^{th} $term, r+1=6 \therefore r=5$
$\Rightarrow 6^{th}$ $term = {}^9C_5(1)^{9-5}(x)^5 = {}^9C_5.x^5 = 126x^5$

8. *Find the last three terms of* $(2x+y)^7$

Solution

$(2x+y)^7$ has $8-terms$
Required terms are: $8^{th}, 7^{th}$ and 6^{th} $term$ each corresponding to $r-values$: $7, 6$ and 5 respectively.
$\therefore 8^{th}-term = {}^7C_7(2x)^{7-7}y^7 = y^7$
$\therefore 7^{th}-term = {}^7C_6(2x)^{7-6}y^6 = 7\times 2xy^6 = 14xy^6$
$6^{th}-term = {}^7C_5(2x)^{7-5}y^5 = 21\times(2x)^2y^5 = 84x^2y^5$

9. *Write down the terms involving:*
   a) $x^4\left(\frac{1}{x}\right)^2$
   b) $x^3\left(\frac{1}{x}\right)^3$, in the expansion of $\left(x+\frac{1}{x}\right)^6$

Solution

For $\left(x+\frac{1}{x}\right)^6$ $, a=x, b=\frac{1}{x}, n=6$
$\Rightarrow U_{r+1} = {}^6C_r.x^{6-r}\left(\frac{1}{x}\right)^r = {}^6C_r.x^{6-r}x^{-r}.(1)^r = {}^6C_r.x^{6-2r}.(1)^r$

a) Now term involving $x^4\left(\frac{1}{x}\right)^2 = x^4.x^{-2}.1^2 = x^2$ is equated to x^{6-2r}
$\Rightarrow x^2 = x^{6-2r} \therefore 2 = 6-2r$ $From$ $which$ $r=2$
$\therefore 3^{rd}-term = {}^6C_2.x^{6-2\times2}.(1)^2 = 15x^2$

b) For term involving $x^3\left(\frac{1}{x}\right)^3 = x^3.x^{-3}(1)^3 = x^0$
Equating $x^0 = x^{6-2r} \Rightarrow 0 = 6-2r$ or $r=3$
$\therefore 4^{th}-term = {}^6C_3.x^{6-2\times3}.(1)^3 = 20$

10. *Write down the constant terms in the expansion of:*
    a. $\left(x-\frac{1}{x}\right)^8$
    b. $\left(2x^2-\frac{1}{x}\right)^6$

Solution

A constant term does not contain x. finding the suitable value for r;

a. $\left(x-\frac{1}{x}\right)^8 = {}^8C_r.x^{8-r}.\left(\frac{1}{x}\right)^r = {}^8C_r.x^{8-r}.x^{-r}.(1)^r = {}^8C_r.x^{8-2r}.(1)^r$
For a constant term, $.x^{8-2r} = x^0 \Rightarrow 8-2r=0 \therefore r=4$
The $constant$ $term = 5^{th}$ $term = {}^8C_4.x^{8-2\times4}.(1)^4 = {}^8C_4 = 70$

b. $\left(2x^2-\frac{1}{x}\right)^6 = {}^6C_r.(2x^2)^{6-r}.\left(-\frac{1}{x}\right)^r = {}^8C_r.2^{6-r}.x^{12-2r}.x^{-r}.(-1)^r$
$= {}^8C_r.2^{6-r}.x^{12-3r}.(-1)^r$
For a constant term, $.x^{12-3r} = x^0 \Rightarrow 12-3r=0 \therefore r=4$
The $constant$ $term = 5^{th}$ $term = {}^6C_4.2^{6-4}.x^{12-3\times4}.(-1)^4$
$= {}^6C_4\times 4 = 60$

11. *Find the coefficients of the terms indicated in the expansions below.*
    a) $\left(x+\frac{1}{x}\right)^6$ $, term$ in x^4
    b) $\left(2x+\frac{1}{x}\right)^7$ $, term$ in $\frac{1}{x^5}$
    c) $\left(x-\frac{2}{x}\right)^8$ $, term$ in x^6

d) $\left(2+\frac{x}{2}\right)^{10}$ $, term\ in\ x^7$

Solution

Using $U_{r+1} = {}^{n}C_r a^{n-r} b^r$ for $(r+1)^{th}\ term\ for\ (a+b)^n$

a) $\left(x+\frac{1}{x}\right)^6 = {}^6C_r.x^{6-r}.\left(\frac{1}{x}\right)^r = {}^6C_r.x^{6-r}.x^{-r}.(1)^r$

$= {}^6C_r.x^{6-2r}.(1)^r$

For term in x^4 a suitable value of r is;

$x^4 = x^{6-2r} \Rightarrow 4 = 6-2r \ \therefore r = 1$

$\therefore term\ in\ x^4 = {}^6C_1.x^{6-2\times1}.(1)^1 = 6x^4$

$\therefore coefficient\ in\ x^4 = 6$

b) $\left(2x+\frac{1}{x}\right)^7 = {}^7C_r.(2x)^{7-r}.\left(\frac{1}{x}\right)^r = {}^7C_r.2^{7-r}.x^{7-r}.x^{-r}.(1)^r$

$= {}^7C_r.2^{7-r}.x^{7-2r}.(1)^r$

For term in $\frac{1}{x^5}$ a suitable value of r is;

$x^{-5} = x^{7-2r} \Rightarrow -5 = 7-2r \ \therefore r = 6$

$\therefore term\ in\ \frac{1}{x^5} = {}^7C_6.x^{7-2\times6}.(1)^6 = 14x^{-5}$

$\therefore coefficient\ in\ \frac{1}{x^5} = 14$

c) $\left(x-\frac{2}{x}\right)^8 = {}^8C_r.x^{8-r}.\left(-\frac{2}{x}\right)^r = {}^8C_r.x^{8-r}.(-2)^r x^{-r}$

$= {}^8C_r.(-2)^r.x^{8-2r}$

For term in x^6 a suitable value of r is;

$x^6 = x^{8-2r} \Rightarrow 6 = 8-2r \ \therefore r = 1$

$\therefore term\ in\ x^6 = {}^8C_1.x^{8-2\times1}.(-2)^1 = -16x^6$

$\therefore coefficient\ in\ x^6 = -16$

d) $\left(2+\frac{x}{2}\right)^{10} = {}^{10}C_r.2^{10-r}.\left(\frac{x}{2}\right)^r = {}^{10}C_r.x^r.2^{10-r}.\left(\frac{1}{2}\right)^r$

$= {}^{10}C_r.2^{10-2r}.(1)^r.x^r$

For term in x^7 a suitable value of r is;

$x^7 = x^r \ \therefore r = 7$

$\therefore term\ in\ x^7 = {}^{10}C_7.x^7.2^{10-2\times7} = {}^{10}C_7.2^{-4}.x^7 = \frac{15}{2}x^7$

$\therefore coefficient\ in\ x^7 = \frac{15}{2}$

The binomial expansion of $(1+x)^n$

From $(a+b)^n = a^n + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots\ldots + b^n\ , if\ a = 1\ , b = x$, then;

$(1+x)^n = 1^n + {}^nC_1(1)^{n-1}x^1 + {}^nC_2(1)^{n-2}x^2 + \ldots\ldots + x^n$

$= 1 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + x^n$

$$\boldsymbol{(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots + x^n} \qquad (1)$$

$$\boldsymbol{= 1 + nx + \frac{n(n-1)}{2!}x^2 + + \ldots\ldots + x^n} \qquad (2)$$

NOTE:

1) The expansion of $(1+x)^n$ has $(n+1) - terms$.
2) The term in x^2 is the $3^{rd} - term$, the term in x^3 is the $4^{th} - term$ and the term in x^r is the $(r+1)^{th} - term$.

$$\boxed{\therefore r^{th} - term = \binom{n}{r-1}.x^{r-1} \;= {}^{n}C_r.x^r}$$

3) The coefficient of $x^r = {}^{n}C_r = \binom{n}{r}$
4) The form of the expansion given in (1) is useful when the coefficient of a larger power of x is required, or when general term is required.
 The form of the expansion given in (2) is useful when the first few terms of an expansion are required.
5) The expansion of $(a+x)^n$ where n is a positive integer can be done as;
 $(a+b)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(1+\frac{x}{a}\right)^n$
 Replacing x *by* $\frac{x}{a}$ in the binomial expansion series gives;
 $(a+x)^n = a^n\left[1+\binom{n}{1}\left(\frac{x}{a}\right)+\binom{n}{2}\left(\frac{x}{a}\right)^2+\cdots\ldots\ldots+\binom{n}{r}\left(\frac{x}{a}\right)^r+\cdots\ldots\ldots\ldots\ldots+\binom{n}{n}\left(\frac{x}{a}\right)^n x^n\right]$
 $= a^n + \binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\cdots\ldots\ldots\ldots+\binom{n}{r}a^{n-r}x^r+\cdots\ldots\ldots+\binom{n}{n}a^n x^n$

$$\boxed{\begin{aligned}(a+x)^n &= a^n + \binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\ldots\ldots\ldots\ldots+\binom{n}{r}a^{n-r}x^r+\ldots\ldots\ldots+\binom{n}{n}a^n x^n\\ (a+x)^n &= a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n\end{aligned}}$$

Example:

1) *Write down the first* $3-$ *terms in the expansion in ascending powers of* x
 a) $\left(1-\frac{x}{2}\right)^{10}$
 b) $(3-2x)^8$

Solution

Using $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n$

Replacing x *by* $-\frac{x}{2}$

a) $\left(1-\frac{x}{2}\right)^{10} = 1+10\left(-\frac{x}{2}\right)+\frac{10\times 9}{2!}\left(-\frac{x}{2}\right)^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
 $= 1-5x+\frac{45}{4}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

b) Using $(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n$
 Replacing x *by* $-2x$
 $\Rightarrow (3-2x)^8 = 3^8+8(3)^7(-2x)+\frac{8(7)}{2!}(3)^6(-2x)^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
 $= 3^8-16(3)^7x+112(3)^6x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2) *Write down the binomial expansion in ascending powers of* x *as far as and including the term in* x^2 *for each of the following;*
 a) $(1+x)(1-x)^9$
 b) $(1-x)(1+2x)^{10}$
 c) $(2+x)\left(1-\frac{x}{20}\right)^{20}$
 d) $(1+x)^2(1-5x)^{14}$
 e) $(1+x-2x^2)^8$
 f) $(1-x+2x^2)^5$

Solution

a) $(1+x)(1-x)^9 = ?$

Expanding $(1-x)^9$ as far as x^2

$(1-x)^9 = [1+(-x)]^9 = 1 + 9(-x) + \frac{9(8)}{2!}(-x)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 - 9x + 36x^2 + \ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = (1+x)(1-9x+36x^2)$

$= 1(1-9x+36x^2) + x(1-9x+36x^2)$

$= 1 - 9x + 36x^2 + x - 9x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = 1 - 8x + 27x^2 \quad [terms\ in\ x^3\ are\ ignored]$

b) $(1-x)(1+2x)^{10}$

Expanding $(1+2x)^{10}$ as far as x^2

$(1+2x)^{10} = 1 + 10(2x) + \frac{10(9)}{2!}(2x)^2 + \ldots\ldots\ldots\ldots\ldots$

$= 1 + 20x + 180x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1-x)(1+2x)^{10} = (1-x)(1+20x+180x^2)$

$= 1(1+20x+180x^2) - x(1+20x+180x^2)$

$= 1 + 20x + 180x^2 - x - 20x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = 1 + 19x + 160x^2 \quad [terms\ in\ x^3\ are\ ignored]$

c) $(2+x)\left(1-\frac{x}{20}\right)^{20}$

Expanding $\left(1-\frac{x}{20}\right)^{20}$ as far as x^2

$\left(1-\frac{x}{20}\right)^{20} = \left[1+\left(-\frac{x}{20}\right)\right]^{20} = 1 + 20\left(-\frac{x}{20}\right) + \frac{20(19)}{2!}\left(-\frac{x}{20}\right)^2 + \ldots\ldots\ldots\ldots\ldots$

$= 1 - x + \frac{19}{40}x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = (2+x)\left(1-x+\frac{19}{40}x^2\right)$

$= 2\left(1-x+\frac{19}{40}x^2\right) - x\left(1-x+\frac{19}{40}x^2\right)$

$= 2 - 2x + \frac{38}{40}x^2 + x - x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = 2 - x - \frac{1}{20}x^2$

d) $(1+x)^2(1-5x)^{14}$

Expanding $(1+x)^2$ as far as x^2

$(1+x)^2 = 1 + 2x + \frac{2(1)}{2!}(x)^2 = 1 + 2x + x^2$

Expanding $(1-5x)^{14}$ as far as x^2

$(1-5x)^{14} = [1+(-5x)]^{14} = 1 + 14(-5x) + \frac{14(13)}{2!}(-5x)^2$

$= 1 - 70x + 2275x^2$

$\therefore (1+x)^2(1-5x)^{14} = (1+2x+x^2)(1-70x+2275x^2)$

$= 1(1-70x+2275x^2) + 2x(1-70x+2275x^2) + x^2(1-70x+2275x^2)$

$= 1 - 70x + 2275x^2 + 2x - 140x^2 + x^2 [terms\ in\ x^3\ and\ higher\ power\ terms\ ignored]$

$\therefore (1+x)^2(1-5x)^{14} = 1 - 68x + 2136x^2$

e) $(1+x-2x^2)^8$

$(1+x-2x^2)^8 = [(1+x)-2x^2]^8 = [(1+x)+(-2x^2)]^8$

Using $(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}a^n x^n$

Replacing a *by* $(1+x)$ and x *by* $(-2x^2)$

$[(1+x)+(-2x^2)]^8 = (1+x)^8 + \binom{8}{1}(1+x)^7(-2x^2) + \binom{8}{2}(1+x)^6(-2x^2)^2 + \ldots\ldots$

Now expanding $(1+x)^8$ and $(1+x)^7$ as far as term in x^2

$(1+x)^8 = 1 + 8x + \frac{8(7)}{2!}x^2 = 1 + 8x + 28x^2$

$(1+x)^7 = 1 + 7x + \frac{7(5)}{2!}x^2 = 1 + 7x + 21x^2$

The $3^{rd} - term$ in the expansion of $(1 + x - 2x^2)^8$ yield higher power terms greater than 2

$\therefore (1 + x - 2x^2)^8 = (1 + 8x + 28x^2\ ) + \binom{8}{1}(1 + 7x + 21x^2)(-2x^2)$

$= 1 + 8x + 28x^2 + \binom{8}{1}(-2x^2)$ $[x^3 \ and\ higher\ power\ terms\ neglected]$

$= 1 + 8x + 28x^2 - 16x^2$

$\therefore (1 + x - 2x^2)^8 = 1 + 8x + 12x^2$

f) $(1 - x + 2x^2)^5$

$(1 - x + 2x^2)^5 = [(1 - x) + 2x^2]^8 = (1-x)^5 + \binom{5}{1}(1-x)^4(2x^2) + \ldots\ldots\ldots$

$[other\ terms\ neglected\ because\ yiel\ unwanted\ tems]$

Now expanding $(1 - x)^5$ and $(1 - x)^4$ as far as term in x^2

$(1 - x)^5 = 1 - 5x + \frac{5(4)}{2!}x^2 = 1 - 5x + 10x^2$

$(1 - x)^4 = 1 - 4x + \frac{4(3)}{2!}x^2 = 1 - 4x + 6x^2$

$\therefore (1 - x + 2x^2)^5 = (1 - 5x + 10x^2\ ) + \binom{5}{1}(1 - 4x + 6x^2)(2x^2)$

$= 1 - 5x + 10x^2 + 10x^2(1 - 4x + 6x^2)$

$= 1 - 5x + 10x^2 + 10x^2$

$[x^3\ and\ higher\ power\ terms\ neglected]$

$\therefore (1 - x + 2x^2)^8 = 1 - 5x + 20x^2$

Task

Expand the following as far as term in x^3

a) $(1 - x - x^2)^4$
b) $(3 + x + x^3)^4$
c) $(3 - 2x + 2x^2)^4$
d) $(1 - x + x^2)^8$

ANSWER:

a) $1 - 4x + 2x^2 + 8x^3$
b) $81 + 108x + 54x^2 + 120x^3$
c) $81 - 216x + 324x^2 - 312x^3$
d) $1 - 8x + 36x^2 - 112x^3$

USING BINOMIAL EXPANSION TO FIND APPROXIMATIONS

1) *If x is so small that x^2 and higher power terms can be neglected, show that*

a) $(1 - x)^5\left(2 + \frac{x}{2}\right)^{10} \cong 2^9(2 - 5x)$
b) $(2x + 3)(1 - 2x)^{10} \cong 3 - 58x + 500x^2$
c) $\frac{x+3}{(1+x)^2} \cong 3 - 5x + 7x^2 - 9x^3$
d) $\frac{1+x}{1-x} \cong 1 + 2x + 2x^2 + 2x^3$

Solution

a) $(1 - x)^5\left(2 + \frac{x}{2}\right)^{10} \cong 2^9(2 - 5x)$

Since the **R.H.S** contains terms with highest power as *one* $i.e.\ x^1$, *then, ignoring terms in x^2 and higher power terms*

$(1 - x)^5 = 1 - 5x$

$\left(2+\frac{x}{2}\right)^{10} = 2^{10} + \binom{10}{1}2^9.\left(\frac{x}{2}\right) = 2^{10} + 5.2^9x$

$\therefore (1-x)^5\left(2+\frac{x}{2}\right)^{10} \cong (1-5x)(2^{10}+5.2^9x)$

$\cong 2^9[(1-5x)(2+5x)] \quad \because [2^9\ is\ factorised\ out]$

$\cong 2^9[2+5x-10x] \quad [x^2 - term\ ignored]$

$\therefore (1-x)^5\left(2+\frac{x}{2}\right)^{10} \cong 2^9(2-5x)$

b) $(2x+3)(1-2x)^{10} \cong 3-58x+500x^2$

Since the **R.H.S** contains terms with highest power as *two*. $i.e. x^2$, *then, ignoring terms in* x^3 *and higher power terms*

$(1-2x)^{10} = 1+10(-2x)+\frac{10(9)}{2!}(-2x)^2$

$= 1-20x+180x^2$

$\therefore (2x+3)(1-2x)^{10} \cong (2x+3)(1-20x+180x^2)$

$\cong 2x-40x^2+3-60x+540x^2$

$\therefore (2x+3)(1-2x)^{10} \cong 3-58x+500x^2$

c) $\frac{x+3}{(1+x)^2} \cong 3-5x+7x^2-9x^3$

$\frac{x+3}{(1+x)^2} = (x+3)(1+x)^{-2}$

Expanding $(1+x)^{-2}$ as far as term in x^3 neglecting higher power terms

$\Rightarrow (1+x)^{-2} = 1-2x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3$

$= 1-2x+3x^2-4x^3$

$\therefore \frac{x+3}{(1+x)^2} \cong (x+3)(1-2x+3x^2-4x^3) \cong x-2x^2+3x^3+3-6x+9x^2-12x^3$

$\therefore \frac{x+3}{(1+x)^2} \cong 3-5x+7x^2-9x^3$

d. $\frac{1+x}{1-x} \cong 1+2x+2x^2+2x^3$

$\frac{1+x}{1-x} = (1+x)(1-x)^{-1} \approx 1+2x+2x^2+2x^3$

Expanding $(1-x)^{-1}$ as far as term in x^3 neglecting higher power terms

$\Rightarrow (1-x)^{-1} = 1+x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3 = 1+x+x^2+x^3$

$\therefore \frac{1+x}{1-x} \cong (1+x)(1+x+x^2+x^3)$

$\cong 1+x+x^2+x^3+x+x^2+x^3 \quad [x^4 - term\ neglected]$

$\therefore \frac{1+x}{1-x} \cong 1+2x+2x^2+2x^3$

NOTE:

In *all* the above examples on binomial theorem, the power on the brackets is an integer not a rational number. $i.e.$ of the form $\frac{p}{q}$.

EXPANSION OF $(1+x)^n\ if\ |x|<1$ **AND n IS RATIONAL**

$(.\boldsymbol{i.e.\ of\ the\ form\ p/q})$

The binomial theorem used is;

$$\boxed{(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ \ldots\ldots\ldots\ldots}$$

Note:

$|x|<1\ means -1<x<+1\ (.i.e. validity\ range\ of\ x\ for\ expansion)$

Application

- Expansion of expressions
- Finding the $n^{th}\ root$ of a number
- Finding errors in numbers

Examples

1) *Expand* $(1-3x)^{\frac{1}{4}}$ *in ascending powers of* x *as far as the term in* x^3. *Hence evaluate* $\sqrt[4]{13}$ *correct to* $3\ s.f.$

Solution

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots$ for $|x| < 1$

$\Rightarrow (1-3x)^{\frac{1}{4}} = 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{3}\right)}{3!}(-3x)^3 + \ldots.$

$= 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3$

$(1-3x)^{\frac{1}{4}}$ is valid for expansion if $|-3x| < 1\ or\ |3x| < 1\ or -1 < 3x < 1$

$-\frac{1}{3} < x < \frac{1}{3}$ is the validity interval for expansion.

Since the value of x to be used to estimate $\sqrt[4]{13}$ in the expansion in the range $-\frac{1}{3} < x < \frac{1}{3}$,then

$\sqrt[4]{13} = \sqrt[4]{(16-3)}$ $[\because 13\ is\ expressed\ as\ a\ differencce\ of\ a\ no.with\ exact\ 4^{th} root]$

$= \sqrt[4]{(16-3)} = \sqrt[4]{16\left(1-\frac{3}{16}\right)} = 2\left(1-\frac{3}{16}\right)^{\frac{1}{4}}$

Now expressing x as a rational for the interval $-\frac{1}{3} < x < \frac{1}{3}$

Comparing $\sqrt[4]{\left(1-\frac{3}{16}\right)}$ $with\ \sqrt[4]{(1-3x)}$

$\Rightarrow \frac{3}{16} \equiv 3x\ \therefore x = \frac{1}{16}$, which is in the range $-\frac{1}{3} < x < \frac{1}{3}$

From $\sqrt[4]{(1-3x)} = 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3$, for $x = \frac{1}{16}$, then

$\sqrt[4]{\left(1-\frac{3}{16}\right)} \cong 1 - \frac{3}{4}\left(\frac{1}{16}\right) - \frac{27}{32}\left(\frac{1}{16}\right)^2 - \frac{189}{128}\left(\frac{1}{16}\right)^3 \cong 0.94946812$

$\therefore \sqrt[4]{13} = 2 \times 0.94946812 \approx 1.90$

2) *Expand* $(1-16x)^{\frac{1}{4}}$ *as far as* x^3. *Hence evaluate* $\sqrt[4]{39}$ *correct to* $5.s.f.$ *Take* $x = \frac{1}{10000}$

Solution

$(1-16x)^{\frac{1}{4}} = 1 + \frac{1}{4}(-16x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-16x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{3}\right)}{3!}(-16x)^3 + \ldots.$

$= 1 - 4x - 24x^2 - 224x^3$

$\Rightarrow \left(1-\frac{16}{10000}\right)^{\frac{1}{4}} = 1 - \frac{4}{10000} - \frac{24}{10000^2} - \frac{224}{10000^3}$

$\left(\frac{9984}{10000}\right)^{\frac{1}{4}} \cong 0.99996$

$\frac{\sqrt[4]{256\times39}}{10} \cong 0.99996\ \therefore \sqrt[4]{39} \cong \frac{0.99996\times10}{\sqrt[4]{254}} \cong \frac{0.99996\times10}{4}$

$\therefore \sqrt[4]{39} \cong 2.4999$

NOTE:

1) In general, if x is small so that successive powers of x quickly become negligible in value, then the sum of the first few terms in the expansion of $(1+x)^n$ gives the approximate value for $(1+x)^n$. The number of terms required to obtain a good approximation depends on;
   a) The value of x (the smaller x is, the fewer are the terms needed to obtain a good approximation)
   b) The accuracy required (an answer correct to 3 s.f needs fewer terms than the answer to 6 s.f)
2) When finding an approximation, the binomial expansion of $(1+x)^n$ and not $(a+x)^n$ should be used if n is rational.$e.g.$ Find an approximate value of $(3.006)^{\frac{1}{5}}$. here use $3^{\frac{1}{5}}(1+0.002)^{\frac{1}{5}}$
3) *Expand* $(1-x)^{\frac{1}{3}}$ *in ascending powers of x as far as the fourth term. Taking* $x=\frac{1}{1000}$ *, find* $\sqrt[3]{37}$ *to* $6.sf$.

Solution

$(1-x)^{\frac{1}{3}}=[1+(-x)]^{\frac{1}{3}}=1+\frac{1}{3}(-x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(-x)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-x)^3$

$=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$

Putting $x=\frac{1}{1000}$

$\Rightarrow\left(1-\frac{1}{1000}\right)^{\frac{1}{3}}=1-\frac{1}{9\times10^6}-\frac{5}{81\times10^9}=\frac{\sqrt[3]{999}}{10}=\frac{2998}{2999}$

$\therefore\sqrt[3]{999}=\sqrt[3]{27\times37}=\frac{2998}{2999}$

$\therefore\sqrt[3]{37}=\frac{29980}{2999\times\sqrt[3]{27}}\cong3.33222$

4) *Expand* $(1+8x)^{\frac{1}{2}}$ *as far as* x^2. *Hence find* $\sqrt{3}$ *to* $3.p$.

Solution

$(1+8x)^{\frac{1}{2}}\cong1+\frac{1}{2}(8x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(8x)^2$

$\cong1+4x-8x^2$

Since the expansion is valid for $|8x|<1\ or-1<8x<+1\ or-\frac{1}{8}<x<\frac{1}{8}$, x' s suitable substitution should be selected to find $\sqrt{3}$ but with in this interval.

From $\sqrt{3}=\sqrt{(4-1)}=2\sqrt{\left(1-\frac{1}{4}\right)}$

Comparing $1+8x\equiv1-\frac{1}{4}\Rightarrow x=-\frac{1}{32}$

From $(1+8x)^{\frac{1}{2}}\cong1+4x-8x^2$, for $x=-\frac{1}{32}$

$\sqrt{\left(1-\frac{1}{4}\right)}\cong1-\frac{1}{8}-\frac{8}{32^2}$

$\therefore\frac{\sqrt{3}}{2}\cong\frac{111}{128}\ or\ \sqrt{3}\cong\frac{111}{64}$

Finding suitable value of x for substitution

Let $x=\frac{1}{a}$, where 'a' has an exact square root

$\Rightarrow1+8x=1+\frac{8}{a}=\frac{a+8}{a}=3N$, N = no. also with exact sq. root

Using try and error method

For $a = 4$, $1 + 8x = \frac{4+8}{4} = \frac{12}{4} = \frac{3\times4}{4}$. This is convenient but $-\frac{1}{8} < x < \frac{1}{8}$, hence $x = \frac{1}{4}$ is not in the range hence rejected.

For $a = 16$, $1 + 8x = \frac{16+8}{16} = \frac{24}{16} = \frac{3\times8}{16}$, 8 has no exact square root

For $a = 25, 1 + 8x = \frac{25+8}{25} = \frac{33}{25} = \frac{3\times11}{25}$, 25 has no exact square root

For $a = 100, 1 + 8x = \frac{100+8}{100} = \frac{108}{100} = \frac{3\times36}{100}$, 36 has an exact square root

$\therefore x = \frac{1}{100}\ for\ -\frac{1}{8} < x < \frac{1}{8}$ can be use.

From $(1+8x)^{\frac{1}{2}} \cong 1 + 4x - 8x^2\ , x = \frac{1}{100}$

$\left(1 + \frac{8}{100}\right)^{\frac{1}{2}} \cong 1 + \frac{4}{100} - \frac{8}{100^2}$

$\frac{\sqrt{108}}{10} \cong 1.0392 \therefore \sqrt{3} \cong \frac{10.392}{\sqrt{36}} \cong 1.732$

5) *Expand* $(1-3x)^{\frac{1}{4}}$ *in ascending powers of x up to the term in* x^3. *Hence evaluate* $\sqrt[4]{13}$

Solution

$(1-3x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-3x)^3$

$\cong 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{819}{128}x^3$

Now to find $\sqrt[4]{13}$, we need a suitable value of x-value in the range $|-3x| < 1\ or\ -\frac{1}{3} < x < \frac{1}{3}$

Br try and error method

Let $x = \frac{1}{a}\ , a = no. with\ exact\ 4^{th} root$

$\Rightarrow 1 - 3x = 1 - \frac{3}{a} = \frac{a-3}{a} = \frac{13N}{a}\ . N = no. with\ exact\ 4^{th} root$

For $a = 16\ , 1 - 3x = \frac{16-3}{16} = \frac{13}{16} = \frac{13(1)^4}{16}$

Hence $x = \frac{1}{16}$ can be used

$(1-3x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-3x)^3$

$\left(1 - \frac{3}{16}\right)^{\frac{1}{4}} \cong 1 - \frac{3}{4}\left(\frac{3}{16}\right) - \frac{27}{32}\left(\frac{3}{16}\right)^2 - \frac{819}{128}\left(\frac{3}{16}\right)^3$

$\cong 0.9495$

$\therefore \sqrt[4]{13} \cong 0.9495 \times \sqrt[4]{16} \cong 1.899$

6) *Expand* $(1-x)^{\frac{1}{3}}$ *in ascending powers of x as far as* 4^{th} *term. By taking the first two terms of x find the value of* $\sqrt[3]{37}$ *, correct to* $6\ s.f.$

Solution

$(1-x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(-x)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-x)^3$

$\cong 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$

Finding a suitable value of x for approximation of $\sqrt[3]{37}$

Let $x = \frac{1}{a}$, $a = no. with\ exact\ cube\ root$

$\Rightarrow 1 - x = 1 - \frac{1}{a} = \frac{a-1}{a} = \frac{37N}{a}$. $N = no. with\ exact\ cube\ root$

For $a = 8$, $1 - x = \frac{8-1}{8} = \frac{7}{8}$, this does not contain 37

For $a = 64$, $1 - x = \frac{63}{64}$, which does not contain 37

⋮

For $a = 1000$, $1 - x = \frac{999}{1000} = \frac{27 \times 37}{1000}$, this holds

Taking $x = \frac{1}{1000}$ in the interval $-1 < x < 1$

$(1-x)^{\frac{1}{3}} \cong 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$

$\left(1 - \frac{1}{1000}\right)^{\frac{1}{3}} \cong 1 - \frac{1}{3}\left(\frac{1}{1000}\right) \cong \frac{2999}{3000} \quad \cong \frac{\sqrt[3]{999}}{10} \cong \frac{2999}{3000}$

$\sqrt[3]{37 \times 27} \cong \frac{2999}{300}$

$\therefore \sqrt[3]{37} \cong \frac{2999}{300} \times \frac{1}{3} \cong 3.33222$

NOTE:

Examples (4), (5) and (6) illustrate how you can use try and error method to obtain a suitable value of x for approximation of n^{th} root using binomial expansion.

See example below.

7) *Expand* $(1+x)^{\frac{1}{3}}$ *up to term in* x^2. *Hence evaluate* $\sqrt[3]{9}$ *correct to* $3\ d.p.$

Solution

$(1+x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2 \quad \cong 1 + \frac{1}{3}x - \frac{1}{9}x^2$

To find $\sqrt[3]{9}$, a suitable x value must be got.

Let $x = \frac{1}{a}$, $a = no. with\ exact\ cube\ root$

$\Rightarrow 1 + x = 1 + \frac{1}{a} = \frac{a+1}{a} = \frac{9N}{a}$. $N = no. with\ exact\ cube\ root$

For $a = 8$, $1 + x = \frac{8+1}{8} = \frac{9}{8} = \frac{9(1)^3}{8}$, hence $x = \frac{1}{8}$ can be used

$(1+x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}x - \frac{1}{9}x^2$

$\left(1 + \frac{1}{8}\right)^{\frac{1}{3}} \cong 1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 \cong \frac{601}{576}$

$\therefore \sqrt[3]{9} \cong \frac{601}{576} \times 2 \cong 2.087$

Task:

Expand $(1-3x)^{\frac{1}{3}}$ in ascending powers of x a far as the term in x^3. Hence evaluate $\sqrt[3]{5}$

Hint: $x = \frac{1}{8}$

More examples on Binomial theorem

1. *Use binomial theorem to find*
   a) $\sqrt{1.001}\ to\ 6.p$
   b) $\frac{1}{(1.02)^2}\ to\ 4d.p$
   c) $\sqrt{0.998}\ to\ 6.p$
   d) $\frac{1}{\sqrt{0.98}}\ to\ 4d.p$

Solution

a) $\sqrt{1.001} = (1+0.001)^{\frac{1}{2}}\,; let\ x = 0.001$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now $x = 0.001$

$\Rightarrow (1+0.001)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(0.001) - \frac{1}{8}(0.001)^2 + \frac{1}{16}(0.001)^3 \cong 1.000500$

b) $\frac{1}{(1.02)^2} = \frac{1}{(1.02)^{-2}} = (1+0.02)^{-2}\,; let\ x = 0.02$

$(1+x)^{-2} \cong 1 - 2x + \frac{-2(-3)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 \cong 1 - 2x + 3x^2 - 4x^3$

Now $x = 0.02$

$\Rightarrow (1+0.02)^{-2} \cong 1 - 2(0.02) + 3(0.02)^2 - 4(0.02)^3$

$\therefore \frac{1}{(1.02)^2} \cong 0.9612$

c) $\sqrt{0.998} = \sqrt{1-0.002} = (1-0.002)^{\frac{1}{2}}$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now $x = -0.002$

$\Rightarrow (1-0.002)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.002) - \frac{1}{8}(-0.002)^2 + \frac{1}{16}(-0.002)^3$

$\therefore \sqrt{0.998} \cong 0.998999$

d) $\frac{1}{\sqrt{0.98}} = (0.98)^{-\frac{1}{2}} = (1-0.02)^{-\frac{1}{2}}\,; \text{let } x = -0.02$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now $x = -0.02$

$\Rightarrow (1-0.02)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.02) - \frac{1}{8}(-0.02)^2 + \frac{1}{16}(-0.02)^3$

$\therefore \frac{1}{\sqrt{0.98}} \cong 1.0102$

Task:

Use binomial theorem to find $\sqrt[3]{1.03}$ *correct to 5s.f*

2. *Use Binomial theorem to evaluate*
   a) $\sqrt{23}$ to 3 d.p
   b) $\sqrt{37}$ to 5 s.f
   c) $\sqrt[3]{8.01}$ to 5 s.f
   d) $\sqrt[3]{64.08}$ to 5 s.f

Solution

a) $\sqrt{23} = \sqrt{25-2} = \sqrt{25\left(1-\frac{2}{25}\right)} = 5\left(1-\frac{2}{25}\right)^{\frac{1}{2}}$

From $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$$\therefore \sqrt{23} \cong 5\left[1+\frac{1}{2}\left(-\frac{2}{25}\right)-\frac{1}{8}\left(-\frac{2}{25}\right)^2\right] \quad [\text{higher power terms ignored}]$$
$$\cong 4.796$$

NOTE:
$\left|-\frac{2}{25}\right| < 1 \,.\, i.e. -1 < \frac{2}{25} < 1$

b) $\sqrt{37} = \sqrt{36+1} = \sqrt{36\left(1+\frac{1}{36}\right)} = 6\left(1+\frac{1}{36}\right)^{\frac{1}{2}}$

$$\therefore \sqrt{37} \cong 6\left[1+\frac{1}{2}\left(\frac{1}{36}\right)-\frac{1}{8}\left(\frac{1}{36}\right)^2\right] \quad [\text{higher power terms ignored}] \cong 6.0828$$

NOTE:
$\sqrt{36+1}$, could not be expanded directly by binomial because $|36| > 1$ yet the required interval of $(1+x)^n$ expansion is $|x| < 1$.

c) $\sqrt[3]{8.01} = \sqrt[3]{8+0.01} = \sqrt[3]{8\left(1+\frac{0.01}{8}\right)} = 2\left(1+\frac{0.01}{8}\right)^{\frac{1}{3}}$

$$\cong 2\left[1+\frac{1}{3}\left(\frac{0.01}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{0.01}{8}\right)^2\right]$$

$\therefore \sqrt[3]{8.01} \cong 2\left[1+\frac{0.01}{24}\right] \cong 2.0008$

d) $\sqrt[3]{64.08} = \sqrt[3]{64+0.08} = \sqrt[3]{64\left(1+\frac{0.08}{64}\right)} = 4\left(1+\frac{0.08}{64}\right)^{\frac{1}{3}}$

$$\therefore \sqrt[3]{64.08} \cong 4\left[1+\frac{0.08}{64\times 3}\right] \quad [\text{higher power terms ignored}] \cong 4.0017$$

Proofs in Binomial expansion

1. *Show that, if x is small enough for terms in x^3 and higher power terms neglected,*

$$\sqrt{\left(\frac{1-x}{1+x}\right)} \cong 1-x+\frac{x^2}{2}$$

By putting $x = \frac{1}{8}$, show that $\sqrt{7} \cong 2\frac{83}{128}$

Solution

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = [(1-x)(1+x)^{-1}]^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$(1-x)^{\frac{1}{2}} \cong 1+\frac{1}{2}(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-x)^2$$
$$\cong 1+\frac{1}{2}x-\frac{1}{8}x^2$$

$$(1+x)^{-\frac{1}{2}} \cong 1-\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2$$
$$\cong 1-\frac{1}{2}x+\frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1-x}{1+x}\right)} \cong \left[1+\frac{1}{2}x-\frac{1}{8}x^2\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2\right]$$
$$\cong 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2 \cong 1-x+\frac{x^2}{2}$$

$$\left[\begin{array}{l}\text{expansion of brackets yield terms in } x^3 \text{ and} \\ \text{higher power terms which have been ignored}\end{array}\right]$$

Now $x = \frac{1}{8}$

$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} \cong 1-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{8}\right)^2 \quad \cong \sqrt{\left(\frac{7}{9}\right)} \cong \frac{113}{128}$

$\therefore \sqrt{7} \cong 3 \times \frac{113}{1128} = 2\frac{83}{128}$

Task:

If x is so small that its cube and higher powers can be neglected, show that $\sqrt{\left(\frac{1+x}{1-x}\right)} \cong 1+x+\frac{1}{2}x^2$ *. By putting* $x=\frac{1}{17}$*, show that* $\sqrt{2} \cong 1\frac{359}{867}$

2. *Show that the first* $3-$ *terms in the expansion in ascending powers of x of* $(1+8x)^{\frac{1}{4}}$ *are the same as the first* $3-$ *terms in the expansion* $\frac{1+5x}{1+3x}$*. Use the corresponding approximation* $(1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x}$ *to obtain approximation to* $(1.16)^{\frac{1}{4}}$ *as a rational fraction in its lowest form.*

Solution

$(1+8x)^{\frac{1}{4}} \cong 1+\frac{1}{4}(8x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(8x)^2$

$\cong 1+2x-6x^2$

$\frac{1+5x}{1+3x} \cong (1+5x)(1+3x)^{-1}$

$\cong \left[1+(-1)(3x)+\frac{(-1)(-2)}{2!}(3x)^2\right](1+5x)$

$\cong (1-3x+9x^2)(1+5x)$

$\cong 1-3x+9x^2+5x-15x^2$

$\cong 1+2x-6x^2$

$\therefore (1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$

Using $(1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$

$(1.16)^{\frac{1}{4}} = (1+0.16)^{\frac{1}{4}}$

By comparison, $8x = 0.16 \ \therefore x = 0.02$

$\Rightarrow (1+0.16)^{\frac{1}{4}} \cong \frac{1+5(0.02)}{1+3(0.02)}$

$\cong \frac{1.1}{1.06} = \frac{59}{53}$

3. *a) It is given that* $f(x)=\frac{1}{(1+x)^2}+\sqrt{4+x}$*. Show that if* x^3 *and higher powers are ignored,* $f(x) \cong a+bx+cx^2$ *, and find the values of a, b and c*

b) Show that $3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$*. Hence evaluate* 3^6

c) Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{\frac{1}{4}}$ *are the same as the first 3-terms in the expansion of* $\frac{1+ax}{1+bx}$*. Find the values of a and b. hence find an approximation to* $(0.6)^{\frac{1}{4}}$ *in the form* $\frac{p}{q}$*.*

Solution

a) $f(x)=\frac{1}{(1+x)^2}+\sqrt{4+x} = (1+x)^{-2}+\sqrt{4\left(1+\frac{x}{4}\right)}$

$(1+x)^{-2} \cong 1+(-2)x+\frac{(-2)(-3)}{2!}x^2 \cong 1-2x+3x^2$

$$\sqrt{4\left(1+\frac{x}{4}\right)} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} \cong 2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2\right]$$

$$\cong 2\left[1+\frac{x}{8}-\frac{1}{128}x^2\right] \cong 2+\frac{x}{4}-\frac{1}{64}x^2$$

$\therefore\ f(x) \cong 1-2x+3x^2+2+\frac{x}{4}-\frac{1}{64}x^2$

$\cong 3-x+\frac{191}{64}x^2$

$3-x+\frac{191}{64}x^2 \equiv a+bx+cx^2$, *comparing corresponding coefficients;*

$\therefore a=3\,, b=-1\,, c=\frac{191}{64}$

b) $3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$

$3^n = (1+2)^n$; By binomial theorem

$3^n = 1+{}^nC_1.(2)^1+{}^nC_2.(2)^2+\ldots\ldots\ldots\ldots\ldots+{}^nC_n(2)^n$

$3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$

For $n=6, 3^6 = \binom{6}{0}+(2)^1\binom{6}{1}+(2)^2\binom{n}{2}+\cdots\ldots\ldots+2^6\binom{6}{6}$

$3^6 = 1+12+60+8.{}^6C_3+16.{}^6C_4+32.{}^6C_5+64.{}^6C_6$

$= 729$

c) $(1-8x)^{\frac{1}{4}} \cong \frac{1+ax}{1+bx}$

$(1-8x)^{\frac{1}{4}} \cong 1+\frac{1}{4}(-8x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(-8x)^2$

$\cong 1-2x-6x^2$

$\frac{1+ax}{1+bx} \cong (1+ax)(1+bx)^{-1}$

$\cong \left[1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2\right](1+ax)$

$\cong (1-bx+b^2x^2)(1+ax)$

$\cong 1-bx+b^2x^2+ax-abx^2$

$[terms\ in\ x^3\ ignored\ \because do\ not\ make\ the\ 1^{st}\ 3-terms]$

$\cong 1+(a-b)x+(b^2-ab)x^2$

$\therefore 1-2x-6x^2 \equiv 1+(a-b)x+(b^2-ab)x^2$

By comparison of coefficients of corresponding terms,

For x: $-2=a-b$ ………………………………………………………… (1)

For x^2: $-6=b^2-ab=-b(a-b)$……………………………… (2)

$Eqn.(1) \div eqn.(2)$

$-\frac{2}{-6}=\frac{(a-b)}{-b(a-b)}=\frac{1}{-b}\ \therefore b=-3$

From (1) $a=b-2=-3-2=-5$

$\therefore (1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2$

$(0.6)^{\frac{1}{4}}=(1-0.4)^{\frac{1}{4}}=(1-8x)^{\frac{1}{4}}$

$\Rightarrow 0.4=8x\ \therefore x=0.05$

From $(1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2$, $For\ x=0.05$

$(0.6)^{\frac{1}{4}} \cong \frac{1-5(0.05)}{1-3(0.05)}=\frac{0.75}{0.85}=\frac{15}{17}$

Task:

1. *Expand* $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ *as a series of ascending powers of x up to and including terms in* x^2. *Using* $x = \frac{1}{100}$, *find an approximation of* $\sqrt{51}$ *stating the number of significant figures to which your answer is accurate.* **ANSWER**: $\mathbf{7.141}$ $(4\ s.f)$
2. *Find the first three terms of the binomial expansion* $\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}}$. *Hence evaluate* $\frac{10}{\sqrt{15}}$ *correct to 3d.p.*

   HINT:

   $$\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}} = 2(1+x)\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \cong 2(1+x)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2\right]$$

   $$\cong 2(1+x)\left[1+\frac{x}{8}+\frac{3}{128}x^2\right]$$

   $$\cong 2\left[1+\frac{x}{8}+\frac{3}{128}x^2+x+\frac{x^2}{8}\right]$$

   $$\cong 2\left(1+\frac{9}{8}x+\frac{19}{128}x^2\right)$$

   Now finding suitable value of x for evaluation of $\frac{10}{\sqrt{15}}$

   $$\Rightarrow \frac{2(1+x)}{\sqrt{1-\frac{x}{4}}} = \frac{10}{\sqrt{15}} \quad or \quad \frac{(1+x)}{\sqrt{1-\frac{x}{4}}} = \frac{5}{\sqrt{15}}$$

   $$\Rightarrow \frac{(1+x)^2}{1-\frac{x}{4}} = \frac{25}{15} = \frac{5}{3} \ or \ \frac{4(1+x)^2}{4-x} = \frac{5}{3}$$

   $$\Rightarrow 12+24x+12x^2 = 20-5x \ or \ 12x^2+29x-8=0$$

   Solving $x = \frac{1}{4}, -\frac{8}{3}$

   Since the expansion $\sqrt{1-\frac{x}{4}}$ is valid for $\left|\frac{x}{4}\right| < 1$ *or* $|x| < 4$, then taking $x = \frac{1}{4}$

   $$\Rightarrow \frac{2\left(1+\frac{1}{4}\right)}{\sqrt{1-\frac{1}{16}}} \cong 2\left[1+\frac{9}{8}\left(\frac{1}{4}\right)+\frac{19}{128}\left(\frac{1}{4}\right)^2\right]$$

   $$\Rightarrow \frac{\frac{10}{4}}{\sqrt{\left(\frac{15}{16}\right)}} \cong \frac{264\times 2}{2048}$$

   $$\therefore \frac{10}{\sqrt{15}} \cong 2.581$$
3. *Determine the binomial expansion of* $\left(1+\frac{x}{2}\right)^4$. *Hence evaluate* $(2.1)^4$ *correct to 2d.p* (***Uneb* 2005**)

   HINT:

   $$\left(1+\frac{x}{2}\right)^4 = 1+2x+\frac{3}{2}x^2+\frac{x^3}{2}$$

   $$(2.1)^4 = (2+0.1)^4 = \left[2\left(1+\frac{0.1}{2}\right)\right]^4 = 16\left(1+\frac{0.1}{2}\right)^4$$

   But $\frac{x}{2} = \frac{0.1}{2} \therefore x = 0.1$

   $$\Rightarrow 16\left(1+\frac{0.1}{2}\right)^4 \cong 16\left(1+2(0.1)+\frac{3}{2}(0.1)^2+\frac{(0.1)^3}{2}\right) \cong 19.448$$
4. *Expand* $\sqrt{\left(\frac{1+5x}{1-5x}\right)}$ *as far as and including the term in* x^3. *Taking the first three terms and* $x = \frac{1}{9}$ *evaluate* $\sqrt{14}$

5. *Show that* $\sqrt{(1+4ax)} \cong 1 + 2ax - 2a^2x^2 + 4a^3x^3 + \ldots\ldots$*, and deduce the expansion for* $\sqrt{(1-4x)}$ *up to the term in* x^4*. Letting* $x = \frac{1}{10}$*, evaluate* $\sqrt{6}$ *to* $5\ d.p.$
   HINT: $compairing\sqrt{(1+4ax)}\ with\ \sqrt{(1-4x)}\ , a = -1$
6. *Expand* $(1-x)^{\frac{1}{3}}$ *in ascending powers of x up to term in* x^3*. Hence evaluate* $\sqrt[3]{998}$*.*
   Hint: $\sqrt[3]{998} = \sqrt[3]{(1000-2)} = \sqrt[3]{1000(1-0.002)} = 10\sqrt[3]{(1-0.002)}$
7. *If x is sufficiently small and allow any terms in* x^5 *or higher power terms be neglected, show that* $(1+x)^6(1-2x^3)^{10} \cong 1 + 6x + 15x^2 - 105x^4$
8. *Expand using binomial theorem* $(1+4x)^{\frac{1}{2}}$ *up to the fourth term. Hence evaluate* $\sqrt{6}$ *to* $3\ d.p$ *using* $x = \frac{1}{100}$
9. *a) Expand* $\sqrt{\left(\frac{1+x}{1-x}\right)}$ *in ascending powers of x to a term in* x^2*.*
   b) (i) Using the expansion of $\sqrt{(1+x)}$ *up to the term in* x^3*, find the value of* $\sqrt{1.08}$ *to* $4\ d.p$
   (ii) Express $\sqrt{1.08}$ *in the form* $\frac{a}{b}\sqrt{c}$*. Hence evaluate* $\sqrt{3}$ *to* $3\ s.f.$ (***Uneb* 2010**)
10. *show that if x is so small in comparison with unity that* x^3 *and higher power terms can be neglected,* $\frac{(1-4x)^{\frac{1}{2}}.(1+3x)^{\frac{1}{3}}}{\sqrt{1+x}} \cong 1 - \frac{3}{2}x - \frac{33}{8}x^2$

PERCENTAGES AND ERRORS IN BINOMIAL APPROXIMATION

The knowledge of measuration is applicable.

Examples

1) *The radius of a cylinder is reduced by 3% and its height increased by 4%. Determine the appropriate percentage change in;*
   a) Its volume
   b) In curved surface area
   (Neglect product of small changes)

Solution

a) Let r and h be the original values of radius and height respectively

Volume of cylinder, $V = \pi r^2 h$

New values of r and h are respectively; $\left(\frac{100-3}{100}\right)r$, $\left(\frac{100+4}{100}\right)h$

New volume, $V' = \pi\left[\left(\left(\frac{100-3}{100}\right)r\right)^2\left(\frac{100+4}{100}\right)h\right]$

$= \pi(1-0.03)^2r^2.(1+0.04)h$

$= \pi r^2h.(1-0.03)^2.(1+0.04)$

But $(1-0.03)^2 \cong 1 + 2(-0.03)$

$\cong 1 - 0.06\ \ (Neglecting\,product\ of\ small\,changes)$

$\therefore V' = \pi r^2h.(1-0.06).(1+0.04)$

$= \pi r^2h.[1 + 0.04 - 0.06]\ \ (Neglecting\,product\ of\ small\,changes)$

$= 0.98\pi r^2h$

$\%ge\ change\ in\ V = \left(\frac{V-V'}{V}\right) \times 100$

$= \left(\frac{\pi r^2h - 0.98\pi r^2h}{\pi r^2h}\right) \times 100 = 2\%$

$\therefore$ *Volume reduced by* 2%

b. Curved surface area of a cylinder, $A = 2\pi rh$

New curved surface area, $A' = 2\pi\left[\left(\frac{100-3}{100}\right)r.\left(\frac{100+4}{100}\right)h\right]$

$\Rightarrow A' = 2\pi rh.(1-0.03).(1+0.04)$

$= 2\pi rh[1 + 0.04 - 0.03]$ [*Products of higher smaller terms ignnored*]

$\therefore A' = 2\pi rh.(1.01)$

$\%ge\ change\ in\ A = \left(\frac{A-A'}{A}\right) \times 100$

$= \left(\frac{2\pi rh.(1.01) - 2\pi rh}{2\pi rh}\right) \times 100 = 1\%$

$\therefore$ *Curved surface area increased by* 1%

2) *Pressure P and volume V are elated by the expression* $PV^3 = C$ *where C is a constant. Find the appropriate percentage change in C when P is increased by 2% and volume decreased by 0.8%.*

Solution

Let P and V be original values of pressure and volume respectively.

New value of $P = \left(\frac{100+2}{100}\right)P = (1+0.02)P$

New value of $V = \left(\frac{100-0.8}{100}\right)V = (1-0.008)V$

Now new value of $C = C' = (1+0.02)P[(1-0.008)V]^3$

$\Rightarrow C' = PV^3.(1+0.02).(1-0.008)^3$

But $(1-0.008)^3 \cong 1 - 3(0.008)$, *higher power terms ignored*

$\therefore C' = PV^3.(1+0.02).\big(1-3(0.008)\big)$

$= PV^3.[1 - 3(0.008) + 0.02]$, *products of smaller terms ignored*

$= PV^3.(0.996)$

$\therefore \%ge\ change\ in\ C = \left(\frac{C-C'}{C}\right) \times 100 = \left(\frac{PV^3 - PV^3.(0.996)}{PV^3}\right) \times 100$

$\therefore \%ge\ change\ in\ C\ (Reduction) = 0.4\%$

3) *An error is made in measuring the radius of a sphere. Find the percentage error in surface area.*

Solution

Let the error lead to increase in radius.

Area of sphere, $A = 4\pi r^2$ where A and r are original values.

New value of $r = \left(1 + \frac{2}{100}\right)r = (1+0.02)r$

New value of $A = A' = 4\pi[(1+0.02)r]^2$

$= 4\pi r^2(1+0.02)^2$

$= 4\pi r^2\big(1+2(0.02)\big) = 4\pi r^2(1.04)$

$\therefore \%ge\ change\ in\ A = \left(\frac{A-A'}{A}\right) \times 100$

$= \left(\frac{4\pi r^2(1.04) - 4\pi r^2}{4\pi r^2}\right) \times 100 = 4\%$

NOTE:

If error was to lead to a decrease in r you get the same percentage change 4%.

4) *An error of* $2\frac{1}{2}\%$ *is made in measurement of the area of a circle. What percentage error results in the radius and circumference?*

Solution

Area $A = \pi r^2$, where A and r are original values of area and radius of circle respectively.

New value of $A = \left(\frac{100+2.5}{100}\right)A = (1+0.025)A$ $[let\ error\ lead\ to\ increase\ in\ A]$

From $A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$

Now new value of $r = \sqrt{\frac{(1+0.025)A}{\pi}} = \sqrt{\frac{A}{\pi}}.(1+0.025)^{\frac{1}{2}}$

$= \sqrt{\frac{A}{\pi}}.\left[1+\frac{1}{2}(0.025)\right]$ $, higher\ power\ terms\ neglected = 1.0125\sqrt{\frac{A}{\pi}}$

$\%ge\ change\ in\ r = \left(\frac{1.0125\sqrt{\frac{A}{\pi}} - \sqrt{\frac{A}{\pi}}}{\sqrt{\frac{A}{\pi}}}\right) \times 100 = 1.25\%$

Circumference $C = 2\pi r$, where C and r are original values of circumference and radius respectively. Since the error of $2\frac{1}{2}\%$ leads to an increase in r by 1.25%, then;

New value of $r = \left(\frac{100+1.25}{100}\right)r = (1+0.0125)r$

$\therefore$ New value of $C = 2\pi(1+0.0125)r = 2\pi r(1+0.0125)$

$\therefore$ $\%ge\ change\ in\ C = \left(\frac{2\pi r(1+0.0125)-2\pi r}{2\pi r}\right) \times 100 = 1.25\%$

Alternatively

$A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$

$C = 2\pi r \Rightarrow C = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi}.\sqrt{A} = 2\sqrt{\pi A}$

New value of A due to error $= (1+0.025)A$

New value of $C = C' = 2\sqrt{\pi}.\sqrt{(1+0.025)A}$

$= 2\sqrt{\pi A}.(1+0.025)^{\frac{1}{2}}$

$\cong 2\sqrt{\pi A}.\left(1+\frac{0.025}{2}\right) = 2\sqrt{\pi A}(1+0.0125)$ $, higher\ power\ terms\ neglected$

$\therefore$ $\%ge\ change\ in\ C = \left(\frac{2\sqrt{\pi A}(1+0.0125)-2\sqrt{\pi A}}{2\sqrt{\pi A}}\right) \times 100 = 1.25\%\ as\ before$

5) *One side of a rectangle is 3-times the other. If the perimeter increases by 2%, what is the percentage increase in area?*

Solution

$length = 3w$

$\boxed{P = perimeter}$ $width = w$ $P = 2(3w+w) = 8w$

New value of $P = P' = \left(\frac{100+2}{100}\right)P = (1+0.02)P$

Now area, $A = 3w^2$

From $P = 8w \Rightarrow w = \frac{P}{8}$

$\therefore A = 3\left(\frac{P}{8}\right)^2 = \frac{3}{64}P^2$

New value of A $(due\ to\ increase\ in\ P) = A' = \frac{3}{64}[(1+0.02)P]^2$

$= \frac{3}{64}P^2\left((1+0.02)\right)^2 = \frac{3}{64}P^2[1+2(0,02)]$ $, higher\ power\ terms\ neglected$

$= 1.04\left(\frac{3}{64}P^2\right)$

$\therefore \%ge\ in\ area = \left[\frac{(1.04-1)\frac{3}{64}P^2}{\frac{3}{64}P^2}\right] \times 100 = 4\%$

Alternatively

$P = 8w \Rightarrow w = \frac{P}{8}$; $New\ P = (1 + 0.02)P$

Finding percentage change in w due to 2% increase in P

New $w = w' = \frac{(1+0.02)P}{8}$

$\%ge\ change\ in\ w = \left(\frac{\frac{(1+0.02)P}{8} - \frac{P}{8}}{\frac{P}{8}} \times\right) 100 = 2\%$

Now area , $A = 3w^2$

New area due to increase in w= $3[(1 + 0.02)w]^2$

$= 3w^2(1 + 0.02)^2 \cong 3w^2[1 + 2(0.02)]$

$= 3w^2.(1.04)$

$\therefore \%ge\ in\ area = \left(\frac{3w^2.(1.04) - 3w^2}{3w^2}\right) \times 100 = 4\%$

6) *The height cylinder 10cm and its radius is 4cm. find an approximate increase in volume when the radius increases to 4.02.*

Solution

Volume, $V = \pi r^2 h$; assuming h is constant

$\Rightarrow r = 4cm\ , h = 10cm$

Finding $\%ge$ change in r= $\left(\frac{4.02-4}{4}\right) \times 100 = 0.5\%$

Now new value of $V - V' = \pi\left[\left(\frac{100+0.5}{100}\right)r\right]^2.h$

$= \pi r^2 h\ (1 + 0.005)^2 \cong \pi r^2 h[1 + 2(0.005)]$

$V' = 1.01\pi r^2 h$

$\therefore Increase\ in\ volume = 1.01\pi r^2 h - \pi r^2 h$

$= 0.01\pi r^2 h = 0.01\pi \times 4^2 \times 10$

$= 1.6\pi\ cm^3$

7) *The base radius of a right circular cone increases and the volume changes by 2%. If the height remains constant, find the percentage increase in the circumference of the base.*(***Uneb* 2011**)

Solution

Let radius increase by $x\%$

Let $V = \frac{1}{3}\pi r^2 h$, where r and V are original values of radius and volume respectively

$\Rightarrow New\ volume, V' = \frac{1}{3}\pi\left[\left(\frac{100+x}{100}\right)r\right]^2 h$

$= \frac{1}{3}\pi r^2 h(1 + 0.01x)^2$

$= \frac{\pi r^2 h}{3}(1 + 0.02x)$ $[Higher\ power\ terms\ neglected]$

Now $\%ge\ change\ in\ volume = \left(\frac{V'-V}{V}\right) \times 100 = 2$

$\therefore \left(\frac{\frac{\pi r^2 h}{3}(1+0.02x) - \frac{\pi r^2 h}{3}}{\frac{\pi r^2 h}{3}}\right) \times 100 = 2$

$\Rightarrow 0.02x = 0.02 \therefore x = 1$

$\therefore Increase\ in\ radius\ is\ by\ 1\%$

Now circumference, $C = 2\pi r$

New value of C after increase in radius by 1% = $C' = 2\pi\left(\frac{100+1}{100}\right)r = 2\pi r(1.01)$

$\therefore \%ge\ increase\ in\ C = \left(\frac{2\pi r(1.01)-2\pi r}{2\pi r}\right)\times 100 = 1\%$

8) *If L is the length of a pendulum and t(s) is the time of one complete swing. It is known that*
$L = kt^2$. *If the length of a pendulum increases by x%, x being small, find the corresponding increase in the time of swing.*

Solution

$L = kt^2$, taking k a constant

Increase in length = $\left(\frac{100+x}{100}\right)L = (1+0.01x)L$

Now $t = \sqrt{\frac{L}{k}}$

$\therefore$ New value of $t = t' = \sqrt{\frac{(1+0.01x)L}{k}} = \sqrt{\frac{L}{k}}.(1+0.01x)^{\frac{1}{2}}$

$$\cong \sqrt{\frac{L}{k}}.\left[1+\frac{1}{2}(0.01x)\right] \cong \sqrt{\frac{L}{k}}.(1+0.005x)$$

$\therefore \%ge\ increase\ in\ t = \left(\frac{t'-t}{t}\right)\times 100$

$$= \left(\frac{\sqrt{\frac{L}{k}}.(1+0.005x)-\sqrt{\frac{L}{k}}}{\sqrt{\frac{L}{k}}}\right)\times 100 = 0.5x\%$$

9) *The period T of a pendulum is calculated from the formula* $T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ *where l is the length of a pendulum and g is acceleration due to gravity. Find the percentage change in the period caused by lengthening the pendulum by 2%.*

Solution

$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$, new value of $l = \left(\frac{100+2}{100}\right)l = (1+0.02)l$

$\therefore New\ T = T' = 2\pi\sqrt{\left(\frac{(1+0.02)l}{g}\right)}$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}.(1+0.02)^{\frac{1}{2}}$$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}.\left(1+\frac{0.02}{2}\right) \quad [higher\ power\ terms\ neglected]$$

$\therefore T' = 2\pi\sqrt{\left(\frac{l}{g}\right)}.(1.01)$

$\therefore \%ge\ change\ in\ T = \left(\frac{T'-T}{T}\right)\times 100$

$$= \left(\frac{2\pi\sqrt{\left(\frac{l}{g}\right)}.(1.01)-2\pi\sqrt{\left(\frac{l}{g}\right)}}{2\pi\sqrt{\left(\frac{l}{g}\right)}}\right)\times 100 = 1\%$$

10) *Find the approximate percentage change in the square of a quantity when the quantity itself changes by 0.1%. hence find the approximate value for* $(10.01)^2$

Solution

Let the quantity be x

$\Rightarrow y = x^2$

Let percentage change in square of quantity be an increase

$\therefore New\ value\ of\ x = \left(\frac{100+0.1}{100}\right)x = (1+0.001)x$

Now percentage change in square of quantity. $i.e. y = \left(\frac{y'-y}{y}\right) \times 100$

But $y' = [(1+0.001)x]^2 = (1+0.001)^2 x^2$

$\cong [1+2(0.001)]x^2 \cong 1.002x^2$

$\therefore$ Required $\%ge$ change in square of a quantity

$= \left(\frac{1.002x^2 - x^2}{x^2}\right) \times 100 = 0.2\%$

Now let $y = (10.01)^2 = (10+0.01)^2$

Hence if $x = 10$, then x increases by $0.01 = 1\%$

Since $y' = x^2 \cong 1.002x^2$, $x = 10$

$\Rightarrow y' = 1.002 \times 10^2 = 100.2$

$$\left[\begin{array}{l} \quad\quad \boldsymbol{Alt}: (10.01)^2 = (10+0.01)^2 = (10.01)^2 = (10+0.01)^2 \\ = \left[10\left(1+\frac{0.01}{10}\right)\right]^2 = 100(1+0.001)^2 = 10^2 \times \left(1+2(0.001)\right) = 100.2 \end{array}\right]$$

NOTE:

The above examples on percentages and errors in binomial expansion can also be done using the knowledge of *differentiation.i.e. Small increments and percentages*

Task:

1. *The modulus of rigidity G is given by* $G = \frac{R^4\theta}{L}$ *where R is radius,* θ *is the angle of twist and L is length. Find the approximate percentage in G when R is measured* 1.5% *too large and* θ *is measured* 5% *too small.* $\boldsymbol{Answer}: \mathbf{1}\%$
2. *The volume of a cone is given by* $V = \frac{1}{3}\pi r^2 h$*. If the volume increases by* $\frac{2\pi}{3}\ cm^3/min$ *and height increase by* $0.03cm/min$ *. Find the rate of change of the radius r when* $r = 10\ cm$ *and* $h = 5\ cm.$ $\boldsymbol{Answer}: \mathbf{5}$
3. *If the radius of a spherical bubble increases from* $1\ cm$ *to* $1.02\ cm$*, find the approximate increase in volume.* $\boldsymbol{Answer}: \mathbf{0.08\pi\ cm^3}$
4. *If* $R = ar^n$ *and an error of* $x\%$ *is made in measuring r, prove that an error of* $nx\%$ *will result in R*
5. *The volume of a sphere is increased by 3%. Find the percentage increase in the radius.* $\boldsymbol{Answer}: \mathbf{1}\%$

VECTORS

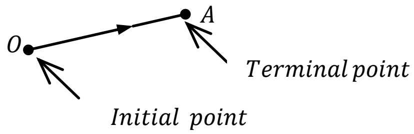
Definition:

A quantity that can be specified with magnitude and direction $e.g.$ force, acceleration, displacement, velocity etc is a vector.

Representation of a vector

A vector is represented symbolically as $\vec{a}$ or **a** or $\underset{\sim}{a}$

Graphically, it is represented by a line



The length of a line is the magnitude and the arrow on the line is the direction of the vector.

Classification of vectors

Vectors are classified depending on their behavior and properties.

a) **Null vector or Zero vector**

A vector with a null/zero magnitude, and is denoted as **O**

b) **Unit vector**

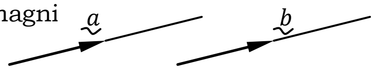
A vector with a unit magnitude is a unit vector. It is denoted with a cap $.i.e.$ for a vector, its unit vector I a

NOTE

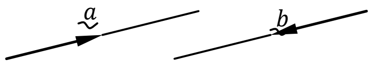
$$\boxed{\frac{VECTOR}{|VECTOR|} = UNIT\,VECTOR}$$

c) **Like/unlike vectors**

Like vectors are vectors with the same direction regardless of their magni

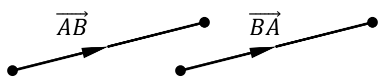


Unlike vectors are vectors with different directions



d) **Negative vectors**

Negative vectors are vectors with same magnitude but opposite in direction. Consider the figure below



$\overrightarrow{AB} = -\overrightarrow{BA}$ **Application:** subtraction of vectors

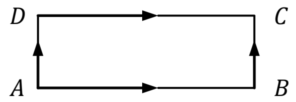
e) **Equal vectors**

Equal vectors arte vectors with same magnitude and direction

Application

- Rhombus
- Parallelogram
- Rectangles
- Squares

Consider the figure below



For a rectangle
$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Note

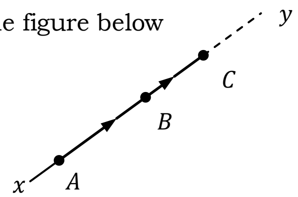
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$, $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus equal vectors are also parallel but with the same magnitude and direction.

f) **Collinear/ parallel vectors**

Are vectors with the same line of action or having line of action parallel to one another.

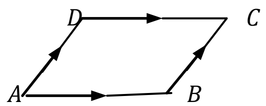
Consider the figure below

a)



$\overrightarrow{AB}$ *and* $\overrightarrow{BC}$, act on point B, but both have xy line of action, hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$

b)



ABCD is a parallelogram.
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$
$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are collinear and also $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are collinear/parallel.

Condition for collinear/parallel vectors

From fig (a)

$$\boxed{\overrightarrow{AB} = k\overrightarrow{BC} \text{ or } \overrightarrow{AB} = \lambda\overrightarrow{AC}}$$

From fig (b)

$$\boxed{\overrightarrow{AB} = k\overrightarrow{DC} \text{ or } \overrightarrow{AD} = \lambda\overrightarrow{BC}}$$

Where k and λ are scalars or constants

g) **Position vectors**

Are vectors that specify position of a point with respect to a fixed point O ($Origin$)

Application

Used mainly in coordinate system, for locating a point in $x - y$ or $x - y - z$ planes

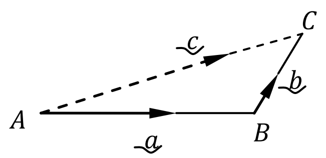
h) **Coplanar vectors**

Are vectors which lie in the same plane or vectors paralleled to the same plane

ADDITION OF VECTORS

Vectors are added by finding their *resultant, a single or equivalent vector.* Consider the following cases of vector addition.

1) **Case I:** <u>*Addition of **TWO** vectors.*</u>



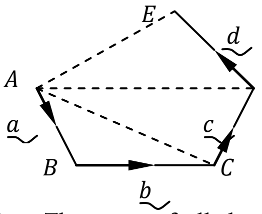
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

or $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$

This is called *triangle law* of addition of vectors.
In general, to find the sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, draw them as a chain, starting the second where the first ends; the sum $\boldsymbol{c}$ is given by the single vector joining the start of the first to the end of the second.

2) **Case II:** <u>*Addition of a number of vectors;* $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+$**.............**</u>

A)



a) Draw the vectors as achain
b) Then;

$\boldsymbol{a}+\boldsymbol{b}=\overrightarrow{AC}, \overrightarrow{AC}+\boldsymbol{c}=\overrightarrow{AD}$

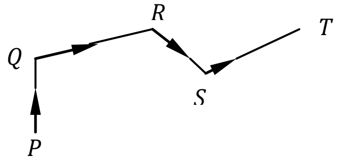
$\therefore \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\overrightarrow{AD}$

$\overrightarrow{AD}+\boldsymbol{d}=\overrightarrow{AE}$

$\therefore \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}=\overrightarrow{AE}$

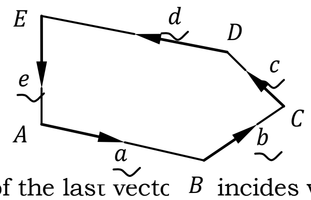
$i.e.$ The sum of all the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ is given by the single vector joining the start of the first to the end of the last . $i.e.\overrightarrow{AE}$

B)



$\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}+\overrightarrow{ST}=\overrightarrow{PT}$

3) **Case III:** *Sum of vectors =* 0 *for a closed figure*
Consider the figure below

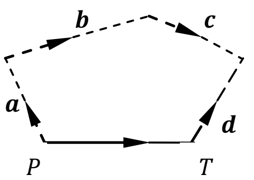


$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}=0$

$i.e..\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}=\boldsymbol{0}$

NOTE
The end of the last vector coincides with the beginning of the first, so the resultant vector is zero with no magnitude.

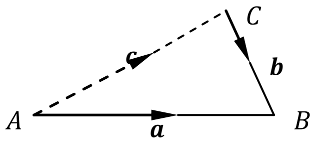
COMPONENTS OF A GIVEN VECTOR
A single vector can be replaced by any number of component vectors so long as they form a chain in the vector diagram, beginning at initial point and ending at the terminal point.



$\overrightarrow{PT}=\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}$
See examples immediately after *subtraction of vectors.*

SUBTRACTION OF VECTORS
During the addition of vectors in a specified direction it may be encountered that the direction of a vector needs to be changed in order for it to be added. Once a direction of vector is changed, a *negative vector* is formed, hence addition, subtraction of a vector occurs.

NOTE
Triangle law is still used for subtraction of *two* vectors.

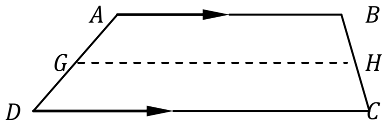


$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
But $\overrightarrow{BC} = -\overrightarrow{CB}$
$\therefore \overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{CB}$ or $\mathbf{c} = \mathbf{a} - \mathbf{b}$

Examples on components of vectors

1. *ABCD is a quadrilateral, with G and H the midpoints of DA and BC respectively. Show that* $\overrightarrow{AB} + \overrightarrow{DC} = 2\overrightarrow{GH}$

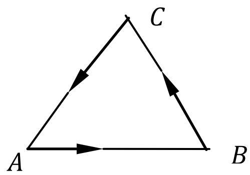
Solution



By chain vectors, $\overrightarrow{AB} = \overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HB}$
Also $\overrightarrow{DC} = \overrightarrow{DG} + \overrightarrow{GH} + \overrightarrow{HC}$
Now $\overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HB} + \overrightarrow{DG} + \overrightarrow{GH} + \overrightarrow{HC}$
$= 2\overrightarrow{GH} + (\overrightarrow{AG} + \overrightarrow{DG}) + (\overrightarrow{HB} + \overrightarrow{HC})$

But G is the midpoint of AD, $\Rightarrow \overrightarrow{AG} = \overrightarrow{GD}$ or $\overrightarrow{AG} = -\overrightarrow{DG}$ $(negative\ vector)$
Also H is the midpoint of BC, $\Rightarrow$ $\overrightarrow{BH} = \overrightarrow{HC}$ or $-\overrightarrow{HB} = \overrightarrow{BH} = \overrightarrow{HC}$
$\therefore \overrightarrow{AB} + \overrightarrow{DC} = 2\overrightarrow{GH} + (-\overrightarrow{DG} + \overrightarrow{DG}) + (-\overrightarrow{HC} + \overrightarrow{HC}) = 2\overrightarrow{GH}$

2. *Points L, M, N are midpoints of the sides AB, BC, CA of the triangle ABC. Show that;*
   a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$
   b. $2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$
   c. $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = 0$

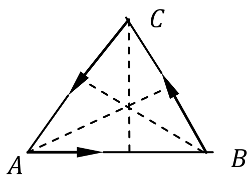


Solution

a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$ $(For\ a\ closed\ figure)$

b. To show that $2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$

$\overrightarrow{CA} = \overrightarrow{CL} + \overrightarrow{LA}$
$\therefore 2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = 4\overrightarrow{AL} + 3\overrightarrow{BL} + 3\overrightarrow{LC} + \overrightarrow{CL} + \overrightarrow{LA}$
$= 4\overrightarrow{AL} + 3(-\overrightarrow{LB}) + 3\overrightarrow{LC} + (-\overrightarrow{LC}) + (-\overrightarrow{AL})$
$= 3\overrightarrow{AL} - 3\overrightarrow{LB} + 2\overrightarrow{LC}$
Since $AL = LB, 3\overrightarrow{AL} - 3\overrightarrow{LB} = 0$
$\therefore 2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$



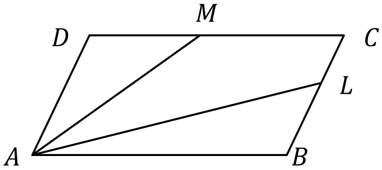
c. To show that $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = 0$
From the figure (b) above
$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$, $\overrightarrow{BN} = \overrightarrow{BC} + \overrightarrow{CN}$, $\overrightarrow{CL} = \overrightarrow{CA} + \overrightarrow{AL}$
$\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = \overrightarrow{AB} + \overrightarrow{BM} + \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AL} + \overrightarrow{CN}$
$= (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) + (\overrightarrow{BM} + \overrightarrow{AL} + \overrightarrow{CN})$
$= 0 + \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{AB} + \overrightarrow{CA}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = 0$

Task

Prove that the line joining he midpoint of sides of a triangle is parallel to the third side and half its length.

3. *ABCD is a parallelogram. If L, M are the midpoints of BC and CD, express* $\overrightarrow{AL}$ *and* $\overrightarrow{AM}$ *in terms of* $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$*. Show that* $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

Solution



$\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL}$

But $\overrightarrow{BL} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AD}$ $[\because \overrightarrow{AD} = \overrightarrow{BC}]$

$\therefore \overrightarrow{AL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM}$

But $\overrightarrow{DM} = \frac{1}{2}\overrightarrow{DC}$, $\overrightarrow{DC} = \overrightarrow{AB}$, $\therefore \overrightarrow{DM} = \frac{1}{2}\overrightarrow{AB}$

$\Rightarrow \overrightarrow{AM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$

Now to show: $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} + \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} = \overrightarrow{AL} + \overrightarrow{AM}$

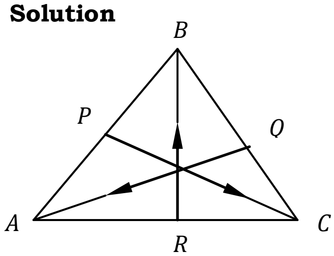
$\Rightarrow \overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{AD} = \frac{3}{2}\left(\overrightarrow{AB} + \overrightarrow{AD}\right)$

But $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

4. *If P, Q, R are midpoints of the sides AB, BC and CA of the triangle ABC and O be a point with in t, prove that* $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$

Solution



$\overrightarrow{OA} = \overrightarrow{OP} + \overrightarrow{PA}$

$\overrightarrow{OB} = \overrightarrow{OQ} + \overrightarrow{QB}$

$\overrightarrow{OC} = \overrightarrow{OR} + \overrightarrow{RC}$

$\therefore \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{OQ} + \overrightarrow{QB} + \overrightarrow{OR} + \overrightarrow{RC}$

$= \left(\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}\right) + \left(\overrightarrow{PA} + \overrightarrow{QB} + \overrightarrow{RC}\right)$

But $\overrightarrow{PA} = \frac{1}{2}\overrightarrow{BA}$, $\overrightarrow{QB} = \frac{1}{2}\overrightarrow{CB}$, $\overrightarrow{RC} = \frac{1}{2}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{PA} + \overrightarrow{QB} + \overrightarrow{RC} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{AC}\right) = \frac{1}{2} \times 0 = 0$

$\therefore \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$

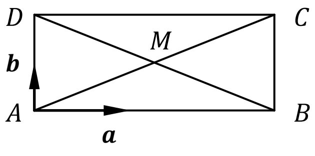
NOTE

Parameters can also be applied in solving the above examples through forming parametric equations.

5. *ABCD is a rectangle. By vectors show that the diagonals bisect each other.*

Solution

Points A, M and C are collinear, and B, M and D are also collinear.



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

Now $\overrightarrow{AM} = \lambda\overrightarrow{AC}$ and $\overrightarrow{BM} = \mu\overrightarrow{BD}$ where λ , μ are constants.

$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \mu\overrightarrow{BD}$

But $BD = BA + AD = -\boldsymbol{a} + \boldsymbol{b}$

$\therefore \overrightarrow{AM} = \boldsymbol{a} + \mu(-\boldsymbol{a} + \boldsymbol{b})$

Also $\overrightarrow{AM} = \lambda\overrightarrow{AC}$; $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$ $\left[\because \overrightarrow{AD} = \overrightarrow{BC} = \boldsymbol{b}\right]$

$\therefore \overrightarrow{AM} = (\boldsymbol{a} + \boldsymbol{b})\lambda$ .... (1)

Also $\overrightarrow{AM} = \boldsymbol{a} + \mu(-\boldsymbol{a} + \boldsymbol{b}) = (1 - \mu)\boldsymbol{a} + \mu\boldsymbol{b}$ .... (2)

Since (1) and (2) are the same

$\Rightarrow (\boldsymbol{a}+\boldsymbol{b})\lambda = \boldsymbol{a}+\mu(-\boldsymbol{a}+\boldsymbol{b}) = (1-\mu)\boldsymbol{a}+\mu\boldsymbol{b}$

Equating coefficients of corresponding vectors

For $\boldsymbol{a}$: $\boldsymbol{\lambda} = 1-\mu$ .... (3)

For $\boldsymbol{b}$: $\lambda = \mu$ .... (4)

Solving (3) and (4) simultaneously

$\mu = 1-\mu$ *or* $\mu = \frac{1}{2} = \lambda$

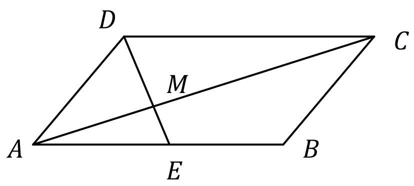
Since$\overrightarrow{AM} = \lambda\overrightarrow{AC}$ and $\overrightarrow{BM} = \mu\overrightarrow{BD}$, then $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BD}$

$\therefore M$ is the midpoint of the diagonals, and hence diagonals bisect.

6. *If ABCD is a parallelogram and E is the midpoint of AB, show by vector method that DE trisects and is trisected by AC.*

Solution

Points A, M and Care collinear and D, M and E are also collinear



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

$\overrightarrow{AM} = \lambda\overrightarrow{AC}$,$\overrightarrow{DM} = k\overrightarrow{DE}$, where k ,λ are constants

$\overrightarrow{AM} = \lambda\overrightarrow{AC}$;$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC}$

$= \overrightarrow{AB}+\overrightarrow{AD} = \boldsymbol{a}+\boldsymbol{b}$ $[\because \overrightarrow{AD} = \overrightarrow{BC} = \boldsymbol{b}]$

$\therefore \overrightarrow{AM} = (\boldsymbol{a}+\boldsymbol{b})\lambda$ .... (1)

$\overrightarrow{DM} = k\overrightarrow{DE}$;$\overrightarrow{DM} = \overrightarrow{DA}+\overrightarrow{AM} = -\boldsymbol{b}+\overrightarrow{AM}$

$\Rightarrow \overrightarrow{DM} = -\boldsymbol{b}+(\boldsymbol{a}+\boldsymbol{b})\lambda$ .... (2)

Now $\overrightarrow{DE} = \overrightarrow{DA}+\overrightarrow{AE} = -\boldsymbol{b}+\frac{1}{2}\overrightarrow{AB} = -\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}$

$\therefore \overrightarrow{DM} = k\overrightarrow{DE}$ gives;

$-\boldsymbol{b}+(\boldsymbol{a}+\boldsymbol{b})\lambda = k\left(-\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}\right)$ *or* $(\lambda-1)\boldsymbol{b}+\lambda\boldsymbol{a} = -k\boldsymbol{b}+\frac{k}{2}\boldsymbol{a}$

Equating coefficients of corresponding vectors

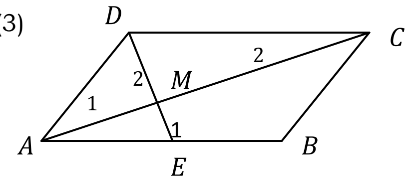
For $\boldsymbol{a}$: $\lambda = \frac{k}{2}$ .... (3)

For $\boldsymbol{b}$: $\lambda - 1 = -k$ .... (4)

$Eqn.$ (3) in to $eqn.$ (4) gives

$\frac{k}{2} - 1 = -k \Rightarrow \frac{3}{2}k = 1$ *or* $k = \frac{2}{3}$,

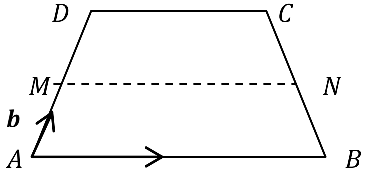
and from (1) $\lambda = \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}$



$\therefore \overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$ *and* $\overrightarrow{DM} = \frac{2}{3}\overrightarrow{DE}$, hence DE trisects and is trisected by AC.

7. *Show that the line joining the midpoints of two non parallel sides of a trapezium is parallel to the other sides and half their sum.*

Solution



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

Since $\overrightarrow{AB}$ // $\overrightarrow{DC}$, then $\overrightarrow{DC} = \lambda\,\overrightarrow{AB} = \lambda\boldsymbol{a}$

Now expressing $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

$\overrightarrow{MN} = \overrightarrow{MA}+\overrightarrow{AB}+\overrightarrow{BN}$

$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{DA}$, $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$

$\Rightarrow \overrightarrow{MN} = {}^{\boldsymbol{a}}{}_{2}\boldsymbol{b}+\boldsymbol{a}+\frac{1}{2}\overrightarrow{BC}$

But $\overrightarrow{BC} = \overrightarrow{BA}+\overrightarrow{AD}+\overrightarrow{DC} = -\boldsymbol{a}+\boldsymbol{b}+\lambda\boldsymbol{a}$

$\Rightarrow \overrightarrow{MN} = -\frac{1}{2}\boldsymbol{b}+\boldsymbol{a}+\frac{1}{2}(-\boldsymbol{a}+\boldsymbol{b}+\lambda\boldsymbol{a})$

$$= -\frac{1}{2}\boldsymbol{b} + \boldsymbol{a} - \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{\mathbf{1}}{\mathbf{2}}\lambda\boldsymbol{a} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\lambda\boldsymbol{a} = \frac{1}{2}(1+\lambda)\boldsymbol{a}$$

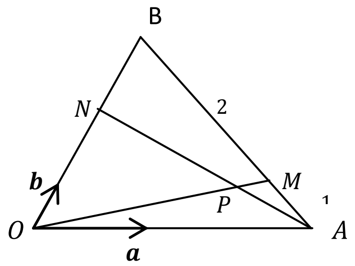
Since $\frac{1}{2}(1+\lambda)$ is a scalar, let $\frac{1}{2}(1+\lambda) = \mu$

$\therefore \overrightarrow{MN} = \mu\boldsymbol{a}$, which shows that $\overrightarrow{MN} \,//\, \overrightarrow{AB}$ and $\overrightarrow{DC}$

From $\overrightarrow{MN} = \frac{1}{2}(1+\lambda)\boldsymbol{a} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\lambda\boldsymbol{a} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right)$, as required.

8. *OAB is a triangle M and N are points on AB and OB respectively, such that $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$. If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$, find the position vector of P, the intersection of AN and OM, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$*

Solution:



$\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB} \Rightarrow \frac{AM}{AB} = \frac{1}{3}$ or $\overrightarrow{AM}:\overrightarrow{AB} = 1:3$

$\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB} \Rightarrow \frac{ON}{OB} = \frac{3}{4}$

Points O, P, M and A, P, N are collinear.

Let $\overrightarrow{OP} = \lambda\overrightarrow{OM}$; $\overrightarrow{AP} = \mu\overrightarrow{AN}$

Now expressing $\overrightarrow{OP}$ in terms of $\lambda, \mu, \boldsymbol{a}, and\ \boldsymbol{b}$

$\overrightarrow{OP} = \lambda\overrightarrow{OM}$; $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$

$\Rightarrow \overrightarrow{OP} = \lambda\left(\overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}\right)$ $But\ \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{b} - \boldsymbol{a}$

$\therefore \overrightarrow{OP} = \lambda\left(\boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})\right) = \frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b}$ .... (1)

Also $\overrightarrow{AP} = \mu\overrightarrow{AN}$; $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{OP} - \boldsymbol{a}$

$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$; $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$

$\therefore \overrightarrow{AN} = \overrightarrow{AO} + \frac{3}{4}\overrightarrow{OB} = -\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

$\Rightarrow \overrightarrow{OP} - \boldsymbol{a} = \mu\left(-\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$

$\therefore OP = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$ .... (2)

$Eqn. 1 = eqn. 2$

$\frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b} = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$

$\frac{2}{3}\lambda = 1 - \mu$ .... (3)

$[Equating\ coefficients\ of\ \boldsymbol{a}]$

$\frac{1}{3}\lambda = \frac{3}{4}\mu$ .... (4)

$[Equating\ coefficients\ of\ \boldsymbol{b}]$

From *eqn. (3)* $\mu = 1 - \frac{2}{3}\lambda$ .... (4)

Eqn. (5) in to eqn. (4) gives

$\frac{1}{3}\lambda = \frac{3}{4}\left(1 - \frac{2}{3}\lambda\right) = \frac{3}{4} - \frac{1}{2}\lambda \Rightarrow \frac{1}{3}\lambda + \frac{1}{2}\lambda = \frac{3}{4}$

$\therefore \frac{5}{6}\lambda = \frac{3}{4} \Rightarrow \lambda = \frac{9}{10}$

From eqn. (4), $\mu = 1 - \frac{2}{3} \times \frac{9}{10} = \frac{2}{5}$

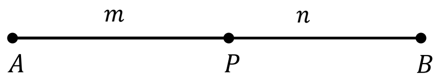
$\overrightarrow{OP} = \frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b} = \frac{2}{3}\left(\frac{9}{10}\right)\boldsymbol{a} + \frac{1}{3}\left(\frac{9}{10}\right)\boldsymbol{b} = \frac{3}{5}\boldsymbol{a} + \frac{3}{10}\boldsymbol{b}$

Task

Show that the diagonals of a square, rhombus and parallelogram bisect each other.

POSITION OF A POINT DIVIDING A LINE *AB* IN THE RATIO $m:n$
(Ratio Theorem)

Let the line AB be divided by a point P in the ratio $m:n$



Ratio $\overrightarrow{AP}:\overrightarrow{PB} = m:n$ or $\frac{m}{n} = \frac{\overrightarrow{AP}}{\overrightarrow{PB}} \Rightarrow n\overrightarrow{AP} = m\overrightarrow{PB}$

By position vectors,

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\overrightarrow{OA} + \overrightarrow{OP}$ $, \overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB} = \overrightarrow{OP} - \overrightarrow{OB}$

$\Rightarrow n(-\overrightarrow{OA} + \overrightarrow{OP}) = m(\overrightarrow{OP} - \overrightarrow{OB})$

$\Rightarrow n\overrightarrow{OP} + m\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA}$ $\therefore \overrightarrow{OP}(m+n) = n\overrightarrow{OA} + m\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$

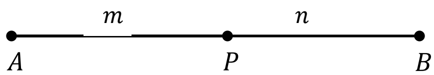
If the position vectors of A and B are respectively $\boldsymbol{a}$ and $\boldsymbol{b}$, then $OA = \boldsymbol{a}$ $, OB = \boldsymbol{b}$**.** The ratio theorem becomes,

$$\boxed{\overrightarrow{OP} = \frac{n}{m+n}\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}}$$

NOTE

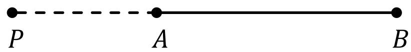
1. P is a point that divides AB either *internally* or *externally* in the ratio the ratio $m:n$

a) For *internal division*, the ratio is represented as
$\overrightarrow{AP}:\overrightarrow{PB} = m:n[Both\, ratio\, values + ve]$

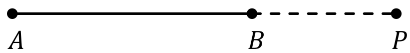


b) For *external division*, the ratio is represented as
$\overrightarrow{AP}:\overrightarrow{PB} = m:-n\; or - m:n$ $[Both\, ratio\, values of\; opposite\; sign]$

i)



ii)



$$\boxed{\overrightarrow{AP}:\overrightarrow{PB} = m:n}$$

2. The knowledge of position vectors is applicable in ratio theorem.

Examples

1. *Find the position vector (p.v) of a point which divides the line segment AB in the ratio*
   a) $8:3$
   b) $5:-4$

Solution

a) Ratio $8:3 = m:n$ $[Internal\; division]$

$\overrightarrow{AP}:\overrightarrow{PB} = 8:3$ or $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{8}{3} \Rightarrow 3\overrightarrow{AP} = 8\overrightarrow{PB}$

By position vectors, $3(\overrightarrow{OP} - \overrightarrow{OA}) = 8(\overrightarrow{OB} - \overrightarrow{OP}) \Rightarrow 3\overrightarrow{OP} - 3\overrightarrow{OA} = 8\overrightarrow{OB} - 8\overrightarrow{OP}$

$11\overrightarrow{OP} = 3\overrightarrow{OA} + 8\overrightarrow{OB} \Rightarrow \overrightarrow{OP} = \frac{3}{11}\overrightarrow{OA} + \frac{8}{11}\overrightarrow{OB}$

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ then $\overrightarrow{OP} = \frac{3}{11}\boldsymbol{a} + \frac{8}{11}\boldsymbol{b}$

b) Ratio $5:-4 = m:n$ $[External\ division]$

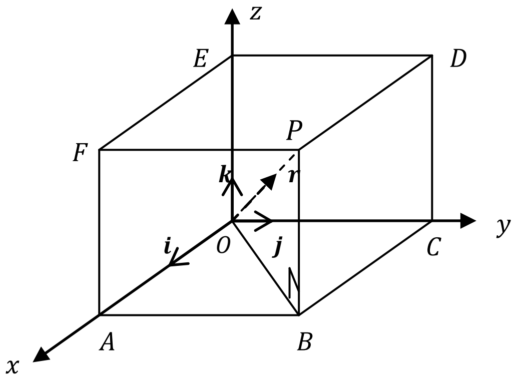
$\overrightarrow{AP}:\overrightarrow{PB} = 5:-4$ or $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{5}{-4}$ $\Rightarrow -4\overrightarrow{AP} = 5\overrightarrow{PB}$

By position vectors, $-4\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = 5\left(\overrightarrow{OB} - \overrightarrow{OP}\right) \Rightarrow -4\overrightarrow{OP} + 4\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OP}$

$\overrightarrow{OP} = 5\overrightarrow{OB} - 4\overrightarrow{OA}$ $\therefore$ $\overrightarrow{OP} = -4\boldsymbol{a} + 5\boldsymbol{b}$

VECTOR "r" IN TERMS OF UNIT VECTORS $\boldsymbol{i}, \boldsymbol{j}$ AND $\boldsymbol{k}$

Consider the figure below



$\overrightarrow{OP} = \boldsymbol{r} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$

Let $\left|\overrightarrow{OA}\right| = x\ units,$

$\left|\overrightarrow{AB}\right| = y\ units$, $\left|\overrightarrow{BP}\right| = z\ units$

From $vector = |vector| \times unit\ vector,$

$\overrightarrow{OA} = x.\boldsymbol{i}$, $\overrightarrow{AB} = z.\boldsymbol{j}$ and $\overrightarrow{BP} = z.\boldsymbol{k}$

$\therefore \overrightarrow{OP} = x.\boldsymbol{i} + y.\boldsymbol{j} + z.\boldsymbol{k}$

A point $P(x, y, z)$ has a p.v

$\overrightarrow{OP} = \boldsymbol{r} = x.\boldsymbol{i} + y.\boldsymbol{j} + z.\boldsymbol{k}$

or $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

MODULUS/MAGNITUDE/LENGTH OF A VECTOR $\boldsymbol{r}$

From ΔOPB, by Pythagoras theorem,

$\left|\overrightarrow{OP}\right|^2 = |\boldsymbol{r}|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{BP}\right|^2$; But $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = x.\boldsymbol{i} + y.\boldsymbol{j}$

$\left|\overrightarrow{OB}\right|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AB}\right|^2$ $[\because \Delta OAB\ is\ right\ angled]$

$= x^2 + y^2$

$\therefore \left|\overrightarrow{OP}\right|^2 = x^2 + y^2 + z^2 \Rightarrow$ $\boxed{\left|\overrightarrow{OP}\right| = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}}$

DISTANCE BETWEEN TWO VECTORS

If $\boldsymbol{r_1} = x_1.\boldsymbol{i} + y_1.\boldsymbol{j} + z_1.\boldsymbol{k}$ and $\boldsymbol{r_2} = x_2.\boldsymbol{i} + y_2.\boldsymbol{j} + z_2.\boldsymbol{k}$, distance between $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ is

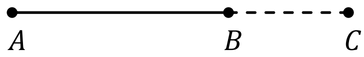
$$\boxed{Distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

More examples on Ratio Theorem

1. *Given the position vector* $\overrightarrow{OA} = (3, -2, 5)$ *and* $OB = (9, 1, -1)$. *Find the position vector of point C such that C divides* $\overline{AB}$ *internally in the ratio* $5: -3$ ($\boldsymbol{Uneb}\ \mathbf{2003}$)

Solution

Method I



$\overrightarrow{AC}:\overrightarrow{CB} = 5: -3 \Rightarrow \frac{\overrightarrow{AC}}{\overrightarrow{CB}} = \frac{5}{-3}$ $or -3\overrightarrow{AC} = 5\overrightarrow{CB}$

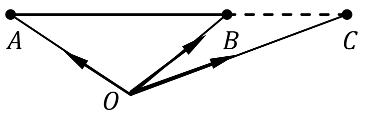
$\Rightarrow -3(\overrightarrow{OC} - \overrightarrow{OA}) = 5(\overrightarrow{OB} - \overrightarrow{OC})$

$\Rightarrow -3\overrightarrow{OC} + 3\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OC}\ \therefore 2\overrightarrow{OC} = 5\overrightarrow{OB} - 3\overrightarrow{OA}$

$\Rightarrow \overrightarrow{OC} = \frac{5}{2}\overrightarrow{OB} - \frac{3}{2}\overrightarrow{OA} = \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix}$ $\therefore \overrightarrow{OC} = 18\boldsymbol{i} + \frac{11}{2}\boldsymbol{j} - 10\boldsymbol{k}$

Method II



By Ratio Theorem $\overrightarrow{AC}:\overrightarrow{CB} = 5: -3 = m: n$

$\overrightarrow{OC} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$

$\therefore \overrightarrow{OC} = -\frac{3}{5+-3}\overrightarrow{OA} + \frac{5}{5+-3}\overrightarrow{OB} = \frac{5}{2}\overrightarrow{OB} - \frac{3}{2}\overrightarrow{OA}$

$= \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$\therefore \overrightarrow{OC} = \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix} \Rightarrow \therefore \overrightarrow{OC} = 18\boldsymbol{i} + \frac{11}{2}\boldsymbol{j} - 10\boldsymbol{k}$

2. *A and B are points whose position vectors are* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{k}$ *and* $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ *respectively. Determine the p.v of the point that divides AB internally in the ratio* $4: -1$ ($\boldsymbol{uneb}\ \mathbf{1993}$)

Solution

Method I

Ratio of division is $\overrightarrow{AP}:\overrightarrow{PB} = 4: -1$ or $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{4}{-1} \Rightarrow -\overrightarrow{AP} = 4\overrightarrow{PB}$

$\Rightarrow -\overrightarrow{OP} + \overrightarrow{OA} = 4\overrightarrow{OB} - 4\overrightarrow{OP}$

$\Rightarrow 3\overrightarrow{OP} = 4\overrightarrow{OB} - \overrightarrow{OA}\ \therefore \overrightarrow{OP} = \frac{4}{3}\overrightarrow{OB} - \frac{1}{3}\overrightarrow{OA} = \frac{4}{3}(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) - \frac{1}{3}(2\boldsymbol{i} + \boldsymbol{k})$

$\Rightarrow \overrightarrow{OP} = \left(\frac{4}{3} - \frac{2}{3}\right)\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \left(\frac{12}{3} - \frac{1}{3}\right)\boldsymbol{k} = \frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{11}{3}\boldsymbol{k}$

Method II

Using Ratio Theorem

$\overrightarrow{OP} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$, $Ratio$: $\overrightarrow{AP}:\overrightarrow{PB} = 4: -1 = m: n$

$\therefore \overrightarrow{OP} = \frac{-1}{4+-1}\overrightarrow{OA} + \frac{4}{4+-1}\overrightarrow{OB} = -\frac{1}{3}\overrightarrow{OA} + \frac{4}{3}\overrightarrow{OB}$

$= -\frac{1}{3}(2\boldsymbol{i} + \boldsymbol{k}) + \frac{4}{3}(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) = \frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{11}{3}\boldsymbol{k}$

NOTE

You can work the vectors in terms of $\boldsymbol{i}, \boldsymbol{j},$ *and* $\boldsymbol{k}$ or write the given vectors in terms of column vectors. *i.e.* For $\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$

3. *Given that the position vectors of A, B and C are*

$\overrightarrow{OA}=\begin{pmatrix}1\\-2\\2\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}7\\10\\-7\end{pmatrix}$

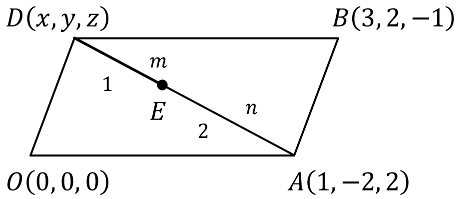
i) Prove that A, B and C are collinear

ii) If OABD is a parallelogram, find the p.v of E and F such that E divides DA in the ratio 1: 2 *and F divides it externally in the ratio* 1: 2 (***Uneb* 2007**)

Solution

i) Shall be tackled later after looking at collinear vectors

ii) For a parallelogram, $\overrightarrow{OD}=\overrightarrow{AB}$ and $\overrightarrow{OA}=\overrightarrow{DB}$



$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}2\\4\\-3\end{pmatrix}$

$\therefore D(x,y,z)=D(2,4,-3)$

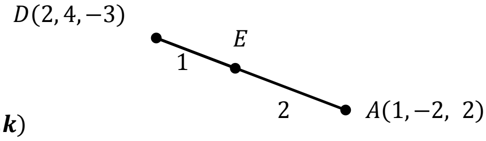
Now finding coordinates of E that divides DA internally in the ratio 1:2

Using Ratio Theorem

$\overrightarrow{OE}=\frac{n}{m+n}\overrightarrow{OD}+\frac{m}{m+n}\overrightarrow{OA}$ $[state\ from\ D\ to\ A]$

$=\frac{2}{2+1}\overrightarrow{OD}+\frac{1}{2+1}\overrightarrow{OA}$

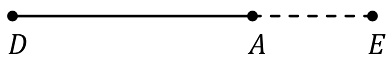
$=\frac{2}{3}\overrightarrow{OD}+\frac{1}{3}\overrightarrow{OA}=\frac{2}{3}\begin{pmatrix}2\\4\\-3\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\-2\\2\end{pmatrix}$



or $\overrightarrow{OE}=\frac{2}{3}(2\boldsymbol{i}+4\boldsymbol{j}-3\boldsymbol{k})+\frac{1}{3}(\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k})$

$\begin{pmatrix}\frac{4}{3}+\frac{1}{3}\\\frac{8}{3}-\frac{2}{3}\\-\frac{6}{3}+\frac{2}{3}\end{pmatrix}=\begin{pmatrix}\frac{5}{3}\\2\\\frac{4}{3}\end{pmatrix}\therefore \overrightarrow{OE}=\frac{5}{3}\boldsymbol{i}+2\boldsymbol{j}+\frac{4}{3}\boldsymbol{k}$

Now finding E, for external division



Ratio $\overrightarrow{DE}:\overrightarrow{EA}=1:-2=m:n$

Using Ratio Theorem

$\overrightarrow{OE}=\frac{n}{m+n}\overrightarrow{OD}+\frac{m}{m+n}\overrightarrow{OA}$

$\Rightarrow \overrightarrow{OE}=\frac{-2}{1+-2}\overrightarrow{OD}+\frac{1}{1+-2}\overrightarrow{OA}=2\overrightarrow{OD}-\overrightarrow{OA}$

$\overrightarrow{OE}=2\begin{pmatrix}2\\4\\-3\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}4-1\\8+2\\-6-2\end{pmatrix}=\begin{pmatrix}5\\10\\-8\end{pmatrix}\therefore \overrightarrow{OE}=5\boldsymbol{i}+10\boldsymbol{j}-8\boldsymbol{k}$

Examples on Distance between two points

1. *Prove that the points* $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{i}-3\boldsymbol{j}-5\boldsymbol{k}$ *and* $3\boldsymbol{i}-4\boldsymbol{j}-4\boldsymbol{k}$ *are vertices of a right angled* Δ.

Solution

Let $P(2,-1,1)$, $Q(1,-3,-5)$, $R(3,-4,-4)$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\\-6\end{pmatrix}\Rightarrow\left|\overrightarrow{PQ}\right|=\sqrt{(-1)^2+2^2+(-6)^2}=\sqrt{41}$$

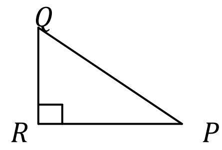
$$\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}3\\-4\\-4\end{pmatrix}-\begin{pmatrix}1\\-3\\-5\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\Rightarrow\left|\overrightarrow{QR}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$$

$$\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}3\\-4\\-4\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}\Rightarrow\left|\overrightarrow{PR}\right|=\sqrt{1^2+(-3)^2+(-5)^2}=\sqrt{35}$$

For a right angled triangle,

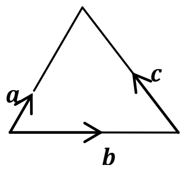
$sum\ of\ squares\ of\ two\ sides=square\ of\ the\ longer\ side$

$\Rightarrow\left|\overrightarrow{QR}\right|^2+\left|\overrightarrow{PR}\right|^2=\left|\overrightarrow{PQ}\right|^2$



2. *Prove that the vectors* $2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}$, $6\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$ *and* $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$ *form the sides of an equilateral triangle.*

Solution



Let $\boldsymbol{a}=6\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$, $\boldsymbol{b}=2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}$ *and* $\boldsymbol{c}=3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$

$|\boldsymbol{a}|=\sqrt{6^2+(-2)^2+3^2}=7\ units$

$|\boldsymbol{b}|=\sqrt{2^2+3^2+(-6)^2}=7\ units$

$|\boldsymbol{c}|=\sqrt{3^2+6^2+(-2)^2}=7\ units$

∴ Vectors form sides of an equilateral triangle.

General examples

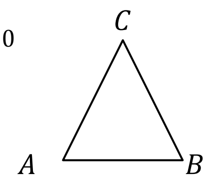
3. *Show that the points A, B and C with position vectors*
   a) $4\boldsymbol{i}-8\boldsymbol{j}-13\boldsymbol{k}, 3\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$ *and* $3\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$
   b) $5\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}, 2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ *and* $7\boldsymbol{i}-3\boldsymbol{j}+10\boldsymbol{k}$
   c) $7\boldsymbol{i}+10\boldsymbol{k}, -\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k}$ *and* $-\boldsymbol{i}+9\boldsymbol{j}+6\boldsymbol{k}$

   , are vertices of a triangle.

   Solution

a) $A(4,-8,-13)$, $B(3,-2,-3)$ $C(3,-1,-2)$

For a closed figure of three sides, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$



$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}3\\-2\\-3\end{pmatrix}-\begin{pmatrix}4\\-8\\-13\end{pmatrix}=\begin{pmatrix}-1\\6\\10\end{pmatrix}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\\-3\end{pmatrix}=\begin{pmatrix}0\\3\\1\end{pmatrix},$$

and $$\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\begin{pmatrix}4\\-8\\-13\end{pmatrix}-\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\-9\\-11\end{pmatrix}$$

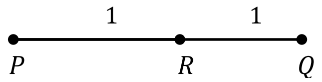
Now $$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\begin{pmatrix}-1\\6\\10\end{pmatrix}+\begin{pmatrix}0\\3\\1\end{pmatrix}+\begin{pmatrix}1\\-9\\-11\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=0$$

b) $Left\ as\ an\ exercise$

c) $Left\ as\ an\ exercise$

4. *Given that the p.vs of P and Q are* $4\boldsymbol{i}-3\boldsymbol{j}+5\boldsymbol{k}$ *and* $\boldsymbol{i}+2\boldsymbol{k}$ *respectively, find the coordinates of the point R such that* $PQ:PR = 2:1$

Solution



Then R is the midpoint of PQ or R divides PQ internally in the ratio $1:1$

Method I

$\overrightarrow{PQ}:\overrightarrow{PR} = 2:1 \Rightarrow \frac{\overrightarrow{PQ}}{\overrightarrow{PR}} = \frac{2}{1}$ *or* $\overrightarrow{PQ} = 2\overrightarrow{PR}$

$\Rightarrow \overrightarrow{OQ} - \overrightarrow{OP} = 2\left(\overrightarrow{OR} - \overrightarrow{OP}\right) = 2\overrightarrow{OR} - 2\overrightarrow{OP}$

$\overrightarrow{OP} = 2\overrightarrow{OR} - \overrightarrow{OQ} \;\Rightarrow \overrightarrow{OR} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right)$

$$\therefore \overrightarrow{OR} = \frac{1}{2}\left[\begin{pmatrix}4\\-3\\5\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix}\right] = \begin{pmatrix}\frac{5}{2}\\-\frac{3}{2}\\\frac{7}{2}\end{pmatrix} \;\therefore R\left(\frac{5}{2}, -\frac{3}{2}, \frac{7}{2}\right)$$

Method II

By ratio theorem, $\overrightarrow{PQ}:\overrightarrow{RQ} = 1:1 \equiv m:n$

$\overrightarrow{OR} = \frac{n}{m+n}\overrightarrow{OP} + \frac{m}{m+n}\overrightarrow{OQ}$

$$= \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) = \frac{1}{2}\frac{1}{2}\left[\begin{pmatrix}4\\-3\\5\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix}\right] = \begin{pmatrix}\frac{5}{2}\\-\frac{3}{2}\\\frac{7}{2}\end{pmatrix}$$

$\therefore R\left(\frac{5}{2}, -\frac{3}{2}, \frac{7}{2}\right)$

5. *The points A and B have p.vs* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *respectively relative to origin, where* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ *and* $\boldsymbol{b} = -4\boldsymbol{i} + s\boldsymbol{j} + t\boldsymbol{k}$*. Find the possible values of s and t if* $\left|\overrightarrow{AB}\right| = 7$ *and* $s = 2t$

Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-4\\s\\t\end{pmatrix} - \begin{pmatrix}2\\1\\-3\end{pmatrix} = \begin{pmatrix}-6\\s-1\\t+3\end{pmatrix}$$

$\left|\overrightarrow{AB}\right| = \sqrt{(-6)^2 + (s-1)^2 + (t+3)^2} = 7$

Squaring both sides

$36 + (s-1)^2 + (t+3)^2 = 49$

$(s-1)^2 + (t+3)^2 = 13$…………………………… (1)

Now

$s = 2t$ ……………………………………………(2)

$eqn, (2)\ in\ to\ (1)$

$(2t-1)^2 + (t+3)^2 = 13$

$4t^2 - 4t + 1 + t^2 + 6t + 9 = 13 \;\Rightarrow\; 5t^2 + 2t - 3 = 0$

$5t^2 + 5t - 3t - 3 = 0$

$5t(t+1) - 3(t+1) = 0 \Rightarrow (5t-3)(t+1) = 0$

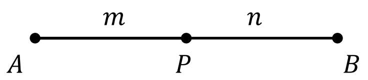
$\therefore t = -1, \frac{3}{5}$

For $t = -1, s = 2t = -2$ and for $t = \frac{3}{5}, s = 2t = \frac{6}{5}$

$\therefore s = \frac{6}{5}, t = \frac{3}{5}; s = -2, t = -1$

6. *A and B are points with p.vs* $\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ *and* $7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ *respectively. Find the coordinates of P and Q which divide AB*
   a) Internally in the ratio $1:2$
   b) Externally in the ratio $3:1$

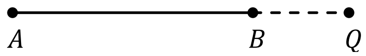
Solution

a)


Ratio: $\overrightarrow{AP}:\overrightarrow{PB}=1:2 \Rightarrow \frac{\overrightarrow{AP}}{\overrightarrow{PB}}=\frac{1}{2}$ or $2\overrightarrow{AP}=\overrightarrow{PB}$

$\Rightarrow 2\left(\overrightarrow{OP}-\overrightarrow{OA}\right)=\overrightarrow{OB}-\overrightarrow{OP}\ \therefore 3\overrightarrow{OP}=\overrightarrow{OB}+2\overrightarrow{OA}$

$\Rightarrow OP=\frac{1}{3}(7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})+\frac{2}{3}(\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})=3\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}\quad \therefore P(3,-1,3)$

b)


Ratio: $\overrightarrow{AQ}:\overrightarrow{QB}=3:-1$ (***check ratio values***)

$\frac{\overrightarrow{AQ}}{\overrightarrow{QB}}=\frac{3}{-1}\Rightarrow\ -\overrightarrow{AQ}=3\overrightarrow{QB}$

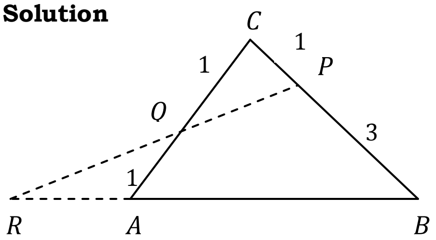
$\Rightarrow 2\left(\overrightarrow{OQ}-\overrightarrow{OA}\right)=3\left(\overrightarrow{OB}-\overrightarrow{OQ}\right)$

$2\overrightarrow{OQ}=3\overrightarrow{OB}-\overrightarrow{OA}\Rightarrow\overrightarrow{OQ}=\frac{3}{2}(7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})-(\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})$

$\overrightarrow{OQ}=\frac{19}{2}\boldsymbol{i}-\frac{1}{2}\boldsymbol{j}-7\boldsymbol{k}\quad\therefore Q\left(\frac{19}{2},-\frac{1}{2},-7\right)$

7. *The vertices A, B and C of a triangle have p.vs* ***a, b,*** *and* ***c*** *respectively. The point P divides BC internally in the ratio 3:1, the point Q is the midpoint of CA and the point R divides AB externally in the ratio 1:3. If the p.vs of P, Q and R are* ***p, q*** *and* ***r*** *respectively, express* ***r*** *in terms of* ***p*** *and* ***q****. find also the ratio* $PR:QR$.

Solution



Given: $\overrightarrow{OA}=\boldsymbol{a}\,,\overrightarrow{OB}=\boldsymbol{b},\overrightarrow{OC}=\boldsymbol{c}$

$\overrightarrow{BP}:\overrightarrow{PC}=3:1\,,\overrightarrow{AQ}:\overrightarrow{QC}=1:1$

$\overrightarrow{AR}:\overrightarrow{RB}=-1:3$

$\overrightarrow{OP}=\boldsymbol{p}\,,\overrightarrow{OQ}=\boldsymbol{q}\,,\overrightarrow{OR}=\boldsymbol{r}$

From $\overrightarrow{AR}:\overrightarrow{RB}=-1:3$

$\Rightarrow\frac{\overrightarrow{AR}}{\overrightarrow{RB}}=-\frac{1}{3}$ or $3\overrightarrow{AR}=-\overrightarrow{RB}$

$\Rightarrow 3\left(\overrightarrow{OR}-\overrightarrow{OA}\right)=-\left(\overrightarrow{OB}-\overrightarrow{OR}\right)=-\overrightarrow{OB}+\overrightarrow{OR}$

$\Rightarrow 2\overrightarrow{OR}=3\overrightarrow{OA}-\overrightarrow{OB}\ \therefore\overrightarrow{OR}=\frac{3}{2}\boldsymbol{a}-\frac{1}{3}\boldsymbol{b}$ .... (1)

From $\overrightarrow{BP}:\overrightarrow{PC}=3:1\Rightarrow\frac{\overrightarrow{BP}}{\overrightarrow{PC}}=\frac{3}{1}$ or $\overrightarrow{BP}=3\overrightarrow{PC}$

$\Rightarrow\overrightarrow{OP}-\overrightarrow{OB}=3\left(\overrightarrow{OC}-\overrightarrow{OP}\right)=3\overrightarrow{OC}-3\overrightarrow{OP}$

$\Rightarrow 4\overrightarrow{OP}=\overrightarrow{OB}+3\overrightarrow{OC}=\boldsymbol{b}+3\boldsymbol{c}$

$\therefore\overrightarrow{OP}=\frac{1}{4}\boldsymbol{b}+\frac{3}{4}\boldsymbol{c}$ .... (2)

From $\overrightarrow{AQ}:\overrightarrow{QC}=1:1\Rightarrow\overrightarrow{AQ}=\overrightarrow{QC}$

$\Rightarrow\overrightarrow{OQ}-\overrightarrow{OA}=\overrightarrow{OC}-\overrightarrow{OQ}$

$\Rightarrow 2\overrightarrow{OQ}=\overrightarrow{OA}+\overrightarrow{OC}\ \therefore\overrightarrow{OQ}=\frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OC}$

$\therefore \boldsymbol{q} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ ………………………………………………………… (3)

Solving (1) , (2) and (3) simultaneously by expressing $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$

$\boldsymbol{p} = \frac{1}{4}\boldsymbol{b} + \frac{3}{4}\boldsymbol{c}$ or $4\boldsymbol{p} = \boldsymbol{b} + 3\boldsymbol{c}$ …………………………… (4)

$\boldsymbol{q} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ or $2\boldsymbol{q} = \boldsymbol{a} + \boldsymbol{c}$ ……………………………… (5)

Eliminating **c** from (4) and (5)

$$\left.\begin{aligned} 4\boldsymbol{p} &= \boldsymbol{b} + 3\boldsymbol{c} \\ 2\boldsymbol{q} &= \boldsymbol{a} + \boldsymbol{c} \end{aligned}\right| \times 3 \qquad \Rightarrow \begin{aligned} 4\boldsymbol{p} &= \boldsymbol{b} + 3\boldsymbol{c} \\ -\quad 6\boldsymbol{q} &= 3\boldsymbol{a} + 3\boldsymbol{c} \\ \hline 4\boldsymbol{p} - 6\boldsymbol{q} &= -3\boldsymbol{a} + \boldsymbol{b} \end{aligned}$$
……………… (6)

From (1) $\overrightarrow{OR} = \boldsymbol{r} = \frac{3}{2}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$ or $2\boldsymbol{r} = 3\boldsymbol{a} - \boldsymbol{b}$ …………………… (7)

Eliminating **a** and **b** from (6) and (7), adding the equations

$\Rightarrow 4\boldsymbol{p} - 6\boldsymbol{q} + 2\boldsymbol{r} = 0$ or $2\boldsymbol{p} - 3\boldsymbol{q} + \boldsymbol{r} = 0$

$\therefore \boldsymbol{r} = 3\boldsymbol{q} - 2\boldsymbol{p}$

Ratio $\overrightarrow{PR}:\overrightarrow{QR} = \frac{\overrightarrow{PR}}{\overrightarrow{QR}}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (3\boldsymbol{q} - 2\boldsymbol{p}) - \boldsymbol{p} = 3(\boldsymbol{q} - \boldsymbol{p})$

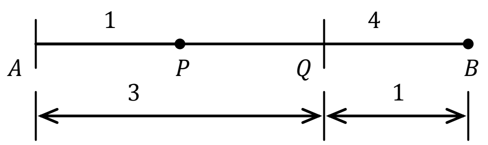
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = (3\boldsymbol{q} - 2\boldsymbol{p}) - \boldsymbol{q} = 2(\boldsymbol{q} - \boldsymbol{p})$

$\therefore \frac{\overrightarrow{PR}}{\overrightarrow{QR}} = \frac{3}{2}\frac{(\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{q}-\boldsymbol{p})} = \frac{3}{2}$

$\therefore \overrightarrow{PR}:\overrightarrow{QR} = 3:2$ $(Internal\ division)$

8. A, P, Q, B are points on a straight line such that $AP:PB = 1:4, AQ:QB = 3:1$. Find;
   a) $AP:PQ:QB$
   b) $AQ:QP$

Solution



Let p.v of A and B be **a** and **b** respectively.

For $AP:PQ = 1:4 \Rightarrow \overrightarrow{OP} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}$

For $AQ:QB = 3:1 \Rightarrow \overrightarrow{OQ} = \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

a) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b} - \boldsymbol{a} = \frac{1}{5}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) - \left(\frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}\right) = -\frac{11}{20}\boldsymbol{a} + \frac{11}{20}\boldsymbol{b} = \frac{11}{20}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{QB} = \overrightarrow{OB} - \overrightarrow{OQ} = \boldsymbol{b} - \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = -\frac{1}{4}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b} = \frac{1}{4}(\boldsymbol{b} - \boldsymbol{a})$

$Now\ \overrightarrow{AP}:\overrightarrow{PQ} = \frac{\overrightarrow{AP}}{\overrightarrow{PQ}} = \frac{\frac{1}{5}(\boldsymbol{b}-\boldsymbol{a})}{\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})} = \frac{4}{11}$ $\quad \therefore \overrightarrow{AP}:\overrightarrow{PQ} = 4:11$

Also $\overrightarrow{PQ}:\overrightarrow{QB} = \frac{\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})}{\frac{1}{4}(\boldsymbol{b}-\boldsymbol{a})} = \frac{11}{5}$ $\quad \therefore \overrightarrow{PQ}:\overrightarrow{QB} = 11:5$

Combining the ratios using the ratio algebra,

$AP:PQ = 4:11$; $PQ:QB = 11:5$ $\therefore AP:PQ:QB = 4:11:5$

b) $\overrightarrow{AQ}:\overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}}$

$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) - \boldsymbol{a} = \frac{3}{4}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \left(\frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}\right) - \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = -\frac{11}{20}(\boldsymbol{b} - \boldsymbol{a})$

$\therefore \overrightarrow{AQ}:\overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}} = \frac{\frac{3}{4}(\boldsymbol{b}-\boldsymbol{a})}{-\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})} = \frac{3}{4} \times \frac{20}{-11} = \frac{15}{-11}$ $\therefore AQ:QP = 15:-11$

NOTE:

Vectors $\overrightarrow{OP} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$ have been deduced using *Ratio Theorem*

Task

1. If the points A and B have p.vs **a** and **b** respectively, find in terms of **a** and **b** the p.vs of
   a) The point P which divides AB internally in the ratio 5:3
   b) The point Q which divides AB externally in the ratio 3:1
   c) The point R such that A is the midpoint of BR

   ANSWER: a) $\frac{1}{8}(3\boldsymbol{a}+5\boldsymbol{b})$, b) $\frac{1}{2}(3\boldsymbol{b}-\boldsymbol{a})$ c) $2\boldsymbol{a}-\boldsymbol{b}$
2. Write down, in terms of **I , j** and **k** the position vectors of the points $A(2,-5,3)$ and $B(7,0,-2)$. Hence find the coordinates of C which divides AB internally in the ratio 2:3 and point D which divides it externally in the ratio 3:8.

   ANSWER: $2\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k}$; $7\boldsymbol{i}-2\boldsymbol{k}$; $(4,-3,1)$, $(-1,-8,6)$
3. The points A, B, C and D have coordinates $A(-7,\ 9)$, $B(3,4)$, $C(1,2)$ and $D(-2.-9)$. Find the position vector of a point P that divides AB in the ratio $2:3$ and Q that divides it in the ratio $1:-4$
4. **ANSWER**:Show that the points A, B and C with p.vs $3\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$, $8\boldsymbol{i}+7\boldsymbol{j}+4\boldsymbol{k}$ and $11\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$ respectively are vertices of a triangle. (***Uneb* 2011**)
5. Given that $\boldsymbol{a} = 2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i}+2\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i}+2\boldsymbol{j}$ and $\boldsymbol{d} = -4\boldsymbol{j}+8\boldsymbol{k}$**.** Express **d** in terms of **a, b** and **c.**

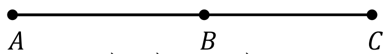
   Answer: *Left as an exercise*

   $[HINT\text{: }Let\ \boldsymbol{d} = \lambda\boldsymbol{a}+\mu\boldsymbol{b}+k\boldsymbol{c}\text{, }Form\ and\ solve\ 3\ simulutaneous\ equations\ for\ the\ scalars]$

COLLINEARITY OF VECTORS

Vectors are either collinear or parallel.

Consider three collinear points A, B and C



Vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are collinear if $\boxed{\lambda\overrightarrow{AB}+\mu\overrightarrow{BC}+\gamma\overrightarrow{AC}=0}$ where λ, μ *and* γ are Scalars/constants and $\boxed{\lambda+\mu+\gamma=0}$

When vectors are given in the form $\boldsymbol{i}-\ \boldsymbol{j}-\ \boldsymbol{k}$**,** 3-equations are formulated containing only scalars. The 3 equations are solved for the scalars using cross multiplication, and when put in , $\boxed{\lambda+\mu+\gamma=0}$ and satisfies then the vectors are collinear.

Alternatively, (commonly used)

For $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ to be collinear/parallel, then $\boxed{\overrightarrow{AB}=\lambda\overrightarrow{BC}=\mu\overrightarrow{AC}}$, where λ, μ are scalars.

Examples

1. *Show that the points with the position vectors*

a) $\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}\,, 3\boldsymbol{i} + 2j + \boldsymbol{k}$ *and* $2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$
b) $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,, 4\boldsymbol{i} + 3\boldsymbol{j}$ *and* $10\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k}$
, are collinear.

Solution

a) Let $A(1\,,2\,,5)\,, B(3\,,2\,,1)\,, C(2\,,2\,,3)$

For A, B and C to be collinear, $\overrightarrow{AB} = \lambda\overrightarrow{AC}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}2\\0\\-4\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}2\\2\\3\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$

Now $\overrightarrow{AB} = \lambda\overrightarrow{AC}$ gives

$\begin{pmatrix}2\\0\\-4\end{pmatrix} = \lambda\begin{pmatrix}1\\0\\-2\end{pmatrix} \Rightarrow 2 = \lambda$ .... (1)

$\Rightarrow -4 = -2\lambda \ or\ \lambda = 2$ .... (2)

Since the $\lambda = 2$ is consistent for the two equations formulated, the points are collinear.

Alternatively (Not commonly used)

Let $A(1\,,2\,,5)\,, B(3\,,2\,,1)\,, C(2\,,2\,,3)$ have position vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

For the vectors to be collinear, then $\lambda\overrightarrow{OA} + \mu\overrightarrow{OB} + \gamma\overrightarrow{OC} = 0$

$\Rightarrow \lambda\begin{pmatrix}1\\2\\5\end{pmatrix} + \mu\begin{pmatrix}3\\2\\1\end{pmatrix} + \gamma\begin{pmatrix}2\\2\\3\end{pmatrix} = 0$

$\Rightarrow \lambda + 3\mu + 2\gamma = 0$ .... (1)

$\Rightarrow 2\lambda + 2\mu + 2\gamma = 0$ .... (2)

$\Rightarrow 5\lambda + \mu + \gamma = 0$............................... (3)

Also $\lambda + \mu + \gamma = 0$ .... (4)

Expressing $\lambda\,, \mu$ in terms of γ only

Considering eqn. (1) and eqn. (2), eliminating λ

$\lambda + 3\mu + 2\gamma = 0 \quad | \times 2$

$\underline{2\lambda + 2\mu + 2\gamma = 0 \quad | \times 1}$

$4\mu + 2\gamma = 0 \Rightarrow \mu = -\frac{1}{2}\gamma$.............. (5)

Eqn. (5) in to *eqn. (1)* gives

$\lambda + 3\left(-\frac{1}{2}\gamma\right) + 2\gamma = 0 \ \Rightarrow \lambda = -\frac{1}{2}\gamma$ .... (6)

Eqn. (5), (6) in to (3) and (4) for confirmation

From eqn. (3),

$5\lambda + \mu + \gamma = 0 \Rightarrow -\frac{5}{2}\gamma - \frac{1}{2}\gamma + 3\gamma = 0\ \ correct$

From eqn. (4) $\lambda + \mu + \gamma = 0$

$\Rightarrow -\frac{1}{2}\gamma - \frac{1}{2}\gamma + \gamma = 0\ \ correct$. Hence points are collinear.

b) *Left as an exercise*

2. *Show that the points R, P and T with position vectors are*

a) $\frac{1}{2}(3\boldsymbol{a} - \boldsymbol{b})\,, \frac{1}{70}(53\boldsymbol{a} + 17\boldsymbol{b})$ *and* $\frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{b})$
b) $7\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}, 10a + 2\boldsymbol{b} + 9\boldsymbol{c}$ *and* $\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}$
c) $\frac{1}{2}(17a + 20\boldsymbol{b} + 3\boldsymbol{c})\,, 10\boldsymbol{a} + 11\boldsymbol{b} + 2\boldsymbol{c}$ *and* $8\boldsymbol{a} + \frac{29}{3}\boldsymbol{b} + \frac{4}{3}\boldsymbol{c}$, are collinear.

Solution

a) For R, P and T to be collinear, $\overrightarrow{RP} = \lambda\overrightarrow{RT}$

$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR} = \frac{1}{70}(53\boldsymbol{a} + 17\boldsymbol{b}) - \frac{1}{2}(3\boldsymbol{a} - \boldsymbol{b})$

$= -\frac{26}{35}\boldsymbol{a} + \frac{26}{35}\boldsymbol{b} = \frac{26}{35}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{b}) - \frac{1}{2}(3\boldsymbol{a} - \boldsymbol{b})$

$= -\frac{13}{10}\boldsymbol{a} + \frac{13}{10}\boldsymbol{b} = \frac{13}{10}(\boldsymbol{b} - \boldsymbol{a})$

Now $\overrightarrow{RP} = \lambda\overrightarrow{RT}$ yields

$\frac{26}{35}(\boldsymbol{b} - \boldsymbol{a}) = \lambda\left[\frac{13}{10}(\boldsymbol{b} - \boldsymbol{a})\right] \Rightarrow \frac{26}{35} = \frac{13}{10}\lambda \quad \therefore \lambda = \frac{2}{3}(scalar)$

$\therefore \overrightarrow{RP} = \frac{2}{3}\overrightarrow{RT}$, if R, P and T are collinear

b) $\overrightarrow{RP} = \lambda\overrightarrow{RT}$

$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR} = (10\boldsymbol{a} + 11\boldsymbol{b} + 2\boldsymbol{c}) - (7\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = 3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c}$

$\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = (\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) - (7\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$

$= -6a - 2\boldsymbol{b} - 4\boldsymbol{c} = -2(3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c})$

$\Rightarrow (3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c}) = -2(3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c})\lambda$

$\Rightarrow 1 = -2\lambda \quad \therefore \lambda = -\frac{1}{2}\ (scalar)$

$\therefore \overrightarrow{RP} = -\frac{1}{2}\overrightarrow{RT}$, if R, P and T are collinear

3. *Given that* $\boldsymbol{a} = 2\boldsymbol{i} + 6\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + t\boldsymbol{j}$ *and* $\boldsymbol{c} = 2\lambda\boldsymbol{i} - \boldsymbol{j}$. *Find the value of* λ*and* t *if*
   a) ***a*** *and* ***b*** *are parallel*
   b) ***a*** *and* ***c*** *are collinear*

Solution

a) For **a** and **b** to be parallel, $\boldsymbol{a} = \mu\boldsymbol{b}$

$\Rightarrow (2\boldsymbol{i} + 6\boldsymbol{j}) = \mu(3\boldsymbol{i} + t\boldsymbol{j}) = 3\mu\boldsymbol{i} + t\mu\boldsymbol{j}$

Equating coefficients of unit vectors

For $\boldsymbol{i}$: $2 = 3\mu \Rightarrow \mu = \frac{2}{3}$

For $\boldsymbol{j}$: $6 = t\mu \Rightarrow 6 = \frac{2}{3}t \Rightarrow t = 9$

b) **a** and **c** are collinear/parallel if $\boldsymbol{a} = k\boldsymbol{c}$

$\Rightarrow 2\boldsymbol{i} + 6\boldsymbol{j} = k(2\lambda\boldsymbol{i} - \boldsymbol{j}) = 2\lambda k\boldsymbol{i} - k\boldsymbol{j}$

Equating coefficients of unit vectors

For $\boldsymbol{i}$: $2 = 2k\lambda$ .... (1)

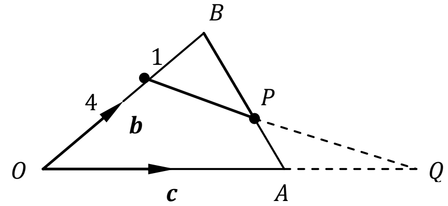
For $\boldsymbol{j}$: $6 = -k \Rightarrow k = -6$ .... (2)

Eqn. (2 in to (1)

$2 = -12\lambda \Rightarrow \lambda = -\frac{1}{6} \quad \therefore k = -6, \lambda = -\frac{1}{6}$

4. *Given that* $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OB} = \boldsymbol{b}$, *point R is on* $\overline{OB}$ *such that* $\overline{OR}:\overline{RB} = 4:1$. *Point P is on BA such that* $\overline{BP}:\overline{PA} = 2:3$, *and when RP and OA are both produced they meet at point Q. Find*
   a) $\overrightarrow{OR}$ and $\overrightarrow{OP}$ in terms of **a** and b
   b) $\overrightarrow{OQ}$ in terms of **a** (***Uneb* 2006**)

Solution



a) From $\overrightarrow{OR}:\overrightarrow{RB} = 4:1 \Rightarrow \frac{\overrightarrow{OR}}{\overrightarrow{RB}} = \frac{4}{1}$

$\overrightarrow{OR} = 4\overrightarrow{RB}$

$\Rightarrow \overrightarrow{OR} = 4(\overrightarrow{OB} - \overrightarrow{OR}) \Rightarrow \overrightarrow{OR} = \frac{4}{5}\overrightarrow{OB}$

$\therefore \overrightarrow{OR} = \frac{4}{5}\boldsymbol{b}$

From $\overrightarrow{BP}:\overrightarrow{PA} = 2:3 \Rightarrow \frac{\overrightarrow{BP}}{\overrightarrow{PA}} = \frac{2}{3} \Rightarrow 3\overrightarrow{BP} = 2\overrightarrow{PA}$

$\Rightarrow 3(\overrightarrow{OP} - \overrightarrow{OB}) = 2(\overrightarrow{OA} - \overrightarrow{OP}) \Rightarrow 5\overrightarrow{OP} = 2\overrightarrow{OA} + 3\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = \frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB} \quad \therefore \overrightarrow{OP} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}$

b) Since O,A and Q are collinear, then

$\overrightarrow{OQ} = \lambda\overrightarrow{OA} = \lambda\boldsymbol{a}$ .... (1)

Since R, P and Q are also collinear, then

$\overrightarrow{RQ} = \mu\overrightarrow{RP} \Rightarrow (\overrightarrow{OQ} - \overrightarrow{OR}) = \mu(\overrightarrow{OP} - \overrightarrow{OR})$

$\therefore \overrightarrow{OQ} = \overrightarrow{OR} + \mu(\overrightarrow{OP} - \overrightarrow{OR}) = (1-\mu)\overrightarrow{OR} + \mu\overrightarrow{OP}$

$= (1-\mu).\frac{4}{5}\boldsymbol{b} + \mu\left(\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}\right) = \left(\frac{4-4\mu}{5} + \frac{3}{5}\mu\right)\boldsymbol{b} + \frac{2}{5}\mu\,\boldsymbol{a}$

$\therefore \overrightarrow{OQ} = \left(\frac{4-\mu}{5}\right)\boldsymbol{b} + \frac{2}{5}\mu\,\boldsymbol{a}$ .... (2)

$eqn.(2) = eqn.(1)$

$\Rightarrow \lambda\boldsymbol{a} = \frac{2}{5}\mu\,\boldsymbol{a} + \left(\frac{4-\mu}{5}\right)\boldsymbol{b}$

Equating corresponding vectors

$\lambda = \frac{2}{5}\mu$ .... (3)

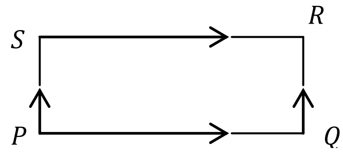
$0 = \frac{4-\mu}{5} \Rightarrow \mu = 4$.................................... (4)

$\therefore \lambda = \frac{8}{5}$

From eqn. (1), $\overrightarrow{OQ} = \lambda\overrightarrow{OA} = \lambda\boldsymbol{a} = \frac{8}{5}\boldsymbol{a}$

5. *The vertices of a triangle are $P(2,-1,5), Q(7,1,-3)$ and $R(2,-2,0)$. Find the coordinates of S if PQRS is a rectangle.*

Solution



using $\overrightarrow{PQ} = \overrightarrow{SR}$ or $\overrightarrow{PS} = \overrightarrow{QR}$

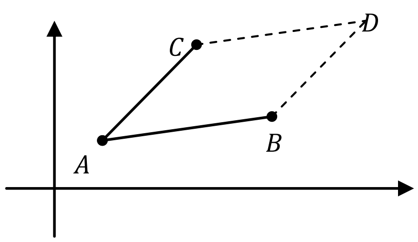
$\overrightarrow{PQ} = \overrightarrow{SR}$, gives $\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OR} - \overrightarrow{OS}$

$\therefore \overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{OP} - \overrightarrow{OQ}$

$= \begin{pmatrix}2\\-2\\0\end{pmatrix} + \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}-3\\-4\\8\end{pmatrix} \quad \therefore S(-3,-4,\ 8)$

6. *The vertices of a parallelogram are $A(1,1), B(5,3), C(3,7)$ and D. find the coordinates of D.*

Solution



For a parallelogram, $\overrightarrow{AB} = \overrightarrow{CD}$ or $\overrightarrow{AC} = \overrightarrow{BD}$

Using $\overrightarrow{AB} = \overrightarrow{CD}$

$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OC} \Rightarrow \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{OD} = \binom{3}{7} + \binom{5}{3} - \binom{1}{1} = \binom{7}{9} \quad \therefore D(7,9)$

PRODUCT OF TWO VECTORS

There are two types of products

- Scalar or Dot product
- Vector or Cross product

SCALAR/DOT PRODUCT

Consider two position vectors $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ making and angle of θ with each other. The scalar product is defined as

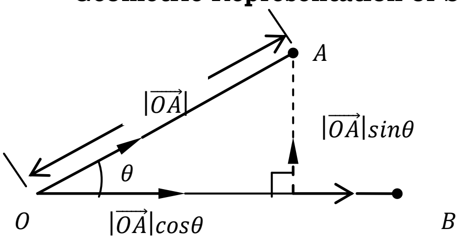
$$\boxed{\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta \ \ where\ 0 \le \theta \le \pi}$$

The product will either be $+ve\ or\ -ve\ or\ zero$ depending on the value of θ.

NOTE

Two vectors can be dot multiplied if they are in *same direction.*

Geometric Representation of Scalar product



$|\overrightarrow{OA}|cos\theta$ is the projection of $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA}cos\theta$ and $\boldsymbol{b}$ are in the same direction.

Multiplying their scalars

$\Rightarrow |\overrightarrow{OA}|cos\theta \times |\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|cos\theta = \boldsymbol{a}.\boldsymbol{b}$

$$\boxed{\therefore \boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta}$$

PROPERTIES OF SCALAR PRODUCT

The properties below are very useful in the computations involving product of vectors.

1) $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$ commutative property
2) $(\boldsymbol{\lambda a}).\boldsymbol{b} = \lambda(\boldsymbol{a}.\boldsymbol{b})$ where $\lambda = scalar$
3) $(\boldsymbol{\lambda a}).(\mu\boldsymbol{b}) = \mu\lambda(\boldsymbol{a}.\boldsymbol{b}) = (\mu\boldsymbol{a}.\lambda\boldsymbol{b})$
4) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$ Distributive property
5) $(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^2 + 2\boldsymbol{a}.\boldsymbol{b} + b^2$ ($a\ and\ b$ are now $scalars$)
6) $(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^2 - b^2$ $[\boldsymbol{a} \neq a]$. Here a and b are scalars
7) $(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = (\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^2 + 2\boldsymbol{a}.\boldsymbol{b} + b^2$ ($a\ and\ b$ are now $scalars$)
8) For $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$
   a) If $\theta = 90^0$ $i.e.\ \boldsymbol{a}\ is\ perpendicular\ to\ \boldsymbol{b}$, then $\boldsymbol{a}.\boldsymbol{b} = 0$ $[\because cos90^0 = 0]$. Hence **a** and **b** are perpendicular if

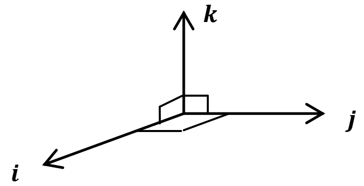
   $$\boxed{\boldsymbol{a}.\boldsymbol{b} = 0}$$

   b) If $\theta = 180^0$. $i.e.\ \boldsymbol{a}\ and\ \boldsymbol{b}\ are\ in\ opposide\ direction$, then $\boldsymbol{a}.\boldsymbol{b} = -|\boldsymbol{a}||\boldsymbol{b}|$ $[\because cos180^0 = -1]$
   c) Angle between **a** and **b** is given from

   $$\boxed{cos\theta = \frac{|\boldsymbol{a}||\boldsymbol{b}|}{\boldsymbol{a}.\boldsymbol{b}}}$$

MULTIPLICATION OF UNIT VECTORS $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$

Unit vectors are mutually perpendicular



$\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{i}^2 = |\boldsymbol{i}||\boldsymbol{i}|cos0 = 1$

$\boldsymbol{j}.\boldsymbol{j} = \boldsymbol{j}^2 = |\boldsymbol{j}||\boldsymbol{j}|cos0 = 1$

$\boldsymbol{k}.\boldsymbol{k} = \boldsymbol{k}^2 = |\boldsymbol{k}||\boldsymbol{k}|cos0 = 1$

$\boldsymbol{k}.\boldsymbol{i} = |\boldsymbol{k}||\boldsymbol{i}|cos90 = 0$

$\boldsymbol{i}.\boldsymbol{j} = |\boldsymbol{i}||\boldsymbol{j}|cos90 = 0$

$\boldsymbol{k}.\boldsymbol{j} = |\boldsymbol{k}||\boldsymbol{j}|cos90 = 0$

The above results in conjunction with the above properties of dot product are useful in vector product computations.

Example

1. *Given that* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ *and* $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{2j} - 3\boldsymbol{k}$. *Find*
   a) $\boldsymbol{a}.\boldsymbol{b}$
   b) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})$
   c) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$

Solution

a) $\boldsymbol{a}.\boldsymbol{b} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 2\boldsymbol{i}.\boldsymbol{i} + \boldsymbol{2i}.\boldsymbol{j} - 6\boldsymbol{i}.\boldsymbol{k}$

$$+\boldsymbol{j}.\boldsymbol{i} + \boldsymbol{j}.\boldsymbol{j} - 3\boldsymbol{j}.\boldsymbol{k} + \boldsymbol{k}.\boldsymbol{i} + \boldsymbol{k}.\boldsymbol{j} - 3\boldsymbol{k}.\boldsymbol{k}$$

$$= 2 + 0 - 0 + 0 + 1 + 0 + 0 + 0 - 3 = 2 + 1 - 3 = 0$$

b) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})$; $\boldsymbol{b} + \boldsymbol{c} = (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + (\boldsymbol{i} + \boldsymbol{2j} - 3\boldsymbol{k}) = 2\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(2\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k})$

$= \boldsymbol{i}.\boldsymbol{i} + \boldsymbol{2i}.\boldsymbol{j} - 3\boldsymbol{i}.\boldsymbol{k} + \boldsymbol{j}.\boldsymbol{i} + \boldsymbol{j}.\boldsymbol{j} - 3\boldsymbol{j}.\boldsymbol{k} - 3\boldsymbol{k}.\boldsymbol{i} - 6\boldsymbol{k}.\boldsymbol{j} + 9\boldsymbol{k}.\boldsymbol{k}$

$= 2 + 0 + 0 + 0 + 2 - 0 + 0 + 0 - 6 = 2 + 2 - 6 = -2$

c) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$; $\boldsymbol{a} - \boldsymbol{b} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = \boldsymbol{i} + 0\boldsymbol{j} + 4\boldsymbol{k}$

But $\boldsymbol{a}.\boldsymbol{b} = 0 \Rightarrow (\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = 0.(\boldsymbol{i} + 0\boldsymbol{j} + 4\boldsymbol{k}) = 0$

NOTE

When doting two vectors in multiplication always dot the corresponding vectors only. $i.e. i^2$, j^2 and k^2, since other products of not similar unit vectors yield zero results

2. *Given that* $\boldsymbol{a} = 4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$, $\boldsymbol{b} = 7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k}$. *Find*
   a) $\boldsymbol{a}.\boldsymbol{b}$
   b) $\boldsymbol{b}.\boldsymbol{c}$
   c) $\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b})$
   d) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{b} + \boldsymbol{c})$

Solution

a) $\boldsymbol{a}.\boldsymbol{b} = (4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}).(7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$$= 28 + 10 - 7 = 31$$

b) $\boldsymbol{b}.\boldsymbol{c} = (7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}).(3\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k})$

$$= 21 - 8 + 9 = 22$$

c) $\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b})$; $\boldsymbol{a} + \boldsymbol{b} = (4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}) + (7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 11\boldsymbol{i} + 7\boldsymbol{j} - 6\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = (4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}).(11\boldsymbol{i}+7\boldsymbol{j}-6\boldsymbol{k}) = 44+35+42 = 121$

Alternatively

Using $\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{a} = (4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}).(4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}) = 16+25+49 = 90$

$\therefore\ \boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = 90+31 = 121$

d) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{b}+\boldsymbol{c}) = 31\times[(7\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})+(3\boldsymbol{i}-4\boldsymbol{j}+9\boldsymbol{k})]$

$= 31(10\boldsymbol{i}-\boldsymbol{2j}+10\boldsymbol{k}) = 310\boldsymbol{i}-62\boldsymbol{j}+310\boldsymbol{k}$

NOTE

When two vectors are dot multiplied the answer is a constant value **not** a vector.

3. *Given that* $|\boldsymbol{a}| = 7, |\boldsymbol{b}| = 4$ *and* $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{8},$ *calculate:*
   a) $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})$ *and* $(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})$
   b) Hence find $|\boldsymbol{a}+\boldsymbol{b}|$ *and* $|\boldsymbol{a}-\boldsymbol{b}|$

Solution

a) $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}+2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b}\ = \boldsymbol{a}.\boldsymbol{a}+\boldsymbol{b}.\boldsymbol{b}+2(8)$

But if $\boldsymbol{a} = x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}$, $|\boldsymbol{a}| = \sqrt{x_1{}^2+x_2{}^2+x_3{}^2}$

, and $\boldsymbol{a}.\boldsymbol{a} = (x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}).(x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}) = x_1{}^2+x_2{}^2+x_3{}^2$

$\therefore |\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a} = 49, |\boldsymbol{b}|^2 = \boldsymbol{b}.\boldsymbol{b} = 16$

$\Rightarrow (\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) = 49+16+16 = 81$

b) $(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}-2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b}$

$= \boldsymbol{a}.\boldsymbol{a}-2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b} = 49-2\times8+16 = 49$

From $|\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a} \Rightarrow |\boldsymbol{a}+\boldsymbol{b}|^{\boldsymbol{2}} = (\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})$

$\therefore |\boldsymbol{a}+\boldsymbol{b}| = \sqrt{(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})} = \sqrt{81} = 9$

Also $|\boldsymbol{a}-\boldsymbol{b}| = \sqrt{(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})} = \sqrt{49} = 7$

Task

Given that $|\boldsymbol{x}| = 5, |\boldsymbol{y}| = 10$ *and* $\boldsymbol{x}.\boldsymbol{y} = 22$, calculate $|\boldsymbol{x}+\boldsymbol{y}|$ and $|\boldsymbol{x}-\boldsymbol{y}|$

***Answer*: 13 , 9**

4. *Vectors* ***a, b*** *and* ***c*** *are such that* $\boldsymbol{a}.\boldsymbol{c} = 3,\ \boldsymbol{b}.\boldsymbol{c} = 4$. *Given that the vector* $\boldsymbol{d} = \boldsymbol{a}+\lambda\boldsymbol{b}$ *is perpendicular to c, find the value of* λ.

Solution

$\boldsymbol{d}$ *is perpendicular to* $\boldsymbol{c}$ *if* $\boldsymbol{d}.\boldsymbol{c} = 0$ [*see properties*]

$\boldsymbol{d} = \boldsymbol{a}+\lambda\boldsymbol{b} = (\boldsymbol{a}+\lambda\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c}+\lambda\boldsymbol{b}.\boldsymbol{c} = 0$ [*see properties*]

$\Rightarrow 3+4\lambda = 0\ \therefore \lambda = -\frac{3}{4}$

5. *Given that* $\boldsymbol{a} = 2\boldsymbol{i}+6\boldsymbol{j}$ *and* $\boldsymbol{b} = 3\boldsymbol{i}+\lambda\boldsymbol{j}$. *Find the value of* λ *such that* ***a*** *and* ***b*** *are perpendicular.*

Solution

For **a** and **b** to be perpendicular $\boldsymbol{a}.\boldsymbol{b} = 0$

$\Rightarrow (2\boldsymbol{i}+6\boldsymbol{j}).(3\boldsymbol{i}+\lambda\boldsymbol{j}) = 0$

$\Rightarrow 6+6\lambda = 0\ \therefore \lambda = 1$

6. *Find the angle between vectors*
   a) $8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ *and* $2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$
   b) $\boldsymbol{j}+7\boldsymbol{k}$ *and* $-5\boldsymbol{i}+4\boldsymbol{j}+3\boldsymbol{k}$
   c) $2\boldsymbol{i}-3\boldsymbol{j}+8\boldsymbol{k}$ *and* $6\boldsymbol{i}-4\boldsymbol{j}-3\boldsymbol{k}$

Solution

a) Let $\boldsymbol{a} = 8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$

From dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

$\boldsymbol{a}.\boldsymbol{b} = (8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}).(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}) = 16-2-4 = 10$

$|\boldsymbol{a}| = \sqrt{8^2 + (-1)^2 + 4^2} = 9$, $|\boldsymbol{b}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$\therefore 10 = 9 \times 3cos\theta \;\; or\; cos\theta = \frac{10}{27}$

$\Rightarrow \theta = cos^{-1}\left(\frac{10}{27}\right) = 68.3^0$

b) $Left\ as\ an\ exercise$ $\quad$ $\boldsymbol{Answer}$: 60^0

c) Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 8\boldsymbol{k}$, $\boldsymbol{b} = 6\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}$

From dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

$\boldsymbol{a}.\boldsymbol{b} = (2\boldsymbol{i} - 3\boldsymbol{j} + 8\boldsymbol{k}).(6\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}) = 12 + 12 - 24 = 0$

$\therefore 0 = |\boldsymbol{a}||\boldsymbol{b}|cos\theta \;\; or\; cos\theta = 0 \Rightarrow \theta = cos^{-1}(0) = 90^0$

Proofs involving dot product

1. *Given that* $|\boldsymbol{a}| = 3, |\boldsymbol{b}| = 8$ *and* $|\boldsymbol{a} - \boldsymbol{b}| = 7$, *show that the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is* 60^0.

Solution

By dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

But $|\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a}$, $|\boldsymbol{b}|^2 = \boldsymbol{b}.\boldsymbol{b}$ and $|\boldsymbol{a} - \boldsymbol{b}|^2 = (\boldsymbol{a} - \boldsymbol{b})^2$

From $|\boldsymbol{a} - \boldsymbol{b}|^2 = 49 = (\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$

$\Rightarrow 49 = 3^2 - 2\boldsymbol{a}.\boldsymbol{b} + 8^2 = 73 - 2\boldsymbol{a}.\boldsymbol{b}$

$\Rightarrow -24 = -2\boldsymbol{a}.\boldsymbol{b} \;\; \therefore \boldsymbol{a}.\boldsymbol{b} = 12$

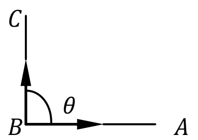
$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta = 12$

$\Rightarrow 12 = 3 \times 8\, cos\theta \;\; or\; cos\theta = \frac{1}{2}$

$\therefore cos^{-1}\left(\frac{1}{2}\right) = 60^0$

2. *The points A, B and C have p.vs* $\boldsymbol{a} = 5\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{c} = 7\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}$ *respectively. Show that* $< ABC$ *is a right angle.*

Solution



NOTE: indicated vectors originate from B

Given: $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}, \overrightarrow{OC} = \boldsymbol{c}$

$\overrightarrow{BA} = \boldsymbol{d} = \overrightarrow{OA} - \overrightarrow{OB}$

$= (5\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

$\overrightarrow{BC} = \boldsymbol{e} = \overrightarrow{OC} - \overrightarrow{OB} = (7\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= 5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$

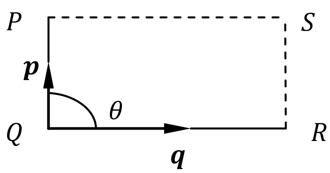
For $\boldsymbol{d}$ and $\boldsymbol{e}$ to be perpendicular, $\boldsymbol{d}.\boldsymbol{e} = \boldsymbol{0}$, by dot product $\boldsymbol{d}.\boldsymbol{e} = |\boldsymbol{d}||\boldsymbol{e}|cos\theta$

$\Rightarrow \boldsymbol{d}.\boldsymbol{e} = (3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}).(5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}) = 15 - 8 - 7 = 0$

$\therefore \boldsymbol{d}.\boldsymbol{e} = 0 \Rightarrow cos\theta = 0 \;\; \therefore \theta = 90^0$

3. *The vertices of a* Δ *are* $P(2,-1,5)$, $Q(7,1,-3)$ *and* $R(13,-2,0)$. *Show that* $< PQR = 90^0$. *Find the coordinates of S if PQRS is a rectangle.* ($\boldsymbol{Uneb}\ \mathbf{2002}$)

Solution



Let $\overrightarrow{PQ} = \boldsymbol{p} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$

$\overrightarrow{QR} = \boldsymbol{q} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix} 13 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$

For **p** and **q** to be perpendicular, $\boldsymbol{p}.\boldsymbol{q} = \mathbf{0}$

$$\Rightarrow \boldsymbol{p}.\boldsymbol{q} = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}.\begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 30 - 6 - 24 = 0$$

$or\ \boldsymbol{p}.\boldsymbol{q} = (5\boldsymbol{i} + 2\boldsymbol{j} - 8\boldsymbol{k}).(6\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}) = 30 - 6 - 24 = 0$

$\therefore cos\theta = 0\ \ or\ \ \theta = 90^0$

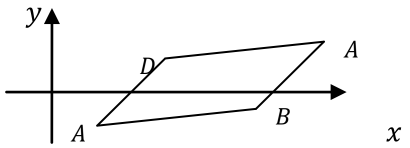
For a rectangle, $\overrightarrow{QR} = \overrightarrow{PS}$

$\Rightarrow \boldsymbol{q} = \overrightarrow{OS} - \overrightarrow{OP}\ \therefore\ \overrightarrow{OS} = \boldsymbol{q} + \overrightarrow{OP}$

$$\Rightarrow \overrightarrow{OS} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix} \therefore S(8, -4, 8)$$

4. *ABCD is a quadrilateral with* $A(2\,,-2)\,, B(5\,,-1)\,, C(6\,,2)$ *and* $D(3\,,1)$. *Show that the quadrilateral is a rhombus.*

Solution



For a rhombus

- All sides are equal
- Diagonals meet at 90^0
- All opposite angles are equal but not right angles

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$|\overrightarrow{AB}| = |\overrightarrow{DC}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$|\overrightarrow{AD}| = |\overrightarrow{BC}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

Hence all sides are equal= $\sqrt{10}\,units$

Now diagonals are $\overrightarrow{BD}$ and $\overrightarrow{AC}$

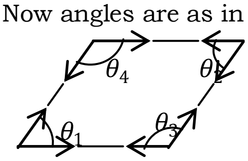
$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

For $\overrightarrow{BD}$ and $\overrightarrow{AC}$ to be perpendicular, $\overrightarrow{BD}.\overrightarrow{AC} = 0$

$\overrightarrow{BD}.\overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 8 - 8 = 0$. Hence diagonals meet at 90^0

Now angles are as in the figure.



$< \theta_1 = angle\ between\ \overrightarrow{AD}\ and\ \overrightarrow{AB}$

$\Rightarrow \overrightarrow{AD}\,.\overrightarrow{AB} = |\overrightarrow{AD}||\overrightarrow{AB}|cos\theta_1$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{10} \times \sqrt{10}\ cos\theta$

$cos\theta_1 = \frac{3+3}{10} \Rightarrow \theta_1 = 53.1^0$

$< \theta_2 = angle\ between\ \overrightarrow{CB}\ and\ \overrightarrow{CD}$

$\Rightarrow \overrightarrow{CB}\,.\overrightarrow{CD} = |\overrightarrow{CB}||\overrightarrow{CD}|cos\theta_2$

But $\overrightarrow{CB} = -\overrightarrow{BC}\,, \overrightarrow{CD} = -\overrightarrow{DC}$

$\therefore |\overrightarrow{CB}| = |\overrightarrow{BC}| = |\overrightarrow{CD}| = |\overrightarrow{DC}| = \sqrt{10}$

$\therefore \overrightarrow{CB}.\overrightarrow{CD} = \overrightarrow{BC}.\overrightarrow{DC} = |\overrightarrow{BC}||\overrightarrow{DC}|cos\theta_2$

$\Rightarrow \begin{pmatrix}1\\3\end{pmatrix}.\begin{pmatrix}3\\1\end{pmatrix} = \sqrt{10} \times \sqrt{10}\ cos\theta_2$

$\therefore \theta_1 = \theta_2 = 53.1^0$

$< \theta_3 = angle\ between\ \overrightarrow{AD}\ and\ \overrightarrow{AB}$

$\Rightarrow \overrightarrow{BC}.\overrightarrow{BA} = |\overrightarrow{BC}||\overrightarrow{BA}|cos\theta_3$

$\Rightarrow \begin{pmatrix}1\\3\end{pmatrix}.\begin{pmatrix}-3\\-1\end{pmatrix} = \sqrt{10} \times \sqrt{10}\ cos\theta_3\ [\because |\overrightarrow{BA}| = |\overrightarrow{AB}|]$

$-3 - 3 = 10cos\theta_3\ \ or\ 10cos\theta_3 = -0.6\ \ \therefore \theta_3 = 126.9^0$

$< \theta_3 = angle\ between\ \overrightarrow{DA}\ and\ \overrightarrow{DC} = 126.9^0$

Hence this is a rhombus.

5. *The points $P(4,-6,1), Q(2,8,4)$ and $R(3,7,14)$ lie in the same plane. Show that the angle formed between $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is $cos^{-1}\left(\frac{14}{\sqrt{209\times102}}\right)$.*

Solution

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}2\\8\\4\end{pmatrix} - \begin{pmatrix}4\\-6\\1\end{pmatrix} = \begin{pmatrix}-2\\14\\5\end{pmatrix}$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix}3\\7\\14\end{pmatrix} - \begin{pmatrix}2\\8\\4\end{pmatrix} = \begin{pmatrix}1\\-1\\10\end{pmatrix}$$

Using dot product $\overrightarrow{PQ}.\overrightarrow{QR} = |\overrightarrow{PQ}||\overrightarrow{QR}|cos\theta$

$$\overrightarrow{PQ}.\overrightarrow{QR} = \begin{pmatrix}-2\\14\\5\end{pmatrix}.\begin{pmatrix}1\\-1\\10\end{pmatrix} = -2 - 14 + 30 = 14$$

$|\overrightarrow{PQ}| = \sqrt{(-2)^2 + 14^2 + 3^2} = \sqrt{209}$

$|\overrightarrow{QR}| = \sqrt{1^2 + (-1)^2 + 10^2} = \sqrt{102}$

$\therefore cos\theta = \left(\frac{14}{\sqrt{209}\times\sqrt{102}}\right) = \left(\frac{14}{\sqrt{209\times102}}\right) \Rightarrow \theta = cos^{-1}\left(\frac{14}{\sqrt{209\times102}}\right)$

6. *Show that A, B and C with p.vs respectively are vertices of a triangle of a ΔABC*

a) $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$ *and* $3\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$

b) $4\boldsymbol{i} - 5\boldsymbol{j} - 2\boldsymbol{k}, 8\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$ *and origin*

c) $7\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$ *and* $3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

Solution

a) Let $A(2,-1,1), B(1,-3,-5), C(3,-4,-4)$

For A, B and C to be vertices of a ΔABC, then

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0\ (for\ a\ closed\ figure)$

or

Sum of angles at A, B and C is 180^0

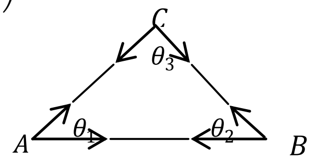
Considering the **first case**

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\-3\\-5\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-6\end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}3\\-4\\4\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} - \begin{pmatrix}1\\-3\\-5\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$$

Now from $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\begin{pmatrix}-1\\-2\\-6\end{pmatrix}+\begin{pmatrix}1\\-3\\-5\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$. Hence the vectors are vertices of a triangle.



Now considering **second case**

$<\theta_1 = angle\ between\ \overrightarrow{AC}\ and\ \overrightarrow{AB}$

By dot product $\overrightarrow{AC}.\overrightarrow{AB}=|\overrightarrow{AC}|.|\overrightarrow{AB}|cos\theta_1$

$\Rightarrow \begin{pmatrix}1\\-3\\-5\end{pmatrix}.\begin{pmatrix}-1\\-2\\-6\end{pmatrix}=\sqrt{1^2+(-3)^2+(-5)^2}.\sqrt{(-1)^2+(-2)^2+(-6)^2}\ cos\theta_1$

$-1+6+30=\sqrt{35}\times\sqrt{41}.cos\theta_1 \Rightarrow\ cos\theta_1=\frac{35}{\sqrt{35\times41}}\ \ \therefore \theta_1=22.49^0$

$<\theta_2 = angle\ between\ \overrightarrow{BA}\ and\ \overrightarrow{BC}$

$\Rightarrow \overrightarrow{BA}.\overrightarrow{BC}=|\overrightarrow{BA}||\overrightarrow{BC}|cos\theta_2\ , but\ \overrightarrow{BA}=-\overrightarrow{AB}$

$\Rightarrow \begin{pmatrix}1\\2\\6\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=\sqrt{1^2+2^2+6^2}\times\sqrt{2^2+(-1)^2+1^2}.cos\theta_2$

$\Rightarrow 2-2+6=\sqrt{41}\times\sqrt{6}.cos\theta_2 \Rightarrow\ cos\theta_2=\frac{6}{\sqrt{41}\times\sqrt{6}}\ \ \therefore \theta_2=67.51^0$

$<\theta_3 = angle\ between\ \overrightarrow{CA}\ and\ \overrightarrow{CB}$

But $\overrightarrow{CA}=-\overrightarrow{AC}\ and\ \overrightarrow{CB}=-\overrightarrow{BC}$

$\Rightarrow (-\overrightarrow{AC}).(-\overrightarrow{BC})=|\overrightarrow{AC}|.|\overrightarrow{BC}|\ cos\theta_3\ \ [\because |\overrightarrow{AC}|=|-\overrightarrow{AC}|, |\overrightarrow{BC}|=|-\overrightarrow{BC}|]$

$\Rightarrow \overrightarrow{AC}.\overrightarrow{BC}=|\overrightarrow{AC}|.|\overrightarrow{BC}|\ cos\theta_3$

$\begin{pmatrix}1\\-3\\-5\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=\sqrt{1^2+(-3)^2+(-5)^2}\times\sqrt{2^2+(-1)^2+1^2}.cos\theta_3$

$2+3-5=\sqrt{35}\times\sqrt{6}.cos\theta_3\ \ or\ cos\theta_3=0\ \ \ \therefore \theta=90^0$

$Now\ \theta_1+\theta_2+\theta_3=22.49^0+67.51^0+90^0=180^0$, which satisfy the condition. Hence points are vertices of a triangle.

b) $Left\ as\ an\ exercise$

c) $Left\ as\ an\ exercise$

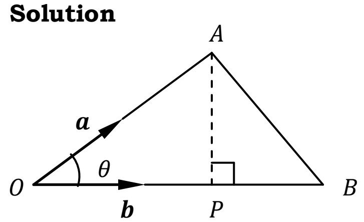
NOTE

The question can also be done using $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

Other Scalar Product Proofs

1. *The points A and B have p.vs* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *with respect to the origin O. show that the area of a triangle OAB is given by*
$\Delta=\frac{1}{2}\sqrt{\{a^2b^2-(\boldsymbol{a}.\boldsymbol{b})^2\}}$, where $a=|\boldsymbol{a}|, b=|\boldsymbol{b}|$

Solution



$Area, \Delta=\frac{1}{2}\times|\overrightarrow{OB}|\times\overline{AP}$

From dot product, $\boldsymbol{a}.\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\ cos\theta$

From ΔOAP, $\overline{AP}=|\overrightarrow{OA}|sin\theta=asin\theta$

$\therefore \Delta=\frac{1}{2}\times b\times asin\theta=\frac{1}{2}absin\theta$

But $sin\theta=\sqrt{1-cos^2\theta}$ $[\because \boldsymbol{cos^2\theta+sin^2\theta=1}]$

From $\boldsymbol{a}.\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|cos\theta=abcos\theta$

$\Rightarrow \frac{\boldsymbol{a}.\boldsymbol{b}}{ab}=cos\theta$

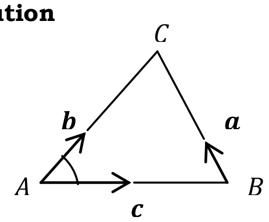
$\Rightarrow sin\theta = \sqrt{1 - \left(\frac{\boldsymbol{a}.\boldsymbol{b}}{ab}\right)^2} = \frac{1}{ab}\sqrt{a^2b^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$

$\therefore Area, \Delta = \frac{1}{2}absin\theta = \frac{1}{2}ab \times \frac{1}{ab}\sqrt{a^2b^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$

$\therefore \Delta = \frac{1}{2}\sqrt{\{a^2b^2 - (\boldsymbol{a}.\boldsymbol{b})^{\boldsymbol{2}}\}}$ *As required*

2. *In a ΔABC with vectors $AC = \boldsymbol{b}$, $AB = \boldsymbol{c}$ and $BC = \boldsymbol{a}$, then $a^2 = b^2 + c^2 - 2bc.cosA$, the cosine rule where $a = |\boldsymbol{a}|$, $b = |\boldsymbol{b}|$*

Solution



$a^2 = \boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c}).(\boldsymbol{b} - \boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= b^2 - \boldsymbol{2b}.\boldsymbol{c} + c^2$

But $\boldsymbol{b}.\boldsymbol{c} = |\boldsymbol{b}||\boldsymbol{c}|cosA = bc\ cosA$

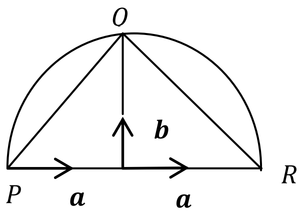
$\therefore a^2 = b^2 - 2bc.cosA + c^2$

$\boldsymbol{or}\ a^2 = b^2 + c^2 - 2bc.cosA$ *As required*

Note: the figure is very important in the derivation.

3. *Show that any angle inscribed in a semi circle is a right angle.*

Solutior



Let O be the centre of the semi circle,

$\overrightarrow{PO} = \overrightarrow{OR} = \boldsymbol{a}$ and

$OQ = \boldsymbol{b}$

Required: $\overrightarrow{QP}.\overrightarrow{QR} = 0$ $.i.e. < P\hat{Q}R = 90^0$

$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = -\boldsymbol{a} - \boldsymbol{b} = -(\boldsymbol{a} + \boldsymbol{b})$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \boldsymbol{a} - \boldsymbol{b}$

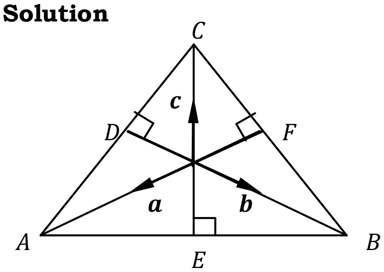
Now $\overrightarrow{QP}.\overrightarrow{QR} = -(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = -(\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b})$

But $\boldsymbol{a}.\boldsymbol{a} = a^2$, $\boldsymbol{b}.\boldsymbol{b} = b^2$ *and* $a^2 = b^2 = radius$

$\Rightarrow \overrightarrow{QP}.\overrightarrow{QR} = -(a^2 - b^2) = (a^2 - a^2) = 0 \therefore < P\hat{Q}R = 90^0$

4. *In ΔABC, the altitudes through A, B and C intersect at point O. show that if $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$, then $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{a}.\boldsymbol{c}$ and $\boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c}$. show also further that $\boldsymbol{a}.\boldsymbol{c} = \boldsymbol{b}.\boldsymbol{c}$*

Solution



From the figure, points A, O, F; B, O, D and C, O, E are collinear.

$\Rightarrow \overrightarrow{AF} = \lambda\overrightarrow{OA}$, $\overrightarrow{BD} = \mu\overrightarrow{OB}$ *and* $\overrightarrow{OE} = k\overrightarrow{OC}$, where λ, μ *and* k are either positive/negative constants. Since

$\overrightarrow{AF} \perp \overrightarrow{BC}$, *then* $\overrightarrow{AF}.\overrightarrow{BC} = 0$ ………………… (1)

$\overrightarrow{AD} \perp \overrightarrow{AC}$, then $\overrightarrow{AD}.\overrightarrow{AC} = 0$…………. (2)

$\overrightarrow{CE} \perp \overrightarrow{AB}$, then $\overrightarrow{CE}.\overrightarrow{AB} = 0$ ………. (3)

From eqn. (1)

$\overrightarrow{AF}.\overrightarrow{BC} = 0$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \boldsymbol{c} - \boldsymbol{b}$

$\Rightarrow \lambda OA.(\boldsymbol{c} - \boldsymbol{b}) = 0$ *or* $\boldsymbol{a}.(\boldsymbol{c} - \boldsymbol{b}) = 0$

$\therefore \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{b} = 0 \Rightarrow \boldsymbol{a}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{b}$

From eqn. (2)

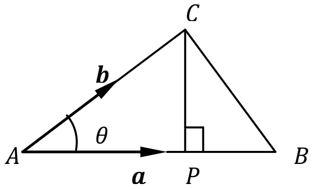
$\overrightarrow{AD}.\overrightarrow{AC} = 0$, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

$\Rightarrow \mu OB.(\boldsymbol{c} - \boldsymbol{a}) = 0$ *or* $\boldsymbol{b}.(\boldsymbol{c} - \boldsymbol{a}) = 0$

$\therefore \boldsymbol{b}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{a} = 0 \Rightarrow \boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c}$
From $\boldsymbol{a}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{b} \Rightarrow \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{b} = 0$ .... (4)
From $\boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c} \Rightarrow \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{c} = 0$ .... (5)
$Eqn.(4) + eqn.(5)$
$\boldsymbol{a}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{c} = 0 \;\therefore \boldsymbol{b}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c}$

AREA OF A TRIANGLE, BY DOT PRODUCT

Consider A, B and C be vertices of a triangle ABC



Let $\overrightarrow{AB} = \boldsymbol{a}, \overrightarrow{AC} = \boldsymbol{b}$
$|\overrightarrow{AC}| = |\boldsymbol{b}|, |\overrightarrow{AB}| = |\boldsymbol{a}|$
$Area, \Delta = \frac{1}{2} \times base \times height = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{AC}|$
But $\overline{CP} = |\overrightarrow{AC}| sin\theta = |\boldsymbol{b}| sin\theta$
Now finding $sin\theta$

By dot product, $\overrightarrow{AC}.\overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}| cos\theta$

$\Rightarrow cos\theta = \frac{\boldsymbol{b}.\boldsymbol{a}}{|\boldsymbol{b}||\boldsymbol{a}|}$

$\Rightarrow Area, \Delta = \frac{1}{2} \times |\boldsymbol{a}| \times |\boldsymbol{b}| \times sin\theta = \frac{1}{2} \times |\boldsymbol{a}| \times \sqrt{(1 - cos^2\theta)}$ $[\because \boldsymbol{cos^2\theta + sin^2\theta = 1}]$

$= \frac{1}{2} \times |\boldsymbol{a}| \times |\boldsymbol{b}| \times \sqrt{\left(1 - \left(\frac{\boldsymbol{b}.\boldsymbol{a}}{|\boldsymbol{b}||\boldsymbol{a}|}\right)^2\right)} = \frac{1}{2}\sqrt{\{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2\}}$

$$\boxed{\therefore Area, \Delta = \frac{1}{2}\sqrt{\{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2\}}}$$

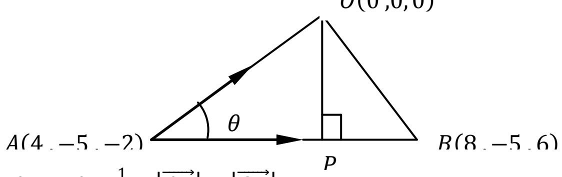
Examples

Find the area of the triangle with vertices

a) $A(4, -5, -2), B(8, -5, 6)$ *and origin*
b) $OA = \boldsymbol{i} - \boldsymbol{j}, OB = 5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}, OC = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$
c) $A(0, -2, 1), B(1, -1, -2)$ *and* $C(-1, 1, 0)$

Solution

a)



$Area, \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}|$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{4^2 + 0^2 + 8^2} = \sqrt{80}$

$|\overrightarrow{OP}| = |\overrightarrow{AO}| sin\theta$; $|\overrightarrow{AO}| = \sqrt{4^2 + (-5)^2 + (-2)^2} = \sqrt{45}$

From dot product, $\overrightarrow{AO}.\overrightarrow{AB} = |\overrightarrow{AO}||\overrightarrow{AB}| cos\theta$

$\overrightarrow{AO}.\overrightarrow{AB} = -\begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}.\begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = -(16 + 0 - 16) = 0$

$\therefore \overrightarrow{AO}.\overrightarrow{AB} = 0 \Rightarrow cos\theta = 0 \;\therefore \theta = 90^0$

$\therefore Area, \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}| = \frac{1}{2} \times \sqrt{80} \times \sqrt{45} \times sin90^0$

$= \frac{1}{2} \times \sqrt{16 \times 5} \times \sqrt{9 \times 5} = \frac{1}{2} \times 60 = 30\, sq.units$

b) $2\sqrt{2}\, sq.units$

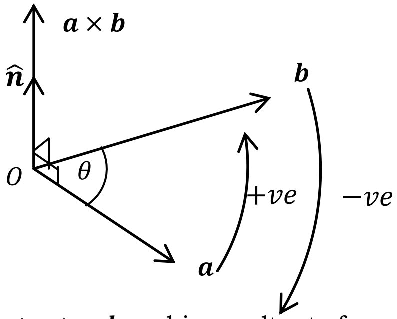
c) ***Answer***: $\sqrt{30}\, sq.units$

VECTOR/CROSS PRODUCT

For two vectors **a** and **b**, the cross product of **a** and **b** is such that

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| sin\theta.\hat{\boldsymbol{n}}\ , where\ 0 \le \theta \le 360^0}$$

Consider the figure below

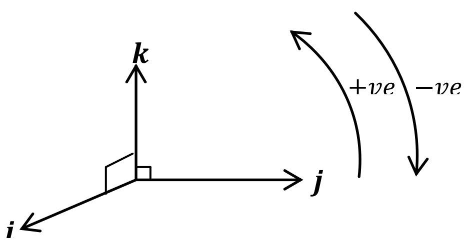


$\hat{\boldsymbol{n}} = Normal\ unit\ vector \perp \boldsymbol{a}\ , \boldsymbol{b}$ and is resultant of $\boldsymbol{a} \times \boldsymbol{b}$

Properties of cross product

1. $\boldsymbol{a} \times \boldsymbol{b} \ne \boldsymbol{b} \times \boldsymbol{a}$ [$\because$ $\boldsymbol{a} \times \boldsymbol{b}$ *is in anticlockwise while* $\boldsymbol{a} \times \boldsymbol{b}$ *is in clokwise direction*]
2. $m(\boldsymbol{a} \times \boldsymbol{b}) = m\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times m\boldsymbol{b}$
3. From the definition $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| sin\theta.\hat{\boldsymbol{n}}$
   a) For $\theta = 0^0, \boldsymbol{a} \times \boldsymbol{b} = 0\ .i.e.\ \boldsymbol{a}\ and\ \boldsymbol{b}\ are\ parallel$. Thus for **a** and **b** to be parallel or collinear, $\boxed{\boldsymbol{a} \times \boldsymbol{b} = 0}$
   b) For $\theta = 90^0, \boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|.\hat{\boldsymbol{n}}\ ,\ \boldsymbol{a} \perp \boldsymbol{b}$. Thus for perpendicular vectors $\boldsymbol{a} \perp \boldsymbol{b}$ $\boxed{\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|.\hat{\boldsymbol{n}}}$
4. $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{c} = 0, since\ \theta = 0$
5. $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \ne (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

MULTIPLICATION OF UNIT VECTORS $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$



$$\left.\begin{aligned} \boldsymbol{i} \times \boldsymbol{j} &= \boldsymbol{k} \\ \boldsymbol{j} \times \boldsymbol{k} &= \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} &= \boldsymbol{j} \end{aligned}\right\}\ Crossed\ in\ Anticlockwise$$

$$\left.\begin{aligned} \boldsymbol{j} \times \boldsymbol{i} &= -\boldsymbol{k} \\ \boldsymbol{i} \times \boldsymbol{k} &= -\boldsymbol{j} \\ \boldsymbol{k} \times \boldsymbol{j} &= -\boldsymbol{i} \end{aligned}\right\}\ Crossed\ in\ Clockwise$$

Examples

Given that $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}\ , \boldsymbol{b} = 4\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}$ *and* $\boldsymbol{c} = 7\boldsymbol{i} + 8\boldsymbol{j} + 9\boldsymbol{k}$ ***,*** *find;*

a) $\boldsymbol{a} \times \boldsymbol{b}$

b) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

Solution

a) $\boldsymbol{a} \times \boldsymbol{b} = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \times (4\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k})$

$$\begin{aligned}
&= 4\boldsymbol{i}\times\boldsymbol{i} + 5\boldsymbol{i}\times\boldsymbol{j} - 6\boldsymbol{i}\times\boldsymbol{k} + \\
&\quad 8\boldsymbol{j}\times\boldsymbol{i} + 10\boldsymbol{j}\times\boldsymbol{j} - 12\boldsymbol{j}\times\boldsymbol{k} + \\
&\quad 12\boldsymbol{k}\times\boldsymbol{i} + 15\boldsymbol{k}\times\boldsymbol{j} - 18\boldsymbol{k}\times\boldsymbol{k} \\
&= 0 + 5\boldsymbol{k} + 6\boldsymbol{j} + \\
&\quad -8\boldsymbol{k} + 0 - 12\boldsymbol{i} + \\
&\quad 12\boldsymbol{j} - 15\boldsymbol{i} - 0 \\
&= -27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k}
\end{aligned}$$

b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (-27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k})\times(7\boldsymbol{i} + 8\boldsymbol{j} + 9\boldsymbol{k})$

$= 186\boldsymbol{i} + 222\boldsymbol{j} - 342\boldsymbol{k}$ ***Left as an exercise***

NOTE:
The same results can still be obtained using multiplication of vectors using *Rectangular Components.*

MULTIPLICATION OF VECTORS BY RECTANGULAR COMPONENTS

Consider $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, where the elements in $|\quad|$ give a matrix whose determinant is of a 2×2 square matrix

Example

Referring to the above example

$\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 4\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}$ and $\boldsymbol{c} = 7\boldsymbol{i} + 8\boldsymbol{j} + 9\boldsymbol{k}$

a) $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & -6 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & 3 \\ 5 & -6 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 3 \\ 4 & -6 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$= (-12 - 15)\boldsymbol{i} + (-6 - 12)\boldsymbol{j} + (5 - 8)\boldsymbol{k}$

$= -27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k}$ *As before*

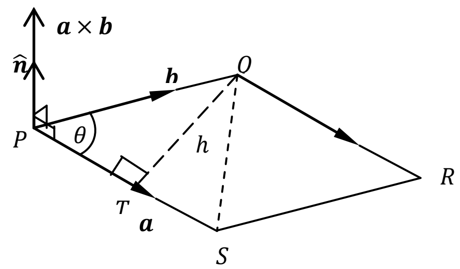
b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -27 & 18 & -3 \\ 7 & 8 & 9 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 18 & -3 \\ 8 & 9 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -27 & -3 \\ 7 & 9 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -27 & 18 \\ 7 & 8 \end{vmatrix}$

$= 186\boldsymbol{i} + 222\boldsymbol{j} - 342\boldsymbol{k}$ *As before*

NOTE:
Using Rectangular Components in multiplication of two vectors is easier to handle to handle than the first method.

GEOMETRIC INTERPRETATION OF VECTOR PRODUCT

The magnitude of a vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the *area* of a parallelogram whose adjacent sides are represented by $\boldsymbol{a}$ and $\boldsymbol{b}$.



From the definition;

$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}$,

where $\hat{\boldsymbol{n}}$ *is a unit vector* $\perp$ *to* $\boldsymbol{a}$ *and* $\boldsymbol{b}$

$\therefore |\boldsymbol{a}\times\boldsymbol{b}| = \big||\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}\big|; since\ |\hat{\boldsymbol{n}}| = 1$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\{|\boldsymbol{b}|sin\theta\}$

From ΔPQT, $h = |\boldsymbol{b}|sin\theta = height\ of\ parallerogram$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\times h$, where $|\boldsymbol{a}| = base\ of\ parallelogram$

$\therefore |\boldsymbol{a} \times \boldsymbol{b}| = base \times height = Area\ of\ parallellogram$
But $\Delta s\ PQS\ and QRS\ are\ equal$

$$\boxed{\therefore |\boldsymbol{a} \times \boldsymbol{b}| = 2 \times Area\ of\ \Delta PQS\ or\ Area\ of\ \Delta = \frac{1}{2} \times |\boldsymbol{a} \times \boldsymbol{b}|}$$

Examples

1. *Find the unit vector perpendicular to each vector* $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ *. find also the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$

Solution

Using $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}} \Rightarrow \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|sin\theta}$
But $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta\ [\because |\hat{\boldsymbol{n}}| = 1]$

$$\boxed{\Rightarrow \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|} = unit\ vector\ perpendicular\ to\ \boldsymbol{a}\ and\ \boldsymbol{b}}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 4 \\ 3 & 1 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = -8\boldsymbol{i} + 8\boldsymbol{j} + 8\boldsymbol{k}$$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(-8)^2 + 8^2 + 8^2} = 8\sqrt{3}$

$\therefore \hat{\boldsymbol{n}} = \frac{-8\boldsymbol{i}+8\boldsymbol{j}+8\boldsymbol{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(= -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\ )$

$Now\ finding\ \theta$

$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta, |\boldsymbol{a}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24},\ |\boldsymbol{b}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$

$\Rightarrow sin\theta = \frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{8\sqrt{3}}{\sqrt{24}\times\sqrt{14}} = 8\sqrt{\left(\frac{3}{24\times14}\right)} = \frac{8}{\sqrt{112}}$

$\therefore \theta = sin^{-1}\left(\frac{8}{\sqrt{112}}\right) = 4.1^0$

2. *Find the unit vector perpendicular to vectors* $\boldsymbol{a} = 6\boldsymbol{i} - 3\boldsymbol{j} - 14\boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$*. Hence also find the angle between them.*

Solution

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 6 & -3 & -14 \\ 3 & 2 & -3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -3 & -14 \\ 2 & -3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 6 & -14 \\ 3 & -3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 6 & -3 \\ 3 & 2 \end{vmatrix} = 37\boldsymbol{i} - 24\boldsymbol{j} + 21\boldsymbol{k}$$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{37^2 + 24^2 + 21^2} = \sqrt{2386}$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|} = \frac{1}{\sqrt{2386}}(37\boldsymbol{i} - 24\boldsymbol{j} + 21\boldsymbol{k}\ )$

$|\boldsymbol{a}| = \sqrt{6^2 + (-3)^2 + (-14)^2} = \sqrt{241},\ |\boldsymbol{b}| = \sqrt{3^2 + 2^2 + (-3)^2} = \sqrt{22}$

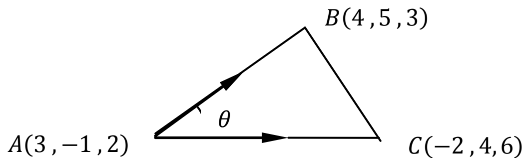
$From\ sin\theta = \frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\sqrt{2386}}{\sqrt{241}\times\sqrt{22}}$

$\therefore \theta = 26.7^0$

3. *Find the area of the triangle with vertices*
   a) $(3, -1, 2), (4,\ 5, 3)\ and\ (-2,\ 4,\ 6)$
   b) $i - 2j + 3k, -3i - 2j + k\ and\ origin$
   c) $(1, -1, 2), (2,\ 1, -1)\ and\ (3, -1,\ 2)$

Solution

a) Let $A(3, -1, 2), B(4,\ 5, 3)\ and\ C(-2\ .\ 4\ .\ 6)$



Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 4 \end{pmatrix}$

$Area\ of \Delta = \frac{1}{2}\ |\boldsymbol{a} \times \boldsymbol{b}|$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 6 & 1 \\ -5 & 5 & 4 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 6 & 1 \\ 5 & 4 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 1 \\ -5 & 4 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 6 \\ -5 & 5 \end{vmatrix} = 19\boldsymbol{i} - 9\boldsymbol{j} + 35\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{19^2 + (-9)^2 + 35^2} = \sqrt{1667}$

$\therefore Area = \frac{1}{2} \times \sqrt{1667} = 20.41\ sq.units$

b) $3\sqrt{5}\ sq.units$Let

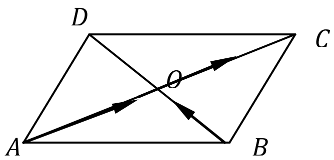
c) Left as an exercise ***Answer***: $\boldsymbol{\sqrt{13}}\ sq.units$

4. *Calculate the area of a parallelogram having diagonals;*

   a) $\overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}\ and\ \overrightarrow{BD} = \boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$

   b) $\overrightarrow{AB} = 2\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}\ and\ \overrightarrow{DC} = 4\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$

Solution

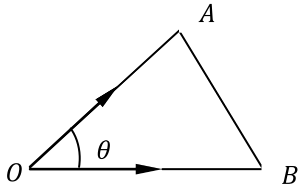


a) Let $O = midpoint\ of\ \overline{AC}\ and\ \overline{BD}$

Now there are $4 - triangles$ of the same area

$\therefore Area\ of\ ABCD = 4 \times area\ of\ one\ triangle$

Consider ΔOCD



$\overrightarrow{OB} = -\overrightarrow{BO}$

Let $\overrightarrow{OC} = \boldsymbol{c}\ and\ \overrightarrow{OB} = \boldsymbol{b}$

But $\overrightarrow{BO} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}(\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k})$

$\Rightarrow \overrightarrow{OB} = \boldsymbol{b} = \frac{1}{2}(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$

Also $\overrightarrow{OC} = \boldsymbol{c} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$\boldsymbol{b} \times \boldsymbol{c} = \frac{1}{2}(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) \times \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= \frac{1}{4}[(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) \times (3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})]\ [\because (\boldsymbol{a}m) \times (\boldsymbol{b}n) = mn(\boldsymbol{a} \times \boldsymbol{b})]$

$\boldsymbol{b} \times \boldsymbol{c} = \frac{1}{4}\begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -1 & 3 & -4 \\ 3 & 1 & -2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -1 & -4 \\ 3 & -2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix}$

$= \frac{1}{4}(-2\boldsymbol{i} + 14\boldsymbol{j} - 10\boldsymbol{k}) = \frac{1}{2}(-\boldsymbol{i} + 7\boldsymbol{j} - 5\boldsymbol{k})$

$Area\ of \Delta = \frac{1}{2}\ |\boldsymbol{b} \times \boldsymbol{c}|$

$|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2}\sqrt{(-1)^2 + 7^2 + (-5)^2} = \frac{1}{2}\sqrt{75} = \frac{1}{2} \times 5\sqrt{3}$

$\therefore Area\ of \Delta = \frac{1}{2}\ |\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2} \times \frac{1}{2} \times 5\sqrt{3} = \frac{5}{4}\sqrt{3}\ sq.units$

$\therefore Area\ of\ parallelogram = 4 \times \frac{5}{4}\sqrt{3} = 5\sqrt{3}\ sq.units$

b) *Left as an exercise.* ***Answer***: $30\ sq.units$

Task

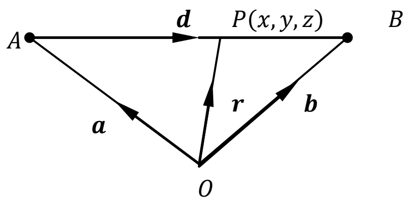
Find the unit vector perpendicular to vectors;

a) $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j}\ and\ \boldsymbol{b} = \boldsymbol{i}$; ***Answer***: $\frac{\sqrt{2}}{2}$

b) $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}\ and\ \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$; ***Answer***: $\frac{\sqrt{2}}{3}$

VECTOR EQUATION OF A STRAIGHT LINE

a) **To find the vector equation of a line passing through a given point A and parallel to a given vector.**



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$; $but\ \overrightarrow{AP}\ is$ // to $\mathbf{d}$

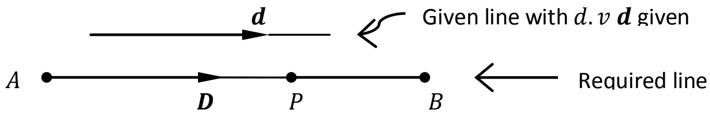
$\therefore \overrightarrow{AP} = \lambda\mathbf{d}$

$\Rightarrow \overrightarrow{OP} = \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

$$\boxed{\therefore \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}}$$

, which is in *parametric* form of equation of a line

b) **To find the Cartesian equation of a line through a given point** $A(a_1\,, a_2\,, a_3)$ **and parallel to the given line whose direction cosines are directly proportional to** $B(b_1\,, b_2\,, b_3)$



From (a) above, the required equation is

$\therefore \mathbf{r} = \mathbf{a} + k\,\mathbf{d}$

But $\mathbf{d}\ is$ // $\mathbf{D}$ $\Rightarrow \mathbf{D} = \mu\mathbf{d}$

$\therefore\ \mathbf{r} = \mathbf{a} + k\,\mu\mathbf{d}$ **,** replacing $k\mu\ with\ \lambda$

$$\boxed{\Rightarrow \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}}$$

Where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\Rightarrow x = a_1 + \lambda\, b_1 \quad \therefore \lambda = \frac{x-a_1}{b_1}$

$\Rightarrow y = a_2 + \lambda\, b_2 \quad \therefore \lambda = \frac{y-a_2}{b_2}$

$\Rightarrow z = a_3 + \lambda\, b_3 \quad \therefore \lambda = \frac{z-a_3}{b_3}$

$\therefore The\ Cartesian\ equation\ of\ a\ line\ is$

$$\boxed{\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}}$$

c) **To find the equation of a line passing through two given points A and B**

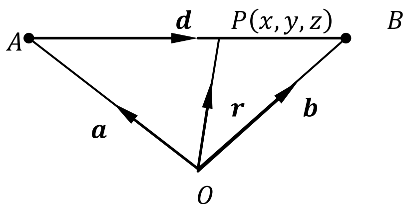
Let $A(a_1\,, a_2\,, a_3)\ and\ B(b_1\,, b_2\,, b_3)$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$; $but\ \overrightarrow{AP}\ is\ //\ \overrightarrow{AB}$

$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

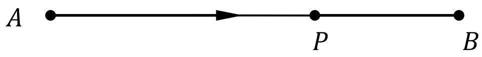
$\therefore \overrightarrow{OP} = \mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$



$$\boxed{\Rightarrow \boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right]}$$

VECTOR EQUATION OF A LINE IN NON-PARAMETRIC FORM

Let $A(a_1\,, a_2\,, a_3)$ *and* $B(b_1\,, b_2\,, b_3)$



$\overrightarrow{AP}$ *is* // $\overrightarrow{AB} \Rightarrow \overrightarrow{AP} \times \overrightarrow{AB} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a}\,, \boldsymbol{a} = \overrightarrow{OA}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{d}$

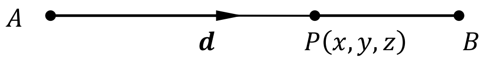
$$\boxed{(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{d} = 0}$$

General examples

1. *Find the vector equation of a line through a point* $4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ *and parallel to the vector* $-2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$. *Express the equation in;*
   a) Cartesian form
   b) Non-parametric form

Solution

Let the point be $A(4\,, 2\,, -3)\,, \boldsymbol{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$



$\overrightarrow{AP}$ *is* // $\boldsymbol{d}\,, \overrightarrow{AP} = \lambda\boldsymbol{d}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$

$\Rightarrow \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

$\therefore \boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

Let $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{r}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

$x = 4 - 2\lambda \Rightarrow \lambda = \frac{x-4}{-2}$

$y = 2 + \lambda \Rightarrow \lambda = \frac{y-2}{1}$

$z = -3 + 6\lambda \Rightarrow \lambda = \frac{z+3}{6}$

$\therefore \frac{x-4}{-2} = \frac{y-2}{1} = \frac{z+3}{6}$ *, cartesian form*

Now expressing in Non-parametric form

$\overrightarrow{AP}$ *is* // $\boldsymbol{d} \Rightarrow \overrightarrow{AP} \times \boldsymbol{d} = 0$ [For parallel vectors]

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix},\ \boldsymbol{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\therefore \left[\boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}\right] \times \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} = 0 \ \ or\ \ [\boldsymbol{r} - (4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})] \times (-2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}) = 0$$

2. *Find the equation of a straight line through points;*
   a) $A(1\,,-1\,,-1)$ *and* $B(3\,,4\,,2)$
   b) $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $4\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$
   c) $A(1\,,2\,,5)$ *and origin*
   Express the equation in Cartesian form

Solution

a) $A(1\,,-1\,,-1)$ *and* $B(3\,,4\,,2)$

$\overrightarrow{AP}$ *is* // $\overrightarrow{AB} \Rightarrow \overrightarrow{AP} = \lambda \overrightarrow{AB}$

$\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$$\therefore \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \left[\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right]$$

$$\therefore \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Now expressing in Cartesian form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$\Rightarrow x = 1 + 2\lambda \ \Rightarrow \lambda = \frac{x-1}{2}$

$\Rightarrow y = -1 + 5\lambda \ \Rightarrow \lambda = \frac{y+1}{5}$

$\Rightarrow z = -1 + 3\lambda \ \Rightarrow \lambda = \frac{z+1}{3}$

$$\therefore \frac{x-1}{2} = \frac{y+1}{5} = \frac{z+1}{3}$$

b) *Left as an exercise.* **ANSWER**: $\frac{x-1}{3} = \frac{y-2}{-5} = \frac{z-3}{-8}$

c) *Left as an exercise.* **ANSWER**: $\frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-5}{-5}$

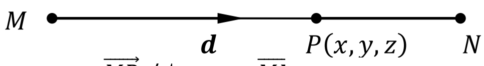
3. *Given the points* $A(-3\,,3\,,4)\,,B(5\,,7\,,2)$ *and* $C(1\,,1\,,4)$. *Find the vector equation of a line which joins the midpoints of* $\overline{AB}$ *and* $\overline{BC}$.

Solution

Let M and N be the points of $\overline{AB}$ *and* $\overline{BC}$ respectively.

$M\left(\frac{-3+5}{2}\,,\frac{3+7}{2}\,,\frac{4+2}{2}\right) = M(1\,,5\,,3)$

$N\left(\frac{5+1}{2}\,,\frac{1+7}{2}\,,\frac{4+2}{2}\right) = N(3\,,4\,,3)$



$\overrightarrow{MP}$ $//MN \ \Rightarrow \overrightarrow{MP} = \lambda MN$

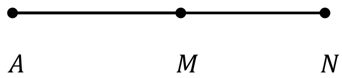
$$\overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-5 \\ z-3 \end{pmatrix}$$

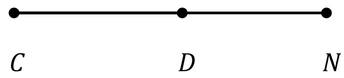
$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x-1 \\ y-5 \\ z-3 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \; or \; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

4. *Given that $A(1,2,3)$, $B(4,3,2)$, $C(-1,-4,-5)$ and $D(-5,-2,-1)$. Find the vector equation of a line joining points M and N such that M is the midpoint of $\overline{AB}$ and N divides $\overline{CD}$ in the ratio $-4:1$.*

Solution





$M\left(\frac{1+4}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right) = M\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$

Finding the coordinates of N

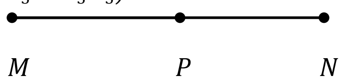
Ratio of division: $CN:ND = -4:1$

$\frac{\overrightarrow{CN}}{\overrightarrow{ND}} = -\frac{4}{1} \Rightarrow \overrightarrow{CN} = -4\overrightarrow{ND}$

$\overrightarrow{ON} - \overrightarrow{OC} = -4\left(\overrightarrow{OD} - \overrightarrow{ON}\right) = -4\overrightarrow{OD} + 4\overrightarrow{ON}$

$$-3\overrightarrow{ON} = \overrightarrow{OC} - 4\overrightarrow{OD} \Rightarrow \overrightarrow{ON} = -\frac{1}{3}\left[\begin{pmatrix} -1 \\ -4 \\ -5 \end{pmatrix} - 4\begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix}\right] = -\frac{1}{3}\begin{pmatrix} 19 \\ 4 \\ -1 \end{pmatrix}$$

$\therefore N\left(-\frac{19}{3}, -\frac{4}{3}, \frac{1}{3}\right)$



$\overrightarrow{MP} = \lambda\overrightarrow{MN} \Rightarrow \overrightarrow{OP} - \overrightarrow{OM} = \lambda\left(\overrightarrow{ON} - \overrightarrow{OM}\right)$

$$\Rightarrow \boldsymbol{r} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{pmatrix} = \lambda \left(\begin{pmatrix} -\frac{19}{3} \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{pmatrix} \right) = \begin{pmatrix} -\frac{53}{6} \\ -\frac{23}{6} \\ -\frac{13}{6} \end{pmatrix} = -\frac{\lambda}{6}\begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix}$$

$$Let \; -\frac{\lambda}{6} = k, \Rightarrow \boldsymbol{r} - \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \end{pmatrix} = k\begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix} \; or \; \boldsymbol{r} = \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \end{pmatrix} + k\begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix}$$

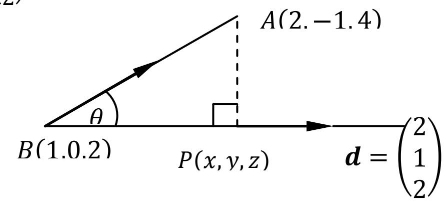
EQUATION OF A PERPENDICULAR AND PERPENDICULAR DISTANCE OF A POINT FROM A GIVEN LINE:

Dot or vector product may be used.

1. *Find the equation of perpendicular line from point* $\boldsymbol{A} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ *on to the line* $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. *Find also perpendicular distance of* $\boldsymbol{A}$ *from* $\boldsymbol{r}$

Solution

The line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ passes through the point $B(1,0,2)$ with direction vector $\boldsymbol{d} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$



<u>Finding the equation of perpendicular AP:</u>

Vectors $\overrightarrow{AP} \perp \boldsymbol{d}$

$\Rightarrow \overrightarrow{AP}.\boldsymbol{d} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0 \text{ or } 2x - 4 + y + 1 + 2z - 8 = 0$

$2x + y + 2z = 11$ ……………………………………………………………… (1)

But $P(x, y, z)$ lies on the line

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow x = 1 + 2\lambda, y = \lambda, z = 2 + 2\lambda$ ……………………………… (2)

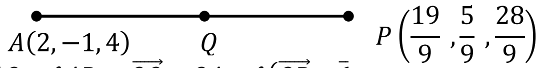
$Eqn. (2) \text{ in to } eqn. (1)$

$\Rightarrow 2(1 + 2\lambda) + \lambda + 2(2 + 2\lambda) = 11$

$\Rightarrow 2 + 4\lambda + \lambda + 4 + 4\lambda = 11 \;\therefore \lambda = \frac{5}{9}$

$P(x, y, z) = P(1 + 2\lambda, \lambda, 2 + 2\lambda) \Rightarrow P\left(1 + \frac{10}{9}, \frac{5}{9}, 2 + \frac{10}{9}\right) = P\left(\frac{19}{9}, \frac{5}{9}, \frac{28}{9}\right)$

Now finding equation of $\overline{AP}$



$AQ = \lambda AP \Rightarrow \overrightarrow{OQ} - OA = \lambda(\overrightarrow{OP} - \bar{C}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \lambda\left[\left(\frac{19}{9}\boldsymbol{i} + \frac{5}{9}\boldsymbol{j} + \frac{28}{9}\boldsymbol{k}\right) - (2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})\right]$

$= \lambda\left(\frac{1}{9}\boldsymbol{i} + \frac{14}{9}\boldsymbol{j} - \frac{8}{9}\boldsymbol{k}\right) = \frac{\lambda}{9}(\boldsymbol{i} + 14\boldsymbol{j} - 8\boldsymbol{k})\text{ ; } let\ \frac{\lambda}{9} = \mu$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$

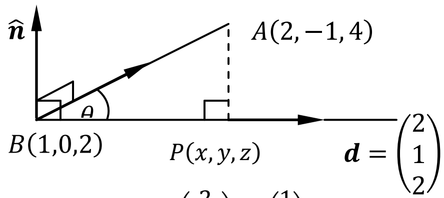
$\therefore \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$

Now finding $|\overrightarrow{AP}| = perpendicular\, distance$

$\overrightarrow{AP} = \frac{1}{9}\boldsymbol{i} + \frac{14}{9}\boldsymbol{j} - \frac{8}{9}\boldsymbol{k}$

$\therefore |\overrightarrow{AP}| = \sqrt{\left(\frac{1}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + \left(-\frac{8}{9}\right)^2} = \frac{\sqrt{29}}{3} units$

Alternative method of finding perpendicular distance of a point from a line:



$|\overrightarrow{AP}| = |\overrightarrow{BA}| sin\theta$

By cross product,

$\boldsymbol{d} \times \overrightarrow{BA} = |\boldsymbol{d}||\overrightarrow{BA}| sin\theta.\hat{\boldsymbol{n}}$

$|\boldsymbol{d} \times \overrightarrow{BA}| = |\boldsymbol{d}||\overrightarrow{BA}| sin\theta \,[\because |\hat{\boldsymbol{n}}| = 1]$

$\therefore |\overrightarrow{AP}| = |\overrightarrow{BA}| sin\theta = \frac{|\boldsymbol{d} \times \overrightarrow{BA}|}{|\boldsymbol{d}|}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \\ -1 \\ 2 \end{pmatrix}$

$\boldsymbol{d} \times \overrightarrow{BA} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 4\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ [*See cross product above how toget this answer*]

$|\boldsymbol{d} \times \overrightarrow{BA}| = \sqrt{4^2 + (-2)^2 + (-3)^2} = \sqrt{29}\,; |\boldsymbol{d}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$\therefore |\overrightarrow{AP}| = \frac{\sqrt{29}}{3} units$

2. *Find the equation and length of perpendicular drawn from a point $R(-2\,,2\,,-3)$ to the line $x - 3 = \frac{y+1}{2} = \frac{z-2}{-4}$*

Solution

Expressing the equation in parametric form

Let $x - 3 = \frac{y+1}{2} = \frac{z-2}{-4} = \lambda$

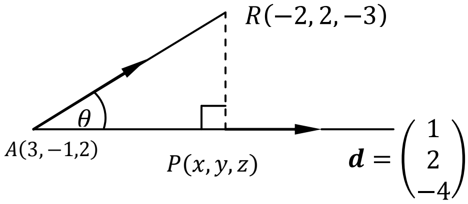
$\Rightarrow x - 3 = \lambda\,, x = 3 + \lambda$

$\Rightarrow \frac{y+1}{2} = \lambda\,, y = -1 + 2\lambda$

$\Rightarrow \frac{z-2}{-4}\lambda\,, z = 2 - 4\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

This line passes through point $A(3\,,-1\,,2)$ and has a direction vector $\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$



Finding the equation of perpendicular PR:

Vectors $\overrightarrow{PR} \perp \boldsymbol{d}$

$\Rightarrow \overrightarrow{PR}.\boldsymbol{d} = 0$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-x \\ 2-y \\ -3-z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2-x \\ 2-y \\ -3-z \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0 \; or \; -2-x+4-2y+12+4z = 0$$

$-x - 2y + 4z = -14$ …………………………………………………………… (1)

But $P(x, y, z)$ lies on the line

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow x = 3 + \lambda\,, y = -1 + 2\lambda\,, z = 2 - 4\lambda$ ……………………………… (2)

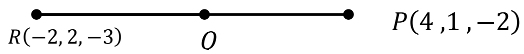
$Eqn. (2)\; in\; to\; eqn. (1)$

$\Rightarrow -(3+\lambda) - 2(-1+2\lambda) + 4(2-4\lambda) = -14$

$\Rightarrow -3 - \lambda + 2 - 4\lambda + 8 - 16\lambda = -14 \; or -21\lambda = -21 \;\therefore \lambda = 1$

$P(x, y, z) = P(3+\lambda\,, -1+2\lambda\,, 2-4\lambda) \Rightarrow P(3+1\,, -1+2\,, 2-4) = P(4\,,1\,,-2)$

Now finding equation of $\overline{RP}$



$\overrightarrow{RQ} = \lambda\overrightarrow{RP} \Rightarrow \overrightarrow{OQ} - \overrightarrow{OR} = \lambda(\overrightarrow{OP} - \overrightarrow{OR})$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = \lambda\left[\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}\right] = \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \; or \; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

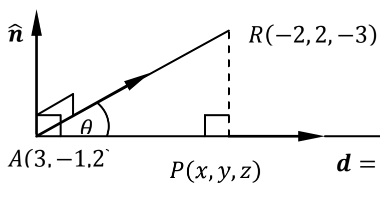
$$\therefore \boldsymbol{r} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

Now finding $|\overrightarrow{PR}| = perpendicular\; distance$

$$\overrightarrow{PR} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore |\overrightarrow{PR}| = \sqrt{(6)^2 + (-1)^2 + (1)^2} = \sqrt{38}\,units$

Alternative method of finding perpendicular distance of a point from a line:



$$\boldsymbol{d} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

$|\overrightarrow{PR}| = |\overrightarrow{AR}|sin\theta$

By cross product,

$\boldsymbol{d} \times \overrightarrow{AR} = |\boldsymbol{d}||\overrightarrow{AR}|sin\theta.\hat{\boldsymbol{n}}$

$|\boldsymbol{d} \times \overrightarrow{AR}| = |\boldsymbol{d}||\overrightarrow{AR}|sin\theta \;[\because |\hat{\boldsymbol{n}}| = 1]$

$\therefore |\overrightarrow{PR}| = |\overrightarrow{AR}|sin\theta = \frac{|\boldsymbol{d} \times \overrightarrow{AR}|}{|\boldsymbol{d}|}$

$\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -5 \end{pmatrix}$; $\boldsymbol{d} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$\boldsymbol{d} \times \overrightarrow{AR} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -4 \\ -5 & 3 & -5 \end{vmatrix} = 2\boldsymbol{i} + 25\boldsymbol{j} + 13\boldsymbol{k}$

$\left|\boldsymbol{d} \times \overrightarrow{AR}\right| = \sqrt{2^2 + (25)^2 + (13)^2} = \sqrt{798}$; $|\boldsymbol{d}| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$

$\therefore \left|\overrightarrow{AP}\right| = \frac{\sqrt{798}}{\sqrt{21}} = \sqrt{38}\, units$

NOTE:
A student must know how to identify the point through which the line is passing If the equation is in Cartesian form. This can easily be deduced from the parametric form.

Examples

Express the following equations in parametric form.

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Solution

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

Let $x - 1 = \lambda \Rightarrow x = 1 + \lambda$

$For\ \frac{y-2}{0}$, $y - 2 = 0 \Rightarrow y = 2 = 2 + 0\lambda$

$\frac{z-1}{3} = \lambda \Rightarrow z = 1 + 3\lambda$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

Let $\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\frac{4-y}{3} = \lambda \Rightarrow y = 4 - 3\lambda$

$For\ \frac{3-z}{0}$, $3 - z = 0 \Rightarrow z = 3 = 3 + 0\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Let $2x + 1 = \lambda \Rightarrow x = -\frac{1}{2} + \frac{1}{2}\lambda$

$\frac{y-2}{4} = \lambda \Rightarrow y = 2 + 4\lambda$

$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 4 \\ -2 \end{pmatrix}$

$or\ \boldsymbol{r} = \begin{pmatrix} -0.5 \\ 2 \\ 3 \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$ $or\ \boldsymbol{r} = \begin{pmatrix} -0.5 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$

INTERSECTION OF TWO LINES

Let the lines be;

$$\boldsymbol{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \; and \; \boldsymbol{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

For two lines to intersect,

$$\boxed{\boldsymbol{r}_1 = \boldsymbol{r}_2}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$\Rightarrow a_1 + \lambda b_1 = c_1 + \mu d_1$ .... (1)

$\Rightarrow a_2 + \lambda b_2 = c_2 + \mu d_2$ .... (2)

$\Rightarrow a_3 + \lambda b_3 = c_3 + \mu d_3$ .... (3)

Solving for the constants $\lambda \; and \; \mu$ in any two equations, the values should satisfy the third equation $.i.e.$ the three equations must be consistent for $\lambda \; and \; \mu.$

Examples

Show that the pair of lines intersect, and find the coordinates of point of intersection.

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \; and \; \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \; and \; \boldsymbol{r} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ and the line passing through $A(2\,,2\,,5)$ and $B(1\,,2\,,3)$.

Solution

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \;;\; \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

$$\boldsymbol{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} \;;\; \boldsymbol{r}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

For intersection $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$$\Rightarrow \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

$\Rightarrow 4 + \lambda = 1 + 2\mu \; or \; \lambda - 2\mu = -3$ .... (1)

$\Rightarrow -3 - 4\lambda = -1 - 3\mu \; or \; -4\lambda + 3\mu = 2$......... (2)

$\Rightarrow -1 + 7\lambda = -10 + 8\mu \; or \; 7\lambda - 8\mu = -9$ .... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$4 \times eqn.(1) + eqn.(2)$

$\Rightarrow \quad 4\lambda - 8\mu = -12$

$\Rightarrow \quad \underline{-4\lambda + 3\mu = 2}$

From eq $\quad -5\mu = -10\,,\mu = 2 \quad\quad 4 = 1$

$\therefore \lambda = 1\,,\mu = 2$ .... (4)

$subst.eqn.(4)in\,to\,eqn.(3)$

$7 - 16 = -9 = \boldsymbol{RHS.}$ Hence lines intersect.

From $\boldsymbol{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ $,for\; \lambda = 1 \; then \; x = 5\,,y = -7\,,z = 6$

$\therefore (5\,,-7\,,6)$ is the point of intersection.

b) $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix}$ *and* $\boldsymbol{r} = \begin{pmatrix}3\\7\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix} = \begin{pmatrix}3\\7\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$$

$\Rightarrow 1 + 2\lambda = 3 + 2\mu \;\; or \;\; 2\lambda - 2\mu = 2$ .... (1)

$\Rightarrow 2 + 5\lambda = 7 + \mu \;\; or \; 5\lambda - \mu = 5$.......................... (2)

$\Rightarrow 3 + 2\lambda = 5 + 2\mu \;\; or \;\; 2\lambda - 2\mu = 2$.................... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$

$\Rightarrow \quad \lambda - \mu = 1$

$\Rightarrow \;\; 5\lambda - \mu = 5$

$-4\lambda = -4\;,\;\lambda = 1$

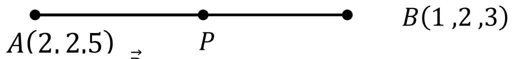
From eqn. (1) $\lambda - \mu = 1\,, \lambda = 1 \Rightarrow \mu = 0$ .... (4)

Eqn. (4) in to eqn. (3) $\Rightarrow 2 - 0 = 2$. Hence lines intersect.

From $\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix}, \lambda = 1 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\7\\5\end{pmatrix}$

$\therefore (3\,,7\,,5)$ is the point of intersection of the lines.

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3} \Rightarrow \boldsymbol{r}_1 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix}$



$\overrightarrow{AP} = \lambda\overrightarrow{AB}$

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$$\Rightarrow \boldsymbol{r} - \begin{pmatrix}2\\2\\5\end{pmatrix} = \lambda\left[\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\2\\5\end{pmatrix}\right] = \begin{pmatrix}-1\\0\\-2\end{pmatrix}$$

$\therefore \boldsymbol{r}_2 = \begin{pmatrix}2\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\-2\end{pmatrix}$ Is the equation through the two given points.

For intersection of lines, $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$$\Rightarrow \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix} = \begin{pmatrix}2\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\-2\end{pmatrix}$$

$\Rightarrow 1 + \mu = 2 - \lambda \;\; or \;\; \mu + \lambda = 1$ .... (1)

$\Rightarrow 1 + 3\mu = 5 - 2\lambda \; or \;\; 3\mu + 2\lambda = 4$ .... (2)

Solving equations simultaneously

$2 \times eqn.(1) - eqn.(2)$

$2\lambda + 2\mu = 2$

$-\;\; 2\lambda + 3\mu = 4$

$-\mu = -2 \Rightarrow \mu = 2$

From eqn. (1) , $\lambda = 1 - 2 = -1$

From $\boldsymbol{r}_1 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix}, \mu = 2 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}2\\0\\6\end{pmatrix} = \begin{pmatrix}3\\2\\7\end{pmatrix}$

$\therefore (3\,,2\,,7)$ is the point of intersection.

Alternative approach to these questions:

Expressing the equations in Cartesian form;

a) $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ .... (1)

$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ .... (2)

Splitting the equations in to two other equations;

For eqn. (1) $\frac{x-4}{1}=\frac{y+3}{-4}$, $\frac{y+3}{-4}=\frac{z+1}{7}$

For eqn. (2) $\frac{x-1}{2}=\frac{y+1}{-3}$, $\frac{y+1}{-3}=\frac{z+10}{8}$

$\Rightarrow \frac{x-4}{1}=\frac{y+3}{-4}$ $or\ -4x+16=y+3$

$\therefore 4x+y=13$ .... (3)

$\Rightarrow \frac{y+3}{-4}=\frac{z+1}{7}$ $or\ \ 7y+21=-4z-4$

$\therefore 7y+4z=-25$ .... (4)

$\Rightarrow \frac{x-1}{2}=\frac{y+1}{-3}$ $or\ -3x+3=2y+2$

$\therefore 3x+2y=1$ .... (5)

$\Rightarrow \frac{y+1}{-3}=\frac{z+10}{8}$ $or\ 8y+8=-3z-30$

$\therefore 8y+3z=-38$ .... (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$2\times\left|\begin{array}{l}4x+y=13\Rightarrow 8x+2y=26\\ 3x+2y=1\Rightarrow\ 3x+2y=1\end{array}\right.$

Subtracting the equations; $5x=25\Rightarrow x=5$

From $4x+y=13$; $x=5\Rightarrow 20+y=13\ \therefore y=-7$

From eqn. (4); $7y+4z=-25$, $y=-7\Rightarrow -49+4z=-25\ \therefore z=6$

$\therefore (5,-7,6)$ is the point of intersection.

This must satisfy the fourth equation if the lines intersect.

$8y+3z=-38$; $y=-7$, $z=6$

$\Rightarrow 8(-7)+3(6)=-38=\boldsymbol{RHS}$. Hence lines intersect at $(5,-7,6)$

b) $\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\5\\2\end{pmatrix}\Rightarrow\frac{x-1}{2}=\frac{y-2}{5}=\frac{z-3}{2}$ .... (1)

$\boldsymbol{r}=\begin{pmatrix}3\\7\\5\end{pmatrix}+\mu\begin{pmatrix}2\\1\\2\end{pmatrix}\Rightarrow\frac{x-3}{2}=\frac{y-7}{1}=\frac{z-5}{2}$ .... (2)

From eqn. (1): $\frac{x-1}{2}=\frac{y-2}{5}\Rightarrow 5x-2y=1$..................... (3)

$\frac{y-7}{1}=\frac{z-5}{2}\Rightarrow 2y-5z=-11$............... (4)

From eqn. (2): $\frac{x-3}{2}=\frac{y-7}{1}\Rightarrow x-2y=-11$ .... (5)

$\frac{y-7}{1}=\frac{z-5}{2}\Rightarrow 2y-z=9$........................ (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$$\begin{array}{rl} - & 5x-2y=1\\ & \underline{x-2y=-11}\\ & 4x\qquad =12\Rightarrow x=3\end{array}$$

From $x-2y=-11$, $x=3\Rightarrow 3-2y=-11\ \therefore y=7$

From eqn. (4): $2y-5z=-11$, $y=7\Rightarrow 14-5z=-11\ \therefore z=5$

$\therefore (3\,,7\,,5)$ is the point of intersection.
This must satisfy eqn. (6)
$2y - z = 9 \Rightarrow 14 - 5 = 9 = \boldsymbol{RHS}$. Hence lines intersect at $(3\,,7\,,5)$

2. *Prove that the lines* $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ *and* $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ *intersect. Find the point of intersection of the lines.* ***Answer***: $(5\,,-7\,,6)$
3. *Find the equation of a straight line perpendicular to both lines* $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$ *and* $\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$ *and passing through their point of intersection.*

Solution

$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}\ ,\ \boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}\ ,\ \boldsymbol{r} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

For intersection to occur, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow 1 + \lambda = -2 + 2\mu\ \ or\ \ \lambda - 2\mu = -3$ ………………………. (1)

$\Rightarrow 1 + 2\lambda = 5 - \mu\ \ or\ 2\lambda + \mu = 4$……………………………… (2)

$\Rightarrow -2 + 3\lambda = -3 + 2\mu\ \ or\ \ 3\lambda - 2\mu = -1$………………. (3)

Solving (1) and (2) simultaneously

$2 \times eqn.(1) - eqn.(2)$

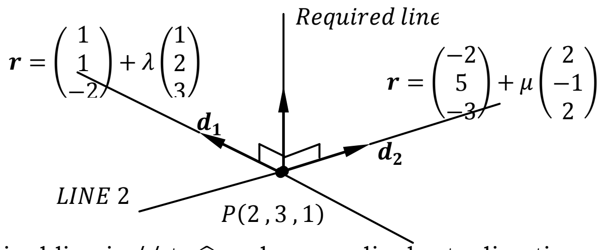
$$\begin{array}{rl} & 2\lambda - 4\mu = -6 \\ - & 2\lambda + \mu = 4 \\ \hline & -5\mu = -10 \Rightarrow \mu = 2 \end{array}$$

From eqn. (1) $\lambda - 2\mu = -3\,, \mu = 2\ solving\ \ \lambda = 1$

$[Confirming, from\ (3), 3\lambda - 2\mu = -1 \Rightarrow 3 - 4 = -1, hence\ lines\ intersect]$

From $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, for\ \lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$\therefore (2\,,3\,,1)$ is the point of intersection.



Required line is // to $\hat{\boldsymbol{n}}$ and perpendicular to direction vectors of the given lines.

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d}_1 \times \boldsymbol{d_2}}{|\boldsymbol{d}_1 \times \boldsymbol{d_2}|}$

$\boldsymbol{d}_1 \times \boldsymbol{d_2} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d}_1 \times \boldsymbol{d_2}}{|\boldsymbol{d}_1 \times \boldsymbol{d_2}|} = \frac{7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}}{\sqrt{90}}$

$\therefore \boldsymbol{d} = d.v\ of\ required\ line = \frac{7\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{90}}.\gamma$ $[\because for\ \boldsymbol{d}\ and\ \hat{\boldsymbol{n}}\ to\ be \quad \boldsymbol{d} = \gamma\hat{\boldsymbol{n}}]$ //

Required line: $\boldsymbol{r} = \boldsymbol{a} + \alpha\, \boldsymbol{d}$

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \frac{\alpha\gamma}{\sqrt{90}}.\begin{pmatrix}7\\4\\-5\end{pmatrix}\ or\ \boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \beta\begin{pmatrix}7\\4\\-5\end{pmatrix}, where\ \frac{\alpha\gamma}{\sqrt{90}} = \beta$$

NOTE:

To avoid over manipulation of constants when finding the direction vector of the required line after crossing $\boldsymbol{d}_1 and\ \boldsymbol{d_2}$, then $\boldsymbol{d}_1 = \hat{\boldsymbol{n}}$ since for parallel lines the ratio of direction vectors for both lines must be the same.

($\boldsymbol{see\ notes\ on\ parallel\ lines}$)

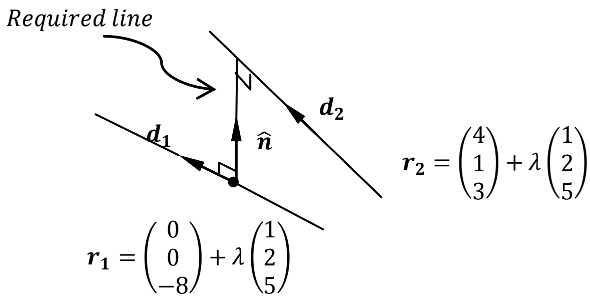
3. *Find the equation of the line through the point* $(1, 3, -2)$ *and perpendicular to the line passing through points* $A(0, 0, -8)$, $B(1, 2, -3)$ *and* $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} + \lambda(-2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

Solution

Equation of line through A and B;

$\overrightarrow{AP} = \mu\overrightarrow{AB}, \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \begin{pmatrix}1\\2\\-3\end{pmatrix} - \begin{pmatrix}0\\0\\-8\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix}$$



$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix}0\\0\\-8\end{pmatrix}$$

$$\therefore\ \boldsymbol{r_1} = \begin{pmatrix}0\\0\\-8\end{pmatrix} + \mu\begin{pmatrix}1\\2\\5\end{pmatrix}$$

$$\boldsymbol{r_2} = \begin{pmatrix}4\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}$$

For $\boldsymbol{r_1}, \boldsymbol{d_1} = \begin{pmatrix}0\\0\\-8\end{pmatrix}, for\ \boldsymbol{r_2}, \boldsymbol{d_2} = \begin{pmatrix}\\2\\5\end{pmatrix}$

Direction vector or perpendicular is

$$\boldsymbol{d_1} \times \boldsymbol{d_2} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\1 & 2 & 5\\-2 & 3 & -1\end{vmatrix} = -17\boldsymbol{i} - 9\boldsymbol{j} + 4\boldsymbol{k}$$

Required line: $\boldsymbol{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + \mu\begin{pmatrix}-17\\-9\\4\end{pmatrix}$

4. *Given the two lines are parametrically given by:*

$\boldsymbol{r_1} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$ *and*

$\boldsymbol{r_2} = -3\boldsymbol{i} + p\boldsymbol{j} + 7\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

If the lines intersect, find the

a. the value of $t, s\ and\ p$.

b. Coordinates of the points of intersection.

Solution

For lines to intersect, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\Rightarrow -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = -3\boldsymbol{i} + p\boldsymbol{j} + 7\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\Rightarrow \begin{pmatrix}-1\\2\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}-3\\p\\7\end{pmatrix}+s\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\Rightarrow -1+t=-3+s \ or \ t-s=-2$ .... (1)

$\Rightarrow 2-2t=p-s \ or \ 2t-s=2-p$ .... (2)

$\Rightarrow 1+3t=7+2s \ or \ 3t-2s=6$ .... (3)

Solving eqn. (1) and eqn. (3) simultaneously

$2\times \left| \begin{array}{l} t-s=-2 \Rightarrow 2t-2s=-4 \\ 3t-2s=6 \Rightarrow 3t-2s=6 \end{array} \right.$

Subtracting; $-t=-10 \ \therefore \ t=10$

From $t-s=-2\,, t=10 \Rightarrow s+10+2=12$

$\therefore s=12\,, t=10$

$From\ 2t-s=2-p \Rightarrow 21-12=2-p \ \therefore p=-6$

From $\boldsymbol{r_1}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\3\end{pmatrix}, t=10 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+10\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}9\\-18\\31\end{pmatrix}$

$\therefore (9\,,-18\,,31)$ is the point of intersection.

5. *Given that the lines* $\boldsymbol{r_1}=\begin{pmatrix}p\\2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\5\\3\end{pmatrix}$ *and* $\boldsymbol{r_2}=\begin{pmatrix}2\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\p\end{pmatrix}$ *intersect, find the value of p. hence find the point of intersection.*

Solution

For point of intersection, $\boldsymbol{r_1}=\boldsymbol{r_2}$

$\Rightarrow \begin{pmatrix}p\\2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\5\\3\end{pmatrix}=\begin{pmatrix}2\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\p\end{pmatrix}$

$\Rightarrow p+\mu=2+\lambda$ .... (1)

$\Rightarrow 2+5\mu=12 \ \Rightarrow \mu=2$ .... (2)

$\Rightarrow 1+3\mu=5+\lambda p$ .... (3)

$eqn.(2)\ in\ to\ eqn,(1)$

$\Rightarrow 1+6=5+\lambda p\,, \lambda p=1 \ or\ \lambda=\frac{1}{p}$ .... (4)

$eqn.(2),(4) in\ to\ eqn.(1)$

$\Rightarrow p+2=2+\lambda\,, p=\lambda$ .... (5)

$eqn.(5)\ in\ to\ eqn,(4)$

$\Rightarrow \lambda=\frac{1}{\lambda}\,, \lambda^2=1 \ \therefore \lambda=\pm 1$

$\therefore p=\pm 1$

6. *Show that the lines* $\boldsymbol{r_1}=\begin{pmatrix}3\\5\\7\end{pmatrix}+m\begin{pmatrix}1\\2\\1\end{pmatrix}$ *and* $\boldsymbol{r_2}=\begin{pmatrix}1\\2\\3\end{pmatrix}+n\begin{pmatrix}2\\3\\5\end{pmatrix}$ *do not intersect and not parallel.* $[i.e. are\ skew]$

Solution

For lines to intersect, $\boldsymbol{r_1}=\boldsymbol{r_2}$

$\begin{pmatrix}3\\5\\7\end{pmatrix}+m\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+n\begin{pmatrix}2\\3\\5\end{pmatrix}$

$\Rightarrow 3+m=1+2n \ or \ m-2n=-2$ .... (1)

$\Rightarrow 5+2m=2+3n \ or \ 2m-3n=-3$ .... (2)

$\Rightarrow 7+m=3+5n \ or \ m-5n=-4$ .... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$m=0\,, n=1$ .... (4)

$eqn.(4) in\ to\ eqn.(5)$
$0-5 \neq -4$. Hence lines do not intersect.
Considering a case of parallel lines
For parallel lines, t*he direction ratio values are similar.*
For $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\7\end{pmatrix} + m\begin{pmatrix}1\\2\\1\end{pmatrix}$ $, direction\,ratio: 1:2:1$
For $\boldsymbol{r_2} = \begin{pmatrix}1\\2\\3\end{pmatrix} + n\begin{pmatrix}2\\3\\5\end{pmatrix}$ $, direction\,ratio: 2:3:5$
Since the direction ratios are not the same, lines are not parallel. Thus are s*kew.*

7. *Show that the lines are parallel*
   a) $\boldsymbol{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\5\end{pmatrix}$ $and\ \boldsymbol{r_2} = \begin{pmatrix}3\\5\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-5\end{pmatrix}$
   b) $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix}$ $and\ \boldsymbol{r_2} = \begin{pmatrix}1\\0\\7\end{pmatrix} + \mu\begin{pmatrix}-2\\3\\5\end{pmatrix}$

Solution

a) For $\boldsymbol{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\5\end{pmatrix}$ $, direction\,ratio: -2:1:5$
For $\boldsymbol{r_2} = \begin{pmatrix}3\\5\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-5\end{pmatrix}$ $, direction\,ratio: 2:-1:-5\,, dividing\ through\ by -1$
$\Rightarrow direction\,ratio\ is -2:1:5$
Since the direction ratios are the same the lines are parallel.

b) For $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix}$ $, direction\,ratio: 2:-3:-5$
For $\boldsymbol{r_2} = \begin{pmatrix}1\\0\\7\end{pmatrix} + \mu\begin{pmatrix}-2\\3\\5\end{pmatrix}$ $, direction\,ratio: -2:3:5\,, dividing\ through\ by -1$
$\Rightarrow direction\,ratio\ is\ 2:-3:-5$
The ratios are the same, hence lines are parallel.

NOTE:
Also for parallel lines $\boldsymbol{d_1} \times \boldsymbol{d_2} = \boldsymbol{0}$ can be applied.

5. *Given the equation of the lines in the form*
$\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$, show that $(8\,,14\,,11)$ lie on this line.

Solution
Let $\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2} = \lambda\,;\ \Rightarrow \frac{x-2}{3} = \lambda\,, \frac{y-4}{5} = \lambda\,, \frac{z-7}{2} = \lambda$
For $(8\,,14\,,11)\,, \lambda$ must be consistent
$\frac{x-2}{3} = \frac{8-2}{3} = 2 = \lambda$
$\frac{y-4}{5} = \frac{14-4}{5} = 2 = \lambda$
$\frac{z-7}{2} == \frac{11-7}{2} = 2 = \lambda$
Since $\lambda = 2$, consistent for $(8\,,14\,,11)$, $hence\ point\ lies\ on\ line$

6. *Given the equation of line:* $\boldsymbol{r} = (2+3\lambda)\boldsymbol{i} + (4+5\lambda)\boldsymbol{j} + (7+2\lambda)\boldsymbol{k}$. *Show that* $(-4\,,-6\,,3)$ *lie on the line.*

Solution

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\Rightarrow x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (2+3\lambda)\boldsymbol{i} + (4+5\lambda)\boldsymbol{j} + (7+2\lambda)\boldsymbol{k}$

Equating unit vectors,

$x = 2+3\lambda\,,\; y = 4+5\lambda\,,\; z = 7+2\lambda$

Point $(-4\,,-6\,,3)$ must give the same value of λ in the equations formulated.

$\Rightarrow -4 = 2+3\lambda \quad \therefore \lambda = -2$

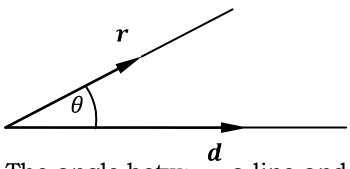
$\Rightarrow -6 = 4+5\lambda \quad \therefore \lambda = -2$

$\Rightarrow 3 = 7+2\lambda \quad \therefore \lambda = -2$

Since $\lambda = 2$, consistent for$(-4\,,-6\,,3)$, *hence point lies on line*

ANGLE BETWEEN A VECTOR AND A LINE

Consider the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ and the vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$



The angle between a line and a vector is the angle between *vector* and *direction vector* of the line. *i.e.*

By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$

Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\boldsymbol{d}.\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = a_1d_1 + a_2d_2 + a_3d_3$

$|\boldsymbol{d}| = \sqrt{{d_1}^2 + {d_2}^2 + {d_3}^2}\,, |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$

$$\boxed{\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|}}$$

Example

1. *Find the angle between*

   a. *Vector* $\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   b. *Vector* $\boldsymbol{r} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ *and line* $\frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2}$

   c. *Vector* $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ *and line* $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

Solution

a. By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$

   Let $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   $\boldsymbol{r}.\boldsymbol{d} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2+6+3 = 11$

$|\boldsymbol{d}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, $|\boldsymbol{r}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|} = \frac{11}{\sqrt{14}\times\sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 38.2^0$

b. $\boldsymbol{r} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{d} = ?$

Let $\frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2} = \lambda$

$\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\frac{4-y}{1} = \lambda \Rightarrow y = 4 - \lambda$

$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-2\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$\Rightarrow \boldsymbol{r}.\boldsymbol{d} = \begin{pmatrix}3\\-2\\1\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix} = 6 + 2 - 2 = 6$

$\Rightarrow |\boldsymbol{d}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$, $|\boldsymbol{r}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|} = \frac{6}{3\times\sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 57.7^0$

c. *Left as an exercise*

2. *Find the value of λ if*

a. Vector $\boldsymbol{r} = 2\boldsymbol{i} + \lambda\boldsymbol{j} + 3\boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}1\\2\\3\end{pmatrix}$ *are perpendicular*

b. Vector $\boldsymbol{r} = \boldsymbol{i} + \lambda\boldsymbol{j} + \boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$ *make an angle of* 60^0

Solution

a. $\boldsymbol{r} = 2\boldsymbol{i} + \lambda\boldsymbol{j} + 3\boldsymbol{k} = \begin{pmatrix}2\\\lambda\\3\end{pmatrix}$, $\boldsymbol{r} = \begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

By dot product,

$\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$, $but\ \theta = 90^0$ $[\because vector\ is\ perpendicular\ to\ line]$

$\therefore \boldsymbol{r}.\boldsymbol{d} = 0 \Rightarrow \begin{pmatrix}2\\\lambda\\3\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix} = 0\ \ or\ \ 2 + 2\lambda + 9 = 0\ \therefore \lambda = -\frac{11}{2}$

b. $\boldsymbol{r} = \boldsymbol{i} + \lambda\boldsymbol{j} + \boldsymbol{k} = \begin{pmatrix}1\\\lambda\\1\end{pmatrix}$, $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\1\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$, $but\ \theta = 60^0$

$\Rightarrow \boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos60^0 = |\boldsymbol{d}||\boldsymbol{r}| \times \frac{1}{2}$ $\left[\because cos60^0 = \frac{1}{2}\right]$

$\therefore \begin{pmatrix}1\\-2\\1\end{pmatrix}.\begin{pmatrix}1\\\lambda\\1\end{pmatrix} = \sqrt{1^2 + (-2)^2 + 1^2} \times \sqrt{1^2 + \lambda^2 + 1^2} \times \frac{1}{2}$

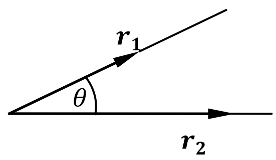
$\Rightarrow 1 - 2\lambda + 1 = \frac{\sqrt{6}}{2} \times \sqrt{2 + \lambda^2}\ \ or\ \ 4 - 4\lambda = \sqrt{12 + 6\lambda^2}$

$squaring\ both\ sides$

$16 - 32\lambda + 16\lambda^2 = 12 + 6\lambda^2 \Rightarrow 10\lambda^2 - 32\lambda + 4 = 0$

Solving, $\lambda = 3.07\ , 0.13$

ANGLE BETWEEN TWO LINES



Consider the lines;

$$\boldsymbol{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \; and \; \boldsymbol{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

The angle between two lines is the angle between the *direction vectors of the lines.*

For $\boldsymbol{r}_1$, $direction\,vector\; \boldsymbol{d}_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $and\,for\; \boldsymbol{r}_2\,, direction\,vector\; \boldsymbol{d}_2 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$$\boxed{\therefore \boldsymbol{d}_1.\boldsymbol{d}_2 = |\boldsymbol{d}_1||\boldsymbol{d}_2|cos\theta}$$

Examples

1. *Find the angle between the pair of lines*

   a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\boldsymbol{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

   b) $\boldsymbol{r} = (1+2\lambda)\boldsymbol{i} + (3+\lambda)\boldsymbol{j} + (5+2\lambda)\boldsymbol{k}$ and $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

   c) $\frac{10-x}{2} = \frac{4-y}{3} = \frac{8-z}{4}$ $and\,\boldsymbol{r} = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

Solution

a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3} \Rightarrow \boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\therefore \boldsymbol{d_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\boldsymbol{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{d_2} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = -3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

From dot product $\boldsymbol{d}_1.\boldsymbol{d}_2 = |\boldsymbol{d}_1||\boldsymbol{d}_2|cos\theta$

$\boldsymbol{d}_1.\boldsymbol{d}_2 = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}).(-3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = -3 - 4 + 3 = -4$

$|\boldsymbol{d}_1| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}\,, |\boldsymbol{d}_2| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{-4}{\sqrt{14}\times\sqrt{14}} \Rightarrow \theta = 106.6^0$

b) $\boldsymbol{r} = (1+2\lambda)\boldsymbol{i} + (3+\lambda)\boldsymbol{j} + (5+2\lambda)\boldsymbol{k}$

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \therefore \boldsymbol{d_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \Rightarrow \boldsymbol{d_2} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 2\boldsymbol{i} + 0\boldsymbol{j} + 5\boldsymbol{k}$

$\boldsymbol{d}_1.\boldsymbol{d}_2 = (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(2\boldsymbol{i} + 0\boldsymbol{j} + 5\boldsymbol{k}) = 4 + 0 + 10 = 14$

$|\boldsymbol{d}_1| = \sqrt{2^2 + 1^2 + 2^2} = 3\,, |\boldsymbol{d}_2| = \sqrt{2^2 + 0^2 + 5^2} = \sqrt{29}$

$\therefore cos\theta = \frac{14}{3\times\sqrt{29}} \Rightarrow \theta = 29.9^0$

c) Left as an exercise

Task

1. *Find the angle between the lines:*

   a) $x - 1 = y - 3 = z - 7$ and $\frac{x-5}{2} = \frac{y+1}{3} = \frac{z}{5}$

      ANSWER: $\theta = 120^0$

   b) $x = 2 - 4t\,, x = 1 + 5t$

$y = 1 - 2t\,, y = -2 - t$
$z = 1 + 2t\,, z = 1 - 2t$

c) $\frac{x-3}{2} = \frac{y-3}{5}\,, z = 0\ ;\ \frac{x+1}{3} = \frac{y+2}{1}\,,\ z = 0$ **ANSWER**: $\theta = 35^0$

$[\boldsymbol{HINT}: \boldsymbol{d}_1 = 2\boldsymbol{i} + 5\boldsymbol{j} + 0\boldsymbol{k}\,,\ \boldsymbol{d}_2 = 3\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}]$ **ANSWER**: $\theta = 50^0$

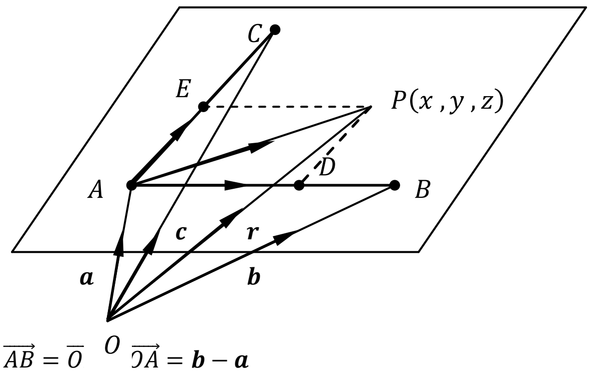
2. *Prove that the vectors* $\boldsymbol{a} = 7\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}\,, \boldsymbol{b} = 3\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ *and* $\boldsymbol{c} = 6\boldsymbol{i} - 16\boldsymbol{j} - 15\boldsymbol{k}$ *are mutually perpendicular.*
3. *a) Find the equation of the line through* $A(2\,,2,5)$ *and* $B(1\,,2\,,3)$
   b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ *at P, find*
   i. coordinates of P
   ii. angle between the lines ($\boldsymbol{Uneb}$ **2004**)

   ANSWER: $\boldsymbol{r} = \begin{pmatrix}2\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\-2\end{pmatrix}\,, P(3\,,2\,,7)\,, \theta = 171.9^0$
4. *Show that vector* $2\boldsymbol{i} - 5\boldsymbol{j} + 3.5\boldsymbol{k}$ *is perpendicular to the line* $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \lambda(4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$
   ($\boldsymbol{Uneb}$ **2000**)
5. *Find the equation of a line through points* $A(2\,,2,5)\ ,B(1\,,2\,,3)$ *and* $C(0\,,2\,,1)$
   $[\boldsymbol{HINT}: 1^{st}$ *check whether A, B and C are collinear, if not different equations must be formed*$]$
6. *Show that the lines* $\boldsymbol{r_1} = (1 + 4\lambda)\boldsymbol{i} + (1 - \lambda)\boldsymbol{j} + 2\lambda\boldsymbol{k}$ *and*
   $\boldsymbol{r_1} = (5 + 3\mu)\boldsymbol{i} + 2\mu\boldsymbol{j} + (2 - 5\mu)\boldsymbol{k}$ cut at right angles. Find the position vector of point of intersection. **ANSWER**: $5\boldsymbol{i} + 2\boldsymbol{k}$
7. *The points A, B, C and D have coordinates* $(3\,,-2\,,0)\,,(-1\,,2\,,4)\,,(0\,,1\,,-1)$ *and* $(5\,,-4\,,4)$ *respectively. Find*
   a) The point of intersection of the lines AB and CD
   b) The angle between the lines in (a) above

   ANSWER: $(2\,,-1\,,1)\ ;\ 70.5^0$
8. *The points A, B and C have p.vs* $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}\,, \boldsymbol{b} = \boldsymbol{j} - 4\boldsymbol{k}$ *and* $\boldsymbol{c} = 6\boldsymbol{i} + \boldsymbol{4j} + 5\boldsymbol{k}$ *respectively. Find the p.v of a point R on* $\overline{BC}$ *such tat* $\overline{AR}$ *is perpendicular to* $\overline{BC}$*. Hence find the perpendicular distance of A from BC.* ***Answer***: $4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}\ ;\ \overline{AR} = \sqrt{21}$
9. *The points A, B and C have p.vs* $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}\,,\ \boldsymbol{i} + 2\boldsymbol{k}$ *and* $4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ *respectively. Find the vector perpendicular to both vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$*.*
   ANSWER: $\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$
10. *Show that the lines* $\boldsymbol{r_1} = 2\lambda\boldsymbol{i} - 3\boldsymbol{j} + (\lambda - 2)\boldsymbol{k}$ *and* $\boldsymbol{r_1} = (1 + \mu)\boldsymbol{i} + (2 - \mu)\boldsymbol{j} + (2\mu - 5)\boldsymbol{k}$ *do not intersect.*

VECTOR EQUATION OF A PLANE

a) **To find the vector equation of a plane through three non-collinear points A, B and C**



Let the $p.vs$ of A, B, C and P be $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{r}$ respectively.

$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP}$. But $\overrightarrow{AD} // \overrightarrow{AB}, \overrightarrow{DP} // \overrightarrow{AC}$

$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$

But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})$

$$\boxed{\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})}$$

NOTE:

If A is the origin, then the equation of a plane will be given by

$\overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} \Rightarrow \overrightarrow{OP} = \lambda\overrightarrow{OB} + \mu\overrightarrow{OC}$

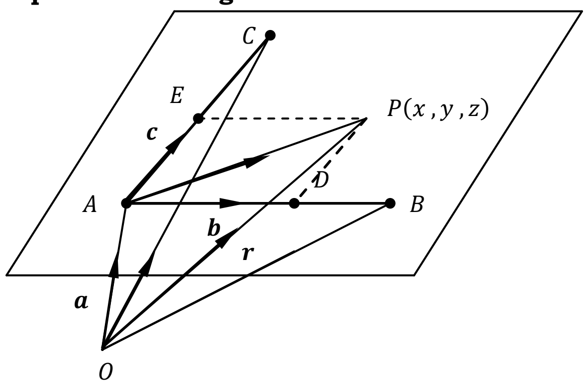
$\therefore \boldsymbol{r} = \boldsymbol{\lambda b} + \mu\boldsymbol{c}$

$$\boxed{\therefore \boldsymbol{r} = \boldsymbol{\lambda b} + \mu\boldsymbol{c}}$$

This equation represents an equation of a plane passing through the origin and parallel to *two* given vectors **b** and **c.**

If given two vectors, then the equation of plane containing the two vectors is of the form above.

b) **To find vector equation of a plane passing through a given point A and parallel to two given vectors $\vec{b}$ and $\vec{c}$.**



Let $p.v$ of P be $\boldsymbol{r}$ and A be $\boldsymbol{a}$

From $\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$; But $\overrightarrow{AE} = \lambda\boldsymbol{c}$, $\overrightarrow{EP} = \mu\boldsymbol{b}$

$\Rightarrow \overrightarrow{AP} = \lambda\boldsymbol{c} + \mu\boldsymbol{b}$

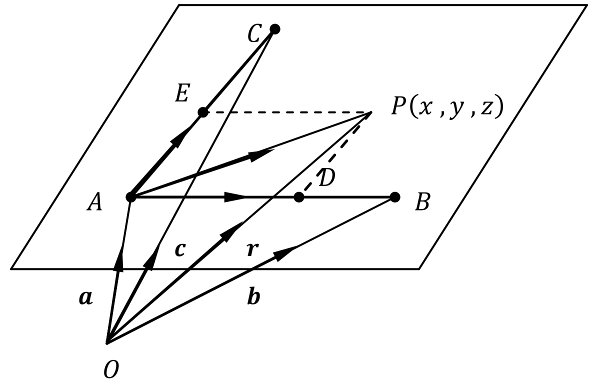
$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda\boldsymbol{c} + \mu\boldsymbol{b}$

$$\boxed{\therefore \boldsymbol{r} = \boldsymbol{a} + \mu\boldsymbol{b} + \lambda\boldsymbol{c}}$$

If A is the Origin. $i.e. A(0, 0, 0)$, then the equation of the plane containing *two* vectors **b** and **c** is

$$\boxed{\therefore \boldsymbol{r} = \mu\boldsymbol{b} + \lambda\boldsymbol{c}}$$

c) **To find the vector equation of a plane through two given points A and B and parallel to a given vector $\vec{c}$.**



Let the $p.v$ of P, A and B be $\boldsymbol{r}, \boldsymbol{a}$ *and* $\boldsymbol{b}$ respectively.
As above, the equation of the plane is

$$\overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\boldsymbol{c}$$

$$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu\boldsymbol{c}$$

or $$\boxed{\therefore \boldsymbol{r} = \boldsymbol{a} + \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu\boldsymbol{c}}$$

NOTE:
The above rectangular enclosed equations are called *parametric equations* of a plane.
When parameters λ *and* μ are eliminated from the equation of a plane, the resulting equation of a plane is called *Cartesian equation of a plane* of the form $Ax + By + Cy + D = 0$.

Examples

1. *Find the Cartesian equation of a plane*

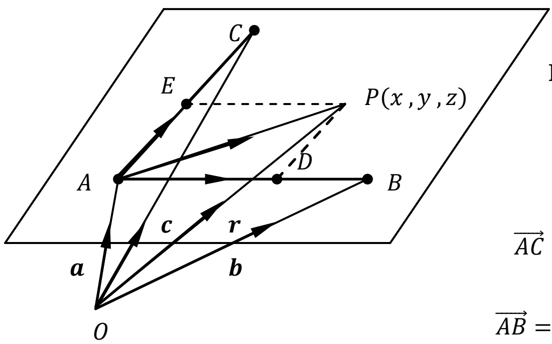
a) *Passing through the points* $A(-1, 1, 2), B(1, -2, 1)$ *and* $C(2, 2, 4)$

b) *Containing vectors* $\begin{pmatrix}0\\3\\0\end{pmatrix}$ *a nd* $\begin{pmatrix}1\\0\\2\end{pmatrix}$

c) *Passing through point* $(1, 2, 3)$ *and parallel to vectors* $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Solution

a)



Let $\overrightarrow{OP} = \boldsymbol{r}$

$$\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$$

But $\overrightarrow{AE}\ //\ \overrightarrow{AC}$ and $\overrightarrow{EP}\ //\ \overrightarrow{AB}$

$$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}2\\2\\4\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\-2\\1\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\-3\\-1\end{pmatrix}$$

$$\therefore \boldsymbol{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\-1\end{pmatrix}$$

Expressing the equation in Cartesian form

$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

$\Rightarrow x = -1 + 3\lambda + 2\mu$ ………………………………………………………… (1)

$y = 1 + \lambda - 3\mu$ ……………………………………………………………………… (2)

$z = 2 + 2\lambda - \mu$ ……………………………………………………………………… (3)

Considering (1) and (2), eliminating λ and μ

From (1) $x + 1 = 3\lambda + 2\mu$ …………………………………………………… (4)

From (2) $y - 1 = \lambda - 3\mu$ ……………………………………………………… (5)

$Eqn. (4) \times 3 - eqn. (3)$

$3y - 3 = 3\lambda - 9\mu$

$\underline{x + 1 = 3\lambda + 2\mu}$

$3y - x - 4 = -11\mu \quad \Rightarrow \mu = \frac{x-3y+4}{11}$ ……………………………………… (6)

$eq. (6) in\ to\ eqn. (4)$

$\Rightarrow \lambda = y - 1 + 3\mu = y - 1 + 3\left[\frac{x-3y+4}{11}\right] = \frac{11y-11+3x-9y+12}{11}$

$\therefore \lambda = \frac{3x+2y+1}{11}$ …………………………………………………………………………… (7)

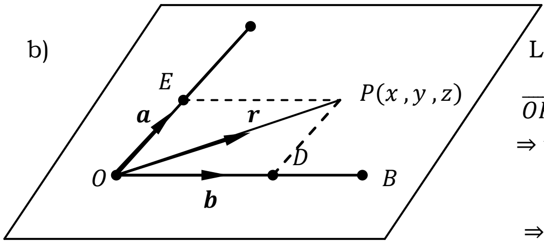
$eqn. (6), (7) in\ to\ (3) to\ conclude\ the\ Cartesian\ plane\ equation$

$\Rightarrow z = 2 + 2\left(\frac{3x+2y+1}{11}\right) - \left(\frac{x-3y+4}{11}\right)$

$11z = 22 + 6x + 4y + 2 - x + 3y - 4$

$\therefore 5x + 7y - 11z + 20 = 0$

b)



Let $\overrightarrow{OP} = \boldsymbol{r}$, $\boldsymbol{a} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}$, $\overrightarrow{OD}$ // $\boldsymbol{b}$ and $\overrightarrow{DP}$ // $\boldsymbol{a}$

$\Rightarrow \boldsymbol{r} = \lambda\boldsymbol{b} + \mu\boldsymbol{a}$

$\Rightarrow \boldsymbol{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

Expressing this equation in Cartesian form

$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

$\Rightarrow x = \lambda$ ………………………………………………… (1)

$\Rightarrow y = 3\mu$ ………………………………………………… (2)

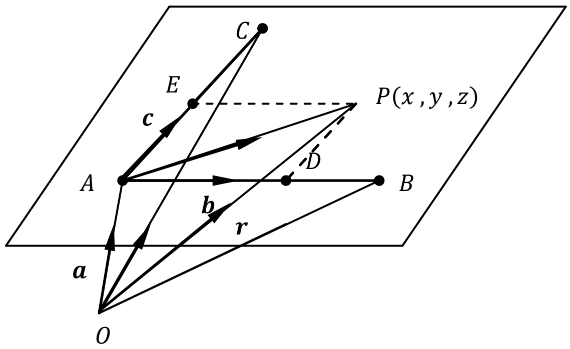
$\Rightarrow z = 2\lambda$ ………………………………………………… (3)

$eqn\ (1) in\ to\ eqn. (3)$

$\therefore z = 2x\ \ or\ \ 2x - z = 0$ is the required equation of plane.

Note: The eqn. (2) cannot be used in any way to find the required equation of plane.

c)



Let $\boldsymbol{b}=\begin{pmatrix}1\\-2\\3\end{pmatrix}, \boldsymbol{c}=\begin{pmatrix}2\\-3\\1\end{pmatrix}$,
$\boldsymbol{r}=\overrightarrow{OP}$
and $\overrightarrow{OA}=\boldsymbol{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$

$\overrightarrow{AP}=\overrightarrow{AD}+\overrightarrow{DP}=\lambda\boldsymbol{b}+\mu\boldsymbol{c}$

$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\boldsymbol{r}-\boldsymbol{a}=\lambda\boldsymbol{b}+\mu\boldsymbol{c}\Rightarrow\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}+\mu\boldsymbol{c}$

$\Rightarrow\boldsymbol{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-3\\1\end{pmatrix}$

$\Rightarrow x=1+\lambda+2\mu\ \ or\ \ x-1=\lambda+2\mu$ .... (1)

$\Rightarrow y=2-2\lambda-3\mu\ \ or\ \ y-2=-2\lambda-3\mu$ .... (2)

$\Rightarrow z=3+3\lambda+\mu\ \ or\ \ z-3=3\lambda+\mu$ .... (3)

Considering eqn. (1) and eqn. (2)

$eqn.(1)\times 2+eqn.(2)$

$2x-2=2\lambda+4\mu$

$y-2=-2\lambda-3\mu$

$2x-2+y-2=\mu\Rightarrow\mu=2x+y-4$ .... (4)

$eqn.(4)\ in\ to\ eqn.(1)$

$x-1=\lambda+2\mu\Rightarrow\lambda=x-1-2\mu$

$\therefore\lambda=x-1-2(2x+y-4)=-3x-2y+7$......................... (5)

$eqn.(4),(5) in\ to\ (3) to\ conclude\ equation\ of\ plane$

$\Rightarrow z-3=3(-3x-2y+7)+2x+y-4$

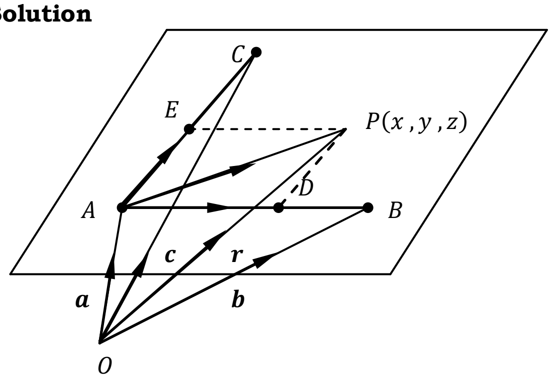
$z-3=-9x-6y+21+2x+y-4=-7x-5y+17$

$\therefore 7x+5y+z-20=0$

2. *Find the Cartesian equation of a plane passing through* $A(0\,,3\,,-4)\,,B(2\,,-1\,,2)$ *and* $C(7\,,4\,,-1)$. *Show that* $Q(10\,,13\,,-10)$ *lies on the plane*

***Uneb* (2001)**

Solution



Let the $p.vs$ of A, B, C and P be $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ *and* $\boldsymbol{r}$ respectively.

$\overrightarrow{AP}=\overrightarrow{AD}+\overrightarrow{DP}$. But $\overrightarrow{AD}//\overrightarrow{AB}\,,\overrightarrow{DP}//\overrightarrow{AC}$

$\therefore\overrightarrow{AP}=\lambda\overrightarrow{AB}+\mu\overrightarrow{AC}$

But $\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}$
$=\boldsymbol{r}-\boldsymbol{a}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a} \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda\left[\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}\right] + \mu\left[\begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$\Rightarrow x = 2\lambda + 7\mu$ .... (1)

$\Rightarrow y = 3 - 4\lambda + \mu \;\; or \;\; y - 3 = -4\lambda + \mu$ .... (2)

$\Rightarrow z = -4 + 6\lambda + 3\mu \;\; or \;\; z + 4 = 6\lambda + 3\mu$ .... (3)

Considering eqn. (1) and (2)

$eqn. (1) \times 2 + eqn. (2)$

$2x = 4\lambda + 14\mu$

$y - 3 = -4\lambda + \mu$

$2x + y - 3 = 15\mu \Rightarrow \mu = \frac{2x+y-3}{15}$ .... (4)

$eqn. (4)\ in\ to\ eqn. (1)$

$x = 2\lambda + 7\mu \Rightarrow \lambda = \frac{x-7\mu}{2}$

$\therefore \lambda = \frac{x-7\left(\frac{2x+y-3}{15}\right)}{2} = \frac{15x-14x-7y+21}{30} = \frac{x-7y+21}{30}$ .... (5)

$eqn. (4), (5) in\ to\ (3) to\ conclude\ equation\ of\ plane$

$\Rightarrow z + 4 = 6\left(\frac{x-7y+21}{30}\right) + 3\left(\frac{2x+y-3}{15}\right)$

$\Rightarrow z + 4 = \left(\frac{x-7y+21}{5}\right) + \left(\frac{2x+y-3}{5}\right)$

$\Rightarrow 5z + 20 = x - 7y + 21 + 2x + y - 3$

$\therefore 3x - 6y - 5z = 2$

For $Q(10, 13, -10)$ to lie on the plane, it must satisfy the equation

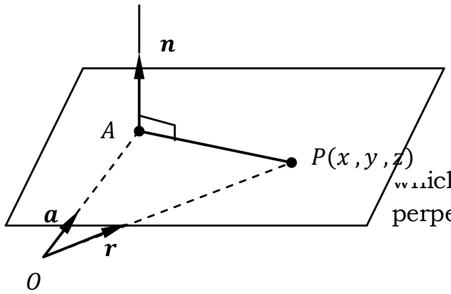
$\Rightarrow 3(10) - 6(13) - 5(-10) = 2 = \boldsymbol{RHS}$ **,** hence point lies on the plane.

VETOR EQUATION OF A PLANE IN NORMAL FORM $\boldsymbol{i.e.}$ **Non-parametric/Scalar/Dot-product form**

Consider a plane which passes through a point A with $p.v\ \boldsymbol{a}$ and which is perpendicular to vector $\boldsymbol{n}$. Let $p.v$ of point $P(x, y, z)$.

$\overrightarrow{AP} \perp \boldsymbol{n}, \therefore \overrightarrow{AP}.\boldsymbol{n} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a} \therefore (\boldsymbol{r} - \boldsymbol{a}).\boldsymbol{n} = 0$



$\Rightarrow \boldsymbol{r}.\boldsymbol{n} - \boldsymbol{a}.\boldsymbol{n} = 0$

$\Rightarrow \boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n},$

$let\ \boldsymbol{a}.\boldsymbol{n} = d\ (scalar)$

$$\boxed{\therefore \boldsymbol{r}.\boldsymbol{n} = d}$$

which is the equation of a plane perpendicular to vector **n.**

NOTE:

From $vector = |vector| \times unit\ vector \Rightarrow \boldsymbol{n} = |\boldsymbol{n}| \times \widehat{\boldsymbol{n}}, \widehat{\boldsymbol{n}} = unit\ vector.$

$\therefore \boldsymbol{r}.\boldsymbol{n} = \boldsymbol{r}.\hat{\boldsymbol{n}}.\ |\boldsymbol{n}| = d$ or $\boldsymbol{r}.\hat{\boldsymbol{n}} = D$,where $D = \frac{d}{|\boldsymbol{n}|} = scalar$.

$\boxed{\therefore\ \boldsymbol{r}.\hat{\boldsymbol{n}} = D}$, also represents equation of a plane perpendicular to $\hat{\boldsymbol{n}}$.

CARTESIAN EQUATION OF A PLANE

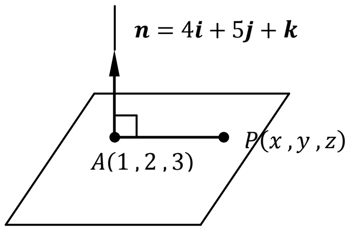
From $\boldsymbol{r}.\boldsymbol{n} = d$, $\boldsymbol{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, let $\boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} A \\ B \\ C \end{pmatrix} = d$

The Cartesian equation of a plane is $\boxed{\therefore Ax + By + Cz = d}$

Examples:

1. *Find the equation of a plane through the point* $(1, 2, 3)$ *and perpendicular to* $\boldsymbol{r} = 4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$.
   (**_Uneb_ 2004**)
   Solution



Let $\boldsymbol{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, let $\boldsymbol{n} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

$\overrightarrow{AP} \perp \boldsymbol{n}$, $\therefore \overrightarrow{AP}.\boldsymbol{n} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \left[\boldsymbol{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right].\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - (4 + 10 + 3) = 0$ *or* $\boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17$

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17 \quad \therefore 4x + 5y + z = 17$

2. *Show that the equation of the plane through point A with* $p.v\ -2\boldsymbol{i} + 4\boldsymbol{k}$ *perpendicular to vector* $\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ *is* $x + 3y - 2z + 10 = 0$. (**_Uneb_ 2000**)
   Solution
   Using $\boldsymbol{r}.\boldsymbol{n} = d$

   $\Rightarrow \boldsymbol{r}.\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d$ *But* $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

   $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d$ *or* $x + 3y - 2z = d$

   But $A(-2, 0, 4)$ lie on plane

   $\Rightarrow (-2) + 3(0) - 2(4) = d \quad \therefore d = -10$

   $\therefore x + 3y - 2z + 10 = 0$

3. *Find the equation of the plane through point A with* $p.v\ 5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *perpendicular to vector* $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$. (**_Uneb_ 1998**)

Solution

Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}1\\3\\-2\end{pmatrix} = d \;\; But\, \boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\4\\-1\end{pmatrix} = d \;\; or\; 3x + 4y - z = d$

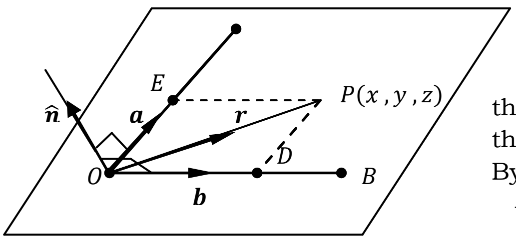
But $(5\,,-2\,,3)$ lie on plane

$\Rightarrow 3(5) + 4(-2) - (3) = d \;\; \therefore d = 4$

$\therefore 3x + 4y - z - 4 = 0$

4. *Given that* $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$. *Determine the equation containing* $\boldsymbol{a}$ *and* $\boldsymbol{b}$.

Solution



Since only two vectors are given, and no other point is given, taking the point as the *origin* through which the plane passes, then finding the unit vector normal to the vectors, By cross product,

using $\hat{\boldsymbol{n}} = \frac{\boldsymbol{b}\times\boldsymbol{a}}{|\boldsymbol{b}\times\boldsymbol{a}|}$

$$\boldsymbol{b}\times\boldsymbol{a} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ -1 & -3 & 2\\ 1 & -3 & 3\end{vmatrix} = -3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$$

$|\boldsymbol{b}\times\boldsymbol{a}| = \sqrt{9 + 25 + 36} = \sqrt{70}$

$\therefore \hat{\boldsymbol{n}} = \frac{1}{\sqrt{70}}(3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k})$

Using $\boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{70}}(3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}3\\5\\6\end{pmatrix} = \sqrt{70}D = k\;(scalar)$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\5\\6\end{pmatrix} = k\,, -3x + 5y + 6z = k$

But $(0\,,0\,,0)$ lies on the plane

$\Rightarrow -3(0) + 5(0) + 6(0) = k$

$\therefore -3x + 5y + 6z = 0$

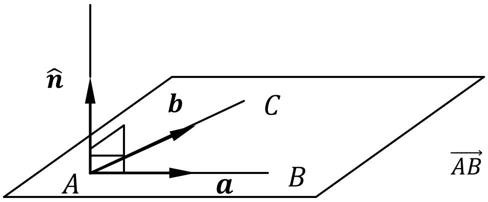
5. *Find the Cartesian equation of a plane through;*

a) $A(0\,,3\,,-4)\,,B(2\,,-1\,,2)$ *and* $C(7\,,4\,,-1)$

b) $C(-1\,,1\,,2)\,,D(1\,,-2\,,1)$ *and* $E(2\,,2,4)$

Solution

a)



Let $\overrightarrow{AB} = \boldsymbol{a}\,,\overrightarrow{AC} = \boldsymbol{b}$

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\-1\\2\end{pmatrix} - \begin{pmatrix}0\\3\\-4\end{pmatrix}$

$= \begin{pmatrix}2\\-4\\6\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 2 \\ 1 & -3 & 3 \end{vmatrix} = -18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(-18)^2 + 36^2 + 30^2} = \sqrt{2520}$

$\widehat{\boldsymbol{n}} = \frac{1}{\sqrt{2520}}(-18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k})$

Using $\boldsymbol{r}.\widehat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{2520}}(-18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = \sqrt{2520}D = k\ (scalar)$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = k\,, -18x + 36y + 30z = k$

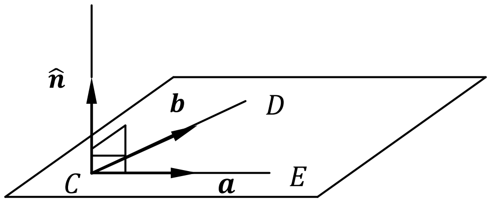
$But\ A\,, B\ and\ C$ lie on the plane. Using point $A(0\,,3\,,-4)$

$\Rightarrow -18(0) + 36(3) + 30(-4) = k \therefore k = -12$

$\therefore -18x + 36y + 30z = -12$ .... $\div (-6)$

$\therefore 3x - 6y - 5z - 2 = 0$

b) $\boldsymbol{Let\ \overrightarrow{CE} = a\,, \overrightarrow{CD} = b}$



$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$

$\boldsymbol{a} = \boldsymbol{\overrightarrow{CE}} = \boldsymbol{\overrightarrow{OE}} - \boldsymbol{\overrightarrow{OC}}$

$= \begin{pmatrix} \boldsymbol{2} \\ \boldsymbol{2} \\ \boldsymbol{4} \end{pmatrix} - \begin{pmatrix} \boldsymbol{-1} \\ \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{3} \\ \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 2 \\ 2 & -3 & -1 \end{vmatrix} = 5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(5)^2 + 7^2 + (-11)^2} = \sqrt{195}$

$\widehat{\boldsymbol{n}} = \frac{1}{\sqrt{195}}(5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k})$

Using $\boldsymbol{r}.\widehat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{195}}(5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = \sqrt{195}D = k\ (scalar)$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = k\,, 5x + 7y - 11z = k$

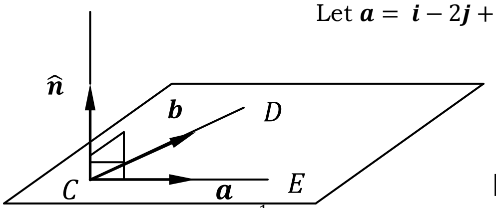
$But\ C\,, D\ and\ E$ lie on the plane. Using point $C(-1\,,1\,,2)$

$\Rightarrow 5(-1) + 7(1) - 11(2) = k \therefore k = -20$

$\therefore 5x + 7y - 11z = -20$

6. *Find the equation of a plane passing through point* $(1\,.2\,,3)$ *and parallel to vectors*
$\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$.

Solution



Let $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}\,, \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}\,, C\big((1\,.2\,,3)\big)$

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = 7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$$

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(7)^2 + (5)^2 + (1)^2} = \sqrt{75}$$

$Using\ \boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{75}}(7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}7\\5\\1\end{pmatrix} = \sqrt{75}D = k\ (scalar)$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}7\\5\\1\end{pmatrix} = k\,, 7x + 5y + z = k$

$But\ A\,, B\ and\ C$ lie on the plane. Using point $C(1\,.2\,,3)$

$\Rightarrow 7(1) + 5(2) + 3 = k \therefore k = 20$

$\therefore 7x + 5y + z = 20$

Deduction from Cartesian equation of a plane $\boldsymbol{Ax + By + Cz = d}$

From $\boldsymbol{r}.\boldsymbol{n} = d\,, \boldsymbol{n} = \begin{pmatrix}A\\B\\C\end{pmatrix}, \boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}A\\B\\C\end{pmatrix} = d\,,\ Ax + By + Cz = d$

From the Cartesian equation of a plane, there exists a *NORMAL VECTOR* to plane whose direction ratios are obtained from coefficients of $x\,, y\,, z$ in the equation of plane. $i.e.\boldsymbol{n} = \begin{pmatrix}A\\B\\C\end{pmatrix}$

Example

State the normal vector to the plane;

a) $2x + 3y + 5z = 7$

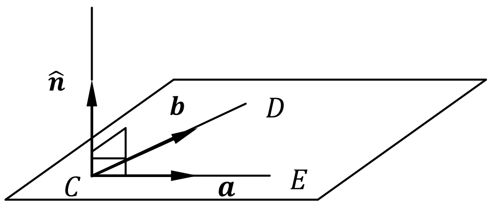
b) $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\-5\\6\end{pmatrix} + \mu\begin{pmatrix}7\\0\\5\end{pmatrix}$

c) $\boldsymbol{r} = (3 + 4\lambda - 3\mu)\boldsymbol{i} + 2\lambda\boldsymbol{j} + (3 - 3\mu)\boldsymbol{k}$

Solution

a) $2x + 3y + 5z = 7\,, \boldsymbol{n} = \begin{pmatrix}2\\3\\5\end{pmatrix}$

b) $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\-5\\6\end{pmatrix} + \mu\begin{pmatrix}7\\0\\5\end{pmatrix}$



Let $\boldsymbol{a} = \begin{pmatrix}4\\-5\\6\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}7\\0\\5\end{pmatrix}$

Using $\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 4 & -5 & 6 \\ 7 & 0 & 5 \end{vmatrix} = -25\boldsymbol{i} + 29\boldsymbol{j} + 35\boldsymbol{k}$$

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(-25)^2 + (29)^2 + (35)^2} = \alpha\ (Scalar)$$

$\Rightarrow\ \hat{\boldsymbol{n}} = \frac{1}{\alpha}(-25\boldsymbol{i} + 29\boldsymbol{j} + 35\boldsymbol{k})$

Direction ratio of normal vector is $-25{:}\,29{:}\,35$ in simplified form

$\therefore \boldsymbol{n} = \begin{pmatrix} -25 \\ 29 \\ 35 \end{pmatrix}$

c) $\boldsymbol{r} = (3 + 4\lambda - 3\mu)\boldsymbol{i} + 2\lambda\boldsymbol{j} + (3 - 3\mu)\boldsymbol{k}$

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

Let $\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 4 & 2 & 0 \\ -3 & 0 & -3 \end{vmatrix} = -6\boldsymbol{i} - 12\boldsymbol{j} + 6\boldsymbol{k}$

Normal vector ratio is $-6{:}-12{:}\,6 = 1{:}\,2{:}-1$

$\therefore \textit{Normal vector is } \boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Alternative approach of obtaining normal vector to plane expressed in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

Find in scalar product form the equation of the plane

$\boldsymbol{r} = (1 + 3\lambda + 2\mu)\boldsymbol{i} + (1 + \lambda + 4\mu)\boldsymbol{j} + (\mu - \lambda)\boldsymbol{k}$

Solution

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

Let the normal vector $\boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Since **n** is perpendicular to **a** and **b**, then by Dot product

$\boldsymbol{n}.\boldsymbol{a} = 0 \Rightarrow (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}).(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 0$

$\therefore 3A + B - C = 0$ ……………………………………………… (1)

$\boldsymbol{n}.\boldsymbol{b} = 0 \Rightarrow (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}).(2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) = 0$

$\therefore 2A + 4B + C = 0$ …………………………………………… (2)

Expressing B and C in terms of A only

$eqn.(1) + eqn.(2)$

$\Rightarrow 5A + 5B = 0\ \therefore A = -B$ ……………………………… (3)

$eqn.(3)\ in\ to\ eqn.(1)$

$\Rightarrow 3A - A - C = 0\ \therefore C = 2A$

$Now\ \boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, Direction\ ratio{:}\ A{:}\,B{:}\,C$

$A{:}-A{:}\,2A = 1{:}-1{:}\,2$

$\therefore \boldsymbol{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

From $\boldsymbol{r}.\boldsymbol{n} = d \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d$. But $(1, 1, 0)$ lies on the plane

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d\ \therefore d = 1 - 1 + 0 = 0$

Equation of plane in dot product form is $\boldsymbol{r}.(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})=0$

NOTE:

The above approach can also be used to determine the normal vector to *two* given vectors.

INTERSECTION OF A LINE AND A PLANE

Consider the line $\boldsymbol{r}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$ and plane in Cartesian form

$Ax+By+Cz=d$

From the line, $\boldsymbol{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$

$\Rightarrow x=a_1+\lambda d_1\ ,y=a_2+\lambda d_2\ ,z=a_3+\lambda d_3$ .... (1)

Substitute eqn. (1)*in to eqn. of plane*

$A(a_1+\lambda d_1\ )+B(a_2+\lambda d_2)+C(a_3+\lambda d_3)=d$

$$\boxed{\Rightarrow (Aa_1+Ba_2+Ca_3)+\lambda(Ad_1+Bd_2+Cd_3)=0}$$

From which the value of λ can be obtained. The point of intersection of the line and plane can be obtained from $eqn.(1)$

$.i.e.\left[\left((a_1+\lambda d_1)\ ,(a_2+\lambda d_2)\ .\right)(a_3+\lambda d_3)\right]$

Deduction from the rectangular enclosed result

The result above is true for every value of λ hence it is an identity. $i.e.$

$$\boxed{\begin{matrix}Aa_1+Ba_2+Ca_3=0\\ \\ Ad_1+Bd_2+Cd_3=0\end{matrix}}$$

, which are the conditions for a line to lie on a plane.

Remark:

1. The condition $\boldsymbol{Ad_1}+\boldsymbol{Bd_2}+\boldsymbol{Cd_3}=\boldsymbol{0}$ shows that the line is perpendicular to the normal of the plane, hence parallel to the plane.
2. The condition $\boldsymbol{Aa_1}+\boldsymbol{Ba_2}+\boldsymbol{Ca_3}=\boldsymbol{0}$ **s**hows that the point $(a_1\ ,a_2\ ,a_3)$ lies on the plane.

Conclusion

For a line to lie in a plane the line must be parallel to the plane and *one* point on the line must also be on plane.

Examples

1. *Find the point of intersection of the given line and plane.*
   a) $\boldsymbol{r}=(2\boldsymbol{i}-\boldsymbol{k})+\lambda(i+3\boldsymbol{j})\ ,plane\!:\boldsymbol{r}.(5\boldsymbol{i}-\boldsymbol{j}+7\boldsymbol{k})=9$
   b) $\boldsymbol{r}=\lambda\boldsymbol{i}-2\boldsymbol{j}+(2\lambda-1)\boldsymbol{k}\ ,plane\!:\boldsymbol{r}.\begin{pmatrix}2\\1\\-3\end{pmatrix}=5$
   c) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\ ,-4x+4y-5z-3=0$

Solution

a) $\boldsymbol{r}=(2\boldsymbol{i}-\boldsymbol{k})+\lambda(i+3\boldsymbol{j})=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}$; $\boldsymbol{r}.(5\boldsymbol{i}-\boldsymbol{j}+7\boldsymbol{k})=9\ \therefore 5x-y+7z=9$

From the line, $\Rightarrow x=2+\lambda\ ,y=3\lambda\ ,z=-1$ .... (1)

$Substitute\ eqn. (1) in\ to\ eqn. of\ plane$
$5(2+\lambda)-(3\lambda)+7(-1)=9 \Rightarrow 10+5\lambda-3\lambda-7=9$
$\therefore \lambda=3$
$From\ eqn. (1), point\ of\ intersection\ is;$
$x=2+3=5\ , y=3(3)=9\ , z=-1$
$(5\ ,9\ ,-1)$ is the point of intersection of the line and plane.

b) $\boldsymbol{r}=\lambda\boldsymbol{i}-2\boldsymbol{j}+(2\lambda-1)\boldsymbol{k} \Rightarrow \boldsymbol{r}=\begin{pmatrix}0\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$
$\Rightarrow\ x=\lambda\ , y=-2\ \ , z=-1+2\lambda$ .... (1)
$\boldsymbol{r}.\begin{pmatrix}2\\1\\-3\end{pmatrix}=5 \Rightarrow 2x+y-3z=5$ .... (2)
$eqn. (1) in\ to\ eqn. (2)$
$\Rightarrow 2\lambda+(-2)-3(-1+2\lambda)=5\ \therefore 2\lambda-2+3-6\lambda=5$
$\therefore \lambda=-1$
$From\ eqn. (1), point\ of\ intersection\ is;$
$x=-1\ , y=-2\ , z=-1-2=-3$
$(-1\ ,-2\ ,-3)$ is the point of intersection of the line and plane.

c) Let $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}=\lambda$
$\Rightarrow\ x=3+2\lambda\ , y=4+3\lambda\ \ , z=5+4\lambda$ .... (1)
$\Rightarrow -4x+4y-5z-3=0$ .... (2)
$eqn. (1) in\ to\ eqn. (2)$
$\Rightarrow -4(3+2\lambda\ )+4(4+3\lambda)-5(5+4\lambda)-3=0$
$\Rightarrow -12-8\lambda+16+12\lambda-25-20\lambda-3=0\ \ or -16\lambda=24$
$\therefore \lambda=-\frac{3}{2}$
$From\ eqn. (1), point\ of\ intersection\ is;$
$x=3-3=0\ , y=4-\frac{9}{2}=-\frac{1}{2}\ , z=-1$
$\left(0\ ,-\frac{1}{2}\ ,-1\right)$ is the point of intersection of the line and plane.

2. *Prove that the line(s) lie on the given plane(s)*
   a) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\ ,\ 4x+4y-5z-3=0$
   b) $\boldsymbol{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}$ $and\ 2x+2y+z=5$

Solution

a) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\ , \boldsymbol{r}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\4\end{pmatrix} \Rightarrow \boldsymbol{d}=\begin{pmatrix}2\\3\\4\end{pmatrix}$
$Plane: 4x+4y-5z-3=0 \Rightarrow \boldsymbol{n}=\begin{pmatrix}4\\4\\-5\end{pmatrix}$

For a line to lie in a plane,
- The line is perpendicular to the normal of the plane $.i.e. \boldsymbol{n}.\boldsymbol{d}=0$
- One point on the line lies on the plane

$\Rightarrow \boldsymbol{n}.\boldsymbol{d}=(4\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{5k}).(2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})=8+12-20=0$
From the line $(3\ ,4\ ,5)$ lies on the line
$\Rightarrow : 4(3)+4(4)-5(5)-3=0$, which satisfies the plane. Hence point lies on the plane.

b) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

$2x + 2y + z = 5 \Rightarrow \boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = (= -\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}).(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = -2 + 2 + 0 = 0$. Hence line is parallel to the plane .i.e. normal $\perp$ plane.

From the line $(2\,,0\,,1)$ lies on the line

$\Rightarrow 2(2) + 2(0) + 1 = 5$, which satisfies the plane. Hence point lies on the plane.

EQUATION OF A PLANE CONTAINING A GIVEN LINE $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ AND SATISFYING ANY OTHER GIVEN CONDITION

Let the line be expressed in the Cartesian form $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$ and let the plane be $Ax + By + Cz = d$.

Since the plane is to pass through the point on the line $.i.e. (a_1\,, a_2\,, a_3)$, then

$Ax + By + Cz = d$ …………………………………………………………………… (1)

$Aa_1 + Ba_2 + Ca_3 = d$ ……………………………………………………………… (2)

$eqn.(1) - eqn.(2)$ give the equation of a plane without d

$\Rightarrow A(x - a_1) + B(y - a_2) + C(z - a_3) = 0$ …………………………… (3)

NOTE:

Plane in equations (3) and (1) are the same.

Equation of plane contains the line if;

$Ad_1 + Bd_2 + Cd_3 = 0$ ……………………………………………………………… (4)

Case I:

From equations (1) and (3)

Let the other condition satisfying the plane be that it also passes through point $(\alpha\,, \alpha\,, \gamma)$

From eqn. (3),

$A(\alpha - a_1) + B(\beta - a_2) + C(\gamma - a_3) = 0$ ……………………………… (5)

Eliminating constants A, B and C from the zeroed equations

$$\boxed{\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ \alpha - a_1 & \beta - a_2 & \gamma - a_3 \end{vmatrix} = 0}$$

Case II

Let the given condition be that the required plane is perpendicular to plane $ax + by + cz = k$

Planes $Ax + By + Cz = d$ and $ax + by + cz = k$ are perpendicular if their normal are perpendicular

$\Rightarrow aA + Bb + Cc = 0$ …………………………………………………………… (6)

Now eliminating constants A, B and C from the zeroed equations (6) , (3) and (4), the required equation of plane is;

$$\boxed{\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ a & b & c \end{vmatrix} = 0}$$

Example

1. *Find the equation of a plane containing the line* $\boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{k}) + \lambda(-\boldsymbol{i} + \boldsymbol{j})$ *and passing through the point with position vector* $\boldsymbol{i} + 3\boldsymbol{k}$.

Solution

Let the equation of the plane be $Ax + By + Cz = d$................................. (1)

From the line: $\boldsymbol{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $(2\,,0\,,1)$ lies on the plane.

$\Rightarrow 2A + B(0) + C = d$

.... (2)

$eqn.(1) - eqn.(2)$ to obtain a zeroed equation

$\Rightarrow A(x-2) + B(y-0) + C(z-1) = 0$ .... (3)

Plane (3) passes through $(1\,,0\,,3)$

$\Rightarrow A(1-2) + B(0-0) + C(3-1) = 0$ *or* $-A + 0B + 2C = 0$ .... (4)

Also eqn. (1) contains line if $\boldsymbol{n}.\boldsymbol{d} = 0$

$\Rightarrow \begin{pmatrix}A\\B\\C\end{pmatrix}.\begin{pmatrix}-1\\1\\0\end{pmatrix} = 0$ *or* $-A + B + 0C = 0$ .... (5)

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

$$\Rightarrow \begin{vmatrix}x-2 & y & z-1\\-1 & 0 & 2\\-1 & 1 & 0\end{vmatrix} = 0$$

$$\Rightarrow (x-2)\begin{vmatrix}0 & 2\\1 & 0\end{vmatrix} - y\begin{vmatrix}-1 & 2\\-1 & 0\end{vmatrix} + (z-1)\begin{vmatrix}-1 & 0\\-1 & 1\end{vmatrix} = 0$$

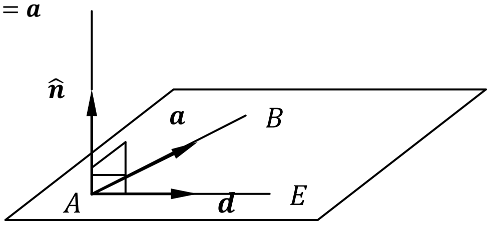
$\Rightarrow -2(x-2) - y(2) + (z-1)(-1) = 0$

$\therefore 2x + 2y + z = 5$

Alternatively

Let $A(2\,,0\,,1)\,, B(1\,,0\,,3)$

$\overrightarrow{AB} = \boldsymbol{a}$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix}$

$= \begin{pmatrix}-1\\0\\-2\end{pmatrix}$

Finding normal vector to plane

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|}$

$$\boldsymbol{d}\times\boldsymbol{a} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\-1 & 1 & 0\\-1 & 0 & 2\end{vmatrix} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \Rightarrow |\boldsymbol{d}\times\boldsymbol{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|} = \frac{1}{3}(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

But normal vector direction ratio is $2:2:1$

$\therefore \boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\Rightarrow \boldsymbol{r}.(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = d$ *or* $2x + 2y + z = d$

This equation is satisfied by either $A(2\,,0\,,1)$ *or* $B(1\,,0\,,3)$

$\Rightarrow 4 + 1 = d \;\therefore d = 5$

The equation of plane now is $2x + 2y + z = 5$

2. *Find the equation of a plane through the line* $\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$ *and perpendicular to the plane* $x-2y+3z=4$.

Solution

Let the plane be $Ax+By+Cz=d$.. (1)

$\Rightarrow \boldsymbol{n}=A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}$

Plane contains the line if the point $(-1,-2,-3)$ on line lies on it

$\Rightarrow A(-1)+B(-2)+C(-3)=d$.. (2)

$eqn.(1)-eqn.(2)$ to obtain a zeroed equation

$A(x+1)+B(y+2)+C(z+3)=0$ .... (3)

Also eqn. (1) contains line if $\boldsymbol{n}.\boldsymbol{d}=0$

$\Rightarrow \begin{pmatrix}A\\B\\C\end{pmatrix}.\begin{pmatrix}2\\3\\4\end{pmatrix}=0\ or\ 2A+3B+4C=0$ .... (4)

For planes to be perpendicular, their normals are at right angles.

From $eqn.(1)$, $\boldsymbol{n}_1=A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}$

From $x-2y+3z=4$, $\boldsymbol{n}_2=\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$

$\therefore\ \boldsymbol{n}_1.\ \boldsymbol{n}_2=0 \Rightarrow A-2B+3B=0$ .... (5)

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

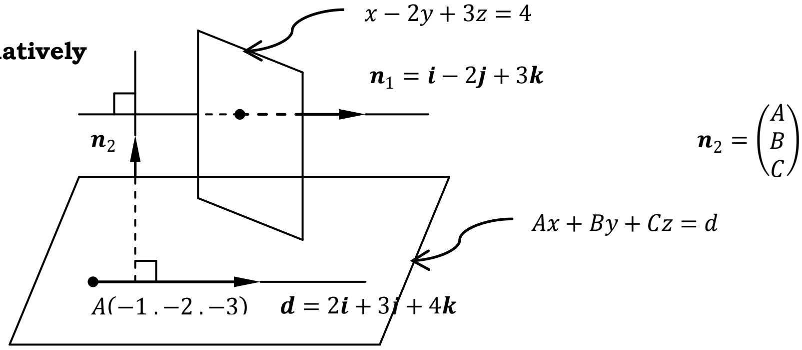
$\Rightarrow \begin{vmatrix}x+1 & y+2 & z+3\\ 2 & 3 & 4\\ 1 & -2 & 3\end{vmatrix}=0$

$\Rightarrow (x+1)\begin{vmatrix}3 & 4\\ -2 & 3\end{vmatrix}-(y+2)\begin{vmatrix}2 & 4\\ 1 & 3\end{vmatrix}+(z+3)\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=0$

$\Rightarrow 17(x+1)-2(y+2)+\ -7(z+3)=0$

$\therefore 17x-2y-7z=8$

Alternatively



From the figure, $\boldsymbol{n}_1//\boldsymbol{d}$ and $\boldsymbol{n}_2$ is perpendicular to the line and $\boldsymbol{n}_1$

Finding $\boldsymbol{n}_2$ using cross product

$\Rightarrow \boldsymbol{n}_2=\begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ 2 & 3 & 4\\ 1 & -2 & 3\end{vmatrix}=17\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n}=k(scalar)$

$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}17\\-2\\-7\end{pmatrix}=k\ ,17x-2y-7z=k$

But $(-1,-2,-3)$ *lies on the plane*

$\Rightarrow -17+4+21=k\ \therefore k=8$

$\therefore 17x-2y-7z=8$ as before.

3. *Find the equation of a plane which passes through the point* $(4\,,5\,,-6)$ *and is;*
   a) *Parallel to the plane* $2x + 3y + 5z + 6 = 0$
   b) *Parallel to the line joining points* $(5\,,2\,,3)$ *and* $(1\,,6\,,4)$
   c) *Perpendicular to planes* $3x + 5y - 6z = 0$ *and* $7x + y + 2z = 8$

Solution

a) Let the plane be $Ax + By + Cz = d$ .... (1)
Eqn. (1) passes through $(4\,,5\,,-6)$
$\Rightarrow 4A + 5B - 6C = d$ .... (2)
$eqn.(1) - eqn.(2)$
$A(x-4) + B(y-5) + C(z+6) = 0$ .... (3)
Plane (1) is parallel to $2x + 3y + 5z + 6 = 0$ if their normals are parallel
$\Rightarrow \boldsymbol{n}_1 = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \;\therefore\; \boldsymbol{n}_1 = \lambda \boldsymbol{n}_2$
$\boldsymbol{n}_1 = \begin{pmatrix} 2\lambda \\ 3\lambda \\ 5\lambda \end{pmatrix} \Rightarrow Direction\, ratio\, of\, \boldsymbol{n}_2\, is\; 2\lambda : 3\lambda : 5\lambda = 2:3:5$
$\therefore \boldsymbol{n}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k} = \boldsymbol{A}i + B\boldsymbol{j} + C\boldsymbol{k}$ .... (4)
$eqn.(4)\; in\, to\; eqn.(3)$
$2(x-4) + 3(y-5) + 5(z+6) = 0 \;\; or \;\; 2x + 3y + 5z + 7 = 0$

b) Direction vector of line joining $(5\,,2\,,3)$ *and* $(1\,,6\,,4)$ *is*
$\begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} = \boldsymbol{d}$
Plane (1) or (3) above is parallel to the line ($.i.e. Direction\; vector\; \boldsymbol{d}$) if the normal of plane and **d** are parallel
$\Rightarrow \boldsymbol{n} = \lambda \boldsymbol{d} = \begin{pmatrix} -4\lambda \\ 4\lambda \\ \lambda \end{pmatrix};\; \begin{pmatrix} -4\lambda \\ 4\lambda \\ \lambda \end{pmatrix} \Rightarrow Direction\, ratio\, of\; \boldsymbol{n}\; is - 4\lambda : 4\lambda : \lambda = -4:4:1$
$\therefore \boldsymbol{n} = -4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} = \boldsymbol{A}i + B\boldsymbol{j} + C\boldsymbol{k}$ .... (5)
$eqn.(5)\; in\, to\; eqn.(3)$
$-4(x-4) + 4(y-5) + 1(z+6) = 0 \;\; or \;\; 4x - 4y - z + 2 = 0$

c) $\boldsymbol{n_1} \perp \boldsymbol{n_2}\,, \boldsymbol{n_2} \perp \boldsymbol{n_3}$
$\Rightarrow (3\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}).(\boldsymbol{A}i + B\boldsymbol{j} + C\boldsymbol{k}) = 0$
$\therefore 3A + 5B - 6C = 0$ .... (6)
$\Rightarrow (7\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(\boldsymbol{A}i + B\boldsymbol{j} + C\boldsymbol{k}) = 0$
$\therefore 3A + 5B - 6C = 0$ .... (7)
Eliminating A, B and C from (3), (6) and (7)
By cross product,
$\Rightarrow \begin{vmatrix} x-4 & y-5 & z+6 \\ 3 & 5 & -6 \\ 7 & 1 & 2 \end{vmatrix} = 0$
$\Rightarrow (x-4)\begin{vmatrix} 5 & -6 \\ 1 & 2 \end{vmatrix} - (y-5)\begin{vmatrix} 3 & -6 \\ 7 & 2 \end{vmatrix} + (z+6)\begin{vmatrix} 3 & 5 \\ 7 & 1 \end{vmatrix} = 0$
$\Rightarrow 16(x-4) - 48(y-5) + -32(z+6) = 0$
$\therefore x - 3y - 2z = 1$

4. *Find the equation of a plane which passes through points* $(6\,,2\,,-4)$ *and* $(3\,,4\,,1)$ *and is;*

a) *Perpendicular to the plane* $x + 5y - 2z = 6$
b) *Parallel to the line joining* $(1\,,0\,,3)$ *and* $(-1\,,2\;,4)$

Solution

a) Let plane be: $Ax + By + Cz = d$ .... (1)

For $(6\,,2\,,-4)\,,6A + 2B - 4C = d$ .... (2)

For $(3\,,4\,,1)\,,3A + 4C + C = d$ .... (3)

$eqn.(1) - eqn.(2)$

$A(x-6) + B(y-2) + C(z+4) = 0$ .... (4)

$eqn.(2) - eqn.(3)$

$3A + 6B - 5C = 0$ .... (5)

For planes to be perpendicular, their normals are also perpendicular

$\Rightarrow (\boldsymbol{A}i + B\boldsymbol{j} + C\boldsymbol{k}).(\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}) = 0$

$\Rightarrow A + 5B - 2C = 0$ .... (6)

Eliminating A, B and C from (3), (6) and (7)

By cross product,

$$\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ 1 & 5 & -2 \end{vmatrix} = 0$$

Simplifying, the plane is

$\therefore 13x + y + 9z = 44$

b) Direction vector of the line is;

$$\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \boldsymbol{d}$$

For a line to be parallel to the plane, the normal vector to plane and direction vector of the line must be perpendicular.

$$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = 0 \ \therefore \begin{pmatrix} A \\ B \\ C \end{pmatrix}.\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = 0\,, -2A + 2B + C = 0 \text{ .... (7)}$$

From (4), (5) and (7), eliminating constants A, B and C

By cross product,

$$\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ -2 & 2 & 1 \end{vmatrix} = 0$$

Simplifying, the plane is

$\therefore 16x + 7y + 18z = 38$

NOTE:

The above question can be done using another alternative as in example (2) above.

5. *Find the equation of a plane containing the line* $\frac{x+1}{-3} = \frac{y-3}{-2} = \frac{z+2}{1}$ *and the point* $(0\,,7\,,-7)$*. Prove that the line* $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ *also lies on the plane.*

Solution

Let plane be: $Ax + By + Cz = d$ .... (1)

$(0\,,7\,,-7)$ lies on the plane

$\Rightarrow A(0) + B(7) + C(-7) = d$ .... (2)

$eqn.(1) - eqn.(2)$

$A(x-0) + B(y-7) + C(z+7) = 0$ .... (3)

$eqn.(1) contains\ line\ if$

$-3A - 2B + C = 0$ .... (4)

The point on line also lies on the plane (3)

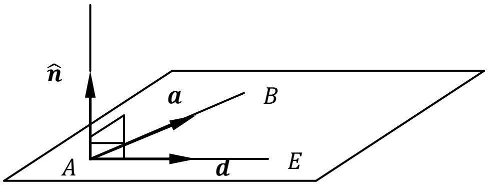
$\Rightarrow A(-1-0) + B(3-7) + C(-2+7) = 0$

$\therefore -A - 4B + 5C = 0$ .... (3)
From (3), (4) and (5), eliminating constants A, B and C
By cross product,

$$\Rightarrow \begin{vmatrix} x & y-2 & z+4 \\ -3 & -2 & 1 \\ -1 & -4 & 5 \end{vmatrix} = 0$$

Simplifying, the plane is
$\therefore 3x - 7y - 5z + 14 = 0$

Alternatively



Let $\overrightarrow{AB} = \boldsymbol{a} = \overrightarrow{OB} - \overrightarrow{OA}$
$A(-1, 3, -2), B(0, 7, -7)$
Let $\boldsymbol{d} = -3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

$$\Rightarrow \boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 7 \\ -7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

Finding normal vector to plane

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|}$

$$\boldsymbol{d} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -3 & -2 & 1 \\ 1 & 4 & -5 \end{vmatrix} = 6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k} \Rightarrow |\boldsymbol{d} \times \boldsymbol{a}| = \sqrt{6^2 + (-14)^2 + (-10)^2} = \sqrt{332}$$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|} = \frac{1}{\sqrt{332}}(6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k})$

But normal vector direction ratio is $\frac{6}{\sqrt{332}} : -\frac{14}{\sqrt{332}} : -\frac{10}{\sqrt{332}} = 3 : -7 : -5$

$\therefore \boldsymbol{n} = 3\boldsymbol{i} - 7\boldsymbol{j} - 5\boldsymbol{k}$
Using $\boldsymbol{r}.\boldsymbol{n} = k$
$\Rightarrow \boldsymbol{r}.(6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k}) = k \; or \; 6x - 14y - 10z = k$
This equation is satisfied by either $A(-1, 3, -2), B(0, 7, -7)$
$\Rightarrow -3 - 21 + 10 = -14 = k = d \; \therefore d = 5$
The equation of plane now is $\therefore 3x - 7y - 5z + 14 = 0$

COPLANAR LINES

Aare lines which lie on the same plane. They can meet/do not meet or parallel.

Consider $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ .... (1)

$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$ .... (2)

Let the plane be $Ax + By + Cz = d$ .... (3)
Eqn. (3) contains (1) if (α, β, γ) lie on the plane
$\Rightarrow A\alpha + B\beta + C\gamma = d$ .... (4)
$eqn.(3) - eqn.(4)$
$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0$ .... (5)
Also Eqn. (3) contains (1) if direction vector of line is perpendicular to plane.
$\Rightarrow Al + Bm + Cn = d$ .... (6)
Now eqn. (5) contains (2) if
$A(\alpha' - \alpha) + B(\beta' - \beta) + C(\gamma' - \gamma) = 0$ .... (7)

Also Eqn. (5) contains (2) if direction vector of line is perpendicular to plane
$\Rightarrow Al' + Bm' + Cn' = d$ .... (8)
From (6), (7) and (8), eliminating constants A, B and C
By cross product,

$$\boxed{\therefore \begin{vmatrix} \alpha' - \alpha & \beta' - \beta & \gamma' - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0}$$

, which is the condition for two lines to be coplanar.
From (5), (6) and (8), eliminating constants A, B and C

$$\boxed{\therefore \begin{vmatrix} x - \alpha & y - \beta & z - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0}$$

, which is the equation of plane containing the two lines.

Example

1. *Prove that the lines* $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ *and* $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ *are coplanar and lie in the plane* $x - 2y + z + 7 = 0$
   Solution
   Equation of a plane containing the first line and parallel to the second line is
   $\begin{vmatrix} x & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \text{ .} i.e.\, x - 2y + z + 7 = 0$, clearly this plane is satisfied by the point $(2, 6, 3)$ which lies on the second line.
2. *Show that the lines* $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ *and* $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ *are coplanar and find the equation of plane containing them.*
   Solution
   Equation of a plane containing the first line and parallel to the second line is
   $\begin{vmatrix} x & y-2 & z+3 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0 \text{ .} i.e.\, 6x - 5y - z = 0$. This plane passes through point $(-1, -1, -1)$ on the second line so that it contains the second line. Thus the two lines are coplanar and the equation of plane containing them is $6x - 5y - z = 0$.

CONDITION FOR 4-POINTS TO BE COPLANAR

Four points will be coplanar if the plane passing through any 3-points passes through the fourth point.

Example

Show that the points below are coplanar

a) $(0, -1, 0), (2, 1, -1), (1, 1, 1)$ *and* $(3, 3, 0)$
b) $(0, 1, 3), (1, 0, -4), (1, 1, -1)$ *and* $(1, 2, 2)$

Solution

a) Let $A(0, -1, 0), B(2, 1, -1), C(1, 1, 1)$ *and* $D(3, 3, 0)$
   Finding the equation of plane through A, B and C
   Using $\boldsymbol{r}.\boldsymbol{n} = d$

Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

Ratio of normal direction vector $= 4{:}-3{:}2 \Rightarrow Normal\ vector\ \boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = d\ \ or\ 4x - 3y + 2z = d$

But $A, B\ and\ C$ lie on this plane

For $A(0\,,-1\,,0)\,, 4(0) - 3(-1) + 2(0) = d\ \therefore d = 3$

The equation of the plane is $4x - 3y + 2z = 3$

Now for $D(3\,,3\,,0)\,, 4(3) - 3(3) + 2(0) = 3 = \boldsymbol{RHS}$ **.** Hence points are coplanar.

b) Let $A(0\,,1\,,3)\,, B(1\,,0\,,-4)\,, C(1\,,1\,,-1)\ and\ D(1\,,2\,,2)$

Finding the equation of plane through A, B and C

Using $\boldsymbol{r}.\boldsymbol{n} = d$

Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & -7 \\ 1 & 0 & -4 \end{vmatrix} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Ratio of normal direction vector $= 4{:}-3{:}1 \Rightarrow Normal\ vector\ \boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = d\ \ or\ 4x - 3y + z = d$

But $A, B\ and\ C$ lie on this plane

For $A(0\,,1\,,3)\,, 4(0) - 3(1) + (3) = d\ \therefore d = 0$

The equation of the plane is $4x - 3y + 2z = 0$

Now for $D(1\,,2\,,2)\,, 4(1) - 3(2) + (2) = 3 = \boldsymbol{RHS}$. Hence points are coplanar.

COPLANAR VECTORS

Vectors lying in the same plane are coplanar vectors.

Let vectors **a, b** and **c** lie on the same plane.

For coplanar vectors $\boxed{\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0}$

Examples

Show that the vectors are coplanar

a) $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{k}\,, \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - 7\boldsymbol{k}\ and\ \boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

b) $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}\,, \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}\ and\ \boldsymbol{c} = 3\boldsymbol{i} + 4\boldsymbol{j}$

Solution

a) $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -4 \\ 1 & -1 & -7 \end{vmatrix} = -4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}).(-4\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}) = -4+6-2=0$. Thus vectors are coplanar.

b) $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\boldsymbol{i}-3\boldsymbol{j}+0\boldsymbol{k}$

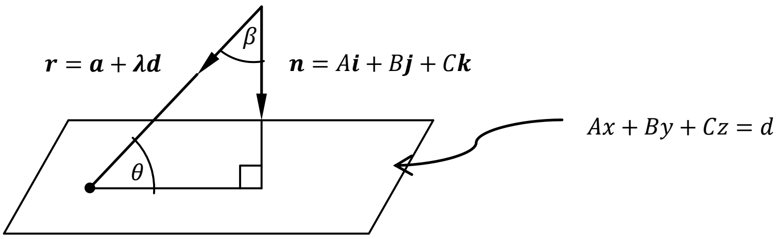
Now $\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b}) = (3\boldsymbol{i}+4\boldsymbol{j}+0\boldsymbol{k}).(4\boldsymbol{i}-3\boldsymbol{j}+0\boldsymbol{k}) = 12-12+0=0$. Thus vectors are coplanar.

NOTE:

A correct combination of vectors can reduce on the unnecessary calculations to satisfy the condition for coplanar vectors.

ANGLE BETWEEN A LINE AND A PLANE

Consider the line $\boldsymbol{r}=\boldsymbol{a}+\boldsymbol{\lambda d}$ and the plane $Ax+By+Cz=d$



Required angle $=\theta = 90-\beta$

Finding angle β between the line and the normal vector to plane

$\boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|cos\beta$

But $\beta = 90-\theta$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|cos(90-\theta)$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|sin\theta$ $[\because cos(90-\theta)=sin\theta]$

$$\boxed{\therefore \theta = sin^{-1}\left(\frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}|.|\boldsymbol{d}|}\right)}$$

In general the angle between a line and a plane is the complement of the angle between the normal vector of plane and direction vector of the line.

Examples

Find the angle between the following line and plane

a) $\boldsymbol{r}=(\boldsymbol{i}-3\boldsymbol{j})+\lambda(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})$, $plane: \boldsymbol{r}.(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k})=10$

b) $\boldsymbol{r}=(2+\lambda)\boldsymbol{i}-3\boldsymbol{j}+(1-\lambda)\boldsymbol{k}$, $plane: \boldsymbol{r}.\begin{pmatrix}4\\1\\-1\end{pmatrix}=6$

c) $\frac{x}{4}=\frac{y-1}{-1}=\frac{z+3}{-5}$, $x-2y+4z+3=0$

d) $\boldsymbol{r}=3\lambda\boldsymbol{i}+2\lambda\boldsymbol{j}-6\lambda\boldsymbol{k}$, $\boldsymbol{r}.(4\boldsymbol{i}-3\boldsymbol{k})=20$

Solution

a) $\boldsymbol{r}=(\boldsymbol{i}-3\boldsymbol{j})+\lambda(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}) \Rightarrow \boldsymbol{d}=2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$

$plane: \boldsymbol{r}.(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k})=10 \Rightarrow \boldsymbol{n}=\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}$

Using $\boldsymbol{n}.\boldsymbol{d}=|\boldsymbol{n}||\boldsymbol{d}|sin\theta$

$\boldsymbol{n}.\boldsymbol{d}=(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}).(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k})=2+2+7=11$

$|\boldsymbol{n}|=\sqrt{1+4+49}=\sqrt{54}$, $|\boldsymbol{d}|=\sqrt{4+1+1}=\sqrt{6}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|}=\frac{11}{\sqrt{54}\times\sqrt{6}}=\frac{11}{18} \Rightarrow \theta = 37.7^0$

b) $\boldsymbol{r} = (2+\lambda)\boldsymbol{i} - 3\boldsymbol{j} + (1-\lambda)\boldsymbol{k}\ , plane: \boldsymbol{r}.\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6$

$\boldsymbol{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, plane: \boldsymbol{r}.\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6 \Rightarrow \boldsymbol{n} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

$\boldsymbol{n}.\boldsymbol{d} = (4\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}).(\boldsymbol{i} + 0\boldsymbol{j} - \boldsymbol{k}) = 4 + 0 + 1 = 5$

$|\boldsymbol{n}| = \sqrt{16+1+1} = \sqrt{18}\ ,\ |\boldsymbol{d}| = \sqrt{1+0+1} = \sqrt{2}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|} = \frac{5}{\sqrt{18}\times\sqrt{2}} = \frac{5}{6} \Rightarrow \theta = 54.4^0$

c) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix} \therefore \boldsymbol{d} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix},\ x - 2y + 4z + 3 = 0 \Rightarrow \boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

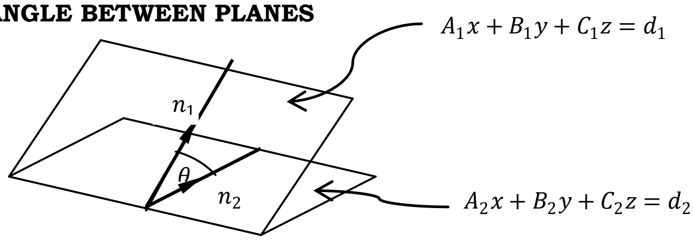
$\boldsymbol{n}.\boldsymbol{d} = (4\boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k}).(\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) = 4 + 2 - 20 = -14$

$|\boldsymbol{n}| = \sqrt{16+1+25} = \sqrt{42}\ ,\ |\boldsymbol{d}| = \sqrt{1+4+16} = \sqrt{21}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|} = \frac{-14}{\sqrt{21}\times\sqrt{42}} \Rightarrow \theta = -28.1^0\ \therefore \theta = 28.1^0\ (Numerical\, value)$

d) Left as an exercise ***Answer***: $sin\theta = \frac{6}{7}\ \therefore \theta = 59^0$

ANGLE BETWEEN PLANES



The angle between planes is the angle between their *normals*.

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$$\boxed{\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right)}$$

Example

Find the angle between the pair of planes

a) $\boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j}) = 4\ and\ \boldsymbol{r}.(\boldsymbol{j} + \boldsymbol{k}) = 1$

b) $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = 5\ and\, \boldsymbol{r}.\boldsymbol{j} = 0$

c) $\boldsymbol{r}.(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 1\, and\, \boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 0$

d) $2x + 3y - 5z = 1\, and\ x - 2y - z = 0$

Solution

a) $\boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j}) = 4\ \Rightarrow \boldsymbol{n_1} = \boldsymbol{i} - \boldsymbol{j} = \boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}\ ;\ \boldsymbol{r}.(\boldsymbol{j} + \boldsymbol{k}) = 1 \Rightarrow \boldsymbol{n_2} = \boldsymbol{j} + \boldsymbol{k} = 0\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}).(0\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0 - 1 + 0 = -1$

$|\boldsymbol{n_1}| = \sqrt{0+1+1} = \sqrt{2}\, ; |\boldsymbol{n_2}| = \sqrt{0+1+1} = \sqrt{2}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{-1}{\sqrt{2}\times\sqrt{2}}\right) = 120^0$

In terms of acute angles, $\theta = 180 - 120 = 60^0$

b) $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = 4\ \Rightarrow \boldsymbol{n_1} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}\, ;\ \boldsymbol{r}.\boldsymbol{j} = 0 \Rightarrow \boldsymbol{n_2} = 0\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}).(0\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}) = 0 - 2 + 0 = -2$

$|\boldsymbol{n_1}| = \sqrt{1+4+4} = 3\, ; |\boldsymbol{n_2}| = \sqrt{0+1+0} = 1$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{-2}{3}\right) = 131.8^0$

In terms of acute angles, $\theta = 180 - 131.8^0 = 48.2^0$

c) $\boldsymbol{r}.(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = 1 \Rightarrow \boldsymbol{n_1} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$; $\boldsymbol{r}.(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) = 0 \Rightarrow \boldsymbol{n_2} = \boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}).(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) = 1-1+1 = 1$

$|\boldsymbol{n_1}| = \sqrt{1+1+1} = \sqrt{3}$; $|\boldsymbol{n_2}| = \sqrt{1+1+1} = \sqrt{3}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{1}{\sqrt{3}\times\sqrt{3}}\right) = cos^{-1}\left(\frac{1}{3}\right) = 70.5^0$

d) $2x+3y-5z = 1 \Rightarrow \boldsymbol{n_1} = 2\boldsymbol{i}+3\boldsymbol{j}-5\boldsymbol{k}$; $x-2y-z = 0 \Rightarrow \boldsymbol{n_2} = \boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (2\boldsymbol{i}+3\boldsymbol{j}-5\boldsymbol{k}).(\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}) = 2-6+5 = 1$

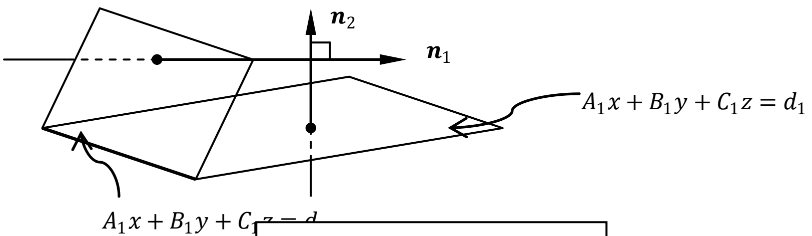
$|\boldsymbol{n_1}| = \sqrt{4+9+25} = \sqrt{38}$; $|\boldsymbol{n_2}| = \sqrt{1+4+1} = \sqrt{6}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{1}{\sqrt{38}\times\sqrt{6}}\right) = 86.2^0$

CONDITION FOR PERPENDICULARITY OF 2-PLANES

If two planes are perpendicular, then their normals are also perpendicular.

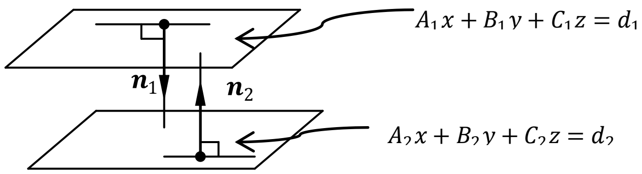
Consider two planes $A_1x+B_1y+C_1z = d_1$ and $A_2x+B_2y+C_2z = d_2$



$\Rightarrow \boldsymbol{n}_1.\boldsymbol{n}_2 = 0$ $\boxed{\Rightarrow A_1A_2 + B_1B_2 + C_1C_2 = 0}$

CONDITION FOR PARALLELLISM OF 2-PLANES

If two planes are parallel, then their *normals* are also parallel.



$\Rightarrow \boldsymbol{n}_1 = \lambda\,\boldsymbol{n}_2$

$\Rightarrow A_1\boldsymbol{i}+B_1\boldsymbol{j}+C_1\boldsymbol{k} = \lambda(A_2\boldsymbol{i}+B_2\boldsymbol{j}+C_2\boldsymbol{k})$

$\Rightarrow A_1 = \lambda A_2 \Rightarrow \lambda = \frac{A_1}{A_2}$

$\Rightarrow B_1 = \lambda B_2 \Rightarrow \lambda = \frac{B_1}{B_2}$

$\Rightarrow C_1 = \lambda C_2 \Rightarrow \lambda = \frac{C_1}{C_2}$

$$\boxed{\Rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \lambda}$$

EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE $ax+by+cz = d$

The required plane is generally given as $\boxed{ax+by+cz = k(constant)}$

, where$k = constant$, that can be obtained from any other condition satisfying the plane.

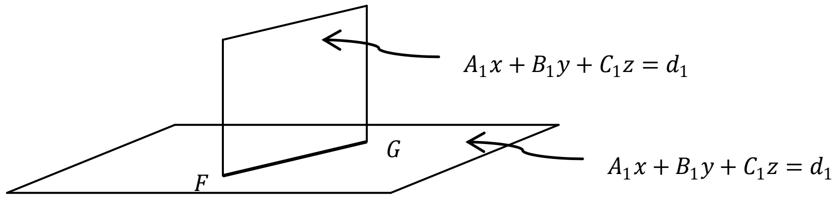
Example

1. *Find the equation of a plane passing through point* $(1\,,2\,,3)$ *and parallel to the plane*
$4x - 5y + 6z = 7$.
Solution
Required plane: $4x - 5y + 6z = k$
But $(1\,,2\,,3)$ lies on the plane
$\Rightarrow 4 - 10 + 18 = k \;\therefore k = 12$
The plane is $4x - 5y + 6z = 12$
2. *Find the equation of a plane parallel to plane* $2x + 3y - 4z = 1$ *and passing through* $(1\,,2\,,3)$. ***Answer***: $2x + 3y - 4z + 4 = 0$

INTERSECTION OF PLANES

Intersection of two planes:

Consider two planes $A_1x + B_1y + C_1z = d_1$ and $A_2x + B_2y + C_2z = d_2$



Two planes intersect in a straight *common line*. FG is the common line of intersection

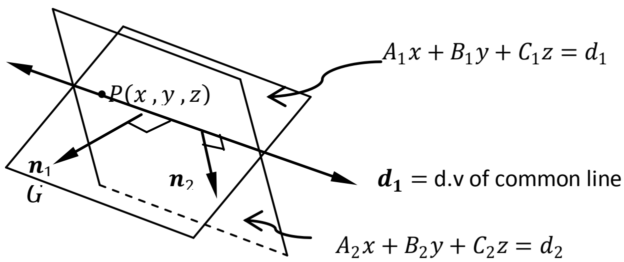
Equation of a plane containing the common line of intersection of two planes:

$$\boxed{(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0}$$

The equation is of the form $U + \lambda V = 0$ *where* $U = V = 0$ and λ = constant to be obtained from the other condition satisfying the plane.

Equation of the common line to 2-intersecting planes:

This is obtained as below.



Direction vector, $\boldsymbol{d}$ is perpendicular to $\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$

$$\therefore \boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$$

Using vector equation of a line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$
This line intersects at least one of the xy, xz or the $yz-plane$. Thus one point on this line has $a\ zero\ x-\ or\ y-\ or\ z-\ coordinate.\ i.e.$ Let x$=0$ or $y = 0$ or $z = 0$
Now solving for x and y if $z = 0$ in the equations of planes, yields the coordinate on the common line. Hence, equation of line FG is of the form;

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \lambda(\boldsymbol{n}_1 \times \boldsymbol{n}_2)\ for\ z = 0$$

Example

1. *Find the equation of a plane passing through the line of intersection of planes* $x + y + z = 6$ *and* $2x + 3y + 4z = -5$, *and is perpendicular to the plane* $4x + 5y - 5z = 8$. *Find also the equation of line of intersection of the planes* $x + y + z = 6$ *and* $2x + 3y + 4z = -5$.

Solution

$x + y + z = 6$ ………………………………………………………… (1)
$2x + 3y + 4z = -5$ …………………………………………… (2)
Required plane: $(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$
$\Rightarrow x(1 + 2\lambda) + y(1 + 3\lambda) + z(1 + 4\lambda) + (5\lambda - 6) = 0$
This plane is perpendicular to $4x + 5y - 5z = 8$ if:
$4(1 + 2\lambda) + 5(1 + 3\lambda) - 5(1 + 4\lambda) = 0$
$\Rightarrow 4 + 8\lambda + 5 + 15\lambda - 5 - 20 = 0\ or\ 3\lambda + 4 = 0\ \therefore \lambda = -\frac{4}{3}$
Equation of plane: $\Rightarrow x\left(1 - \frac{8}{3}\right) + y(1 - 4) + z\left(1 - \frac{16}{3}\right) + \left(-\frac{20}{3} - 6\right) = 0$
$-5x - 3y - 13z - 38 = 0\ or\ 5x + 3y + 13 + 38 = 0$
Now finding the equation of the common line:
$x + y + z = 6$, $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, and $2x + 3y + 4z = -5$, $\boldsymbol{n}_2 = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$
Direction vector of line: $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$

$$\boldsymbol{d} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

Finding a point on the required line:
Fixing $z = 0$
$\Rightarrow x + y + z = 6\ becomes\ x + y = 6$ ………………………… (3)
$\Rightarrow 2x + 3y + 4z = -5\ becomes\ 2x + 3y = -5$ ………… (4)
Solving eqns. (3) and (4) simultaneously,
$x = 23\ and\ y = -17$
$\Rightarrow (23, -17, 0)$ is the point on the required line.

$$\therefore \boldsymbol{r} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2. *Find the parametric equations of the line which is the intersection of* $x + 2y + 3z - 8 = 0$ *and* $3x + 3y + z + 3 = 0$. *Find the equation of the plane containing this line and perpendicular to the plane* $2x + 3y - 4z = -2$.

Solution

Fixing $z = 0$ and solving the equations of planes simultaneously,
$x + 2y + 3z - 8 = 0 \Rightarrow x + 2y = 8$ …………………………………… (1)

$3x + 3y + z + 3 = 0 \Rightarrow 3x + 3y = -3$ *or* $x - y = -1$ .... (2)
Solving eqns. (1) and (2) simultaneously,
$x = 2$ *and* $y = 3$
$\Rightarrow (2\,, 3\,, 0)$ is the point on the required line.
Direction vectors of the lines are respectively $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{2j} + 3\boldsymbol{k}\,, \boldsymbol{n}_2 = 3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$
Direction vector of line: $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$

$$\boldsymbol{d} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 3 & -3 & 1 \end{vmatrix} = 11\boldsymbol{i} + 8\boldsymbol{j} - 9\boldsymbol{k}$$

The required line is;

$$\therefore \boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 8 \\ -9 \end{pmatrix}$$

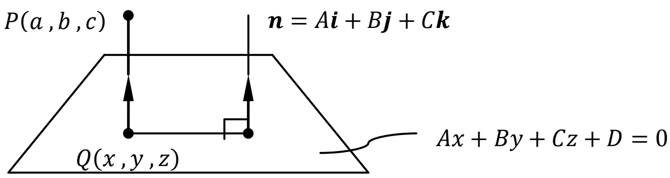
Required plane:
$(x + 2y + 3z - 8) + k(3x + 3y + z + 3) = 0$
$x(1 + 3k) + y(2 - 3k) + z(3 + k) + 3k - 8 = 0$
This plane is perpendicular to plane $2x + 3y - 4z = -2$ if:
$2(1 + 3k) + 3(2 - 3k) - 4(3 + k) = 0$ [$\because \boldsymbol{n}_1 \,.\, \boldsymbol{n}_2 = 0$]
$\Rightarrow 2 + 6k + 6 - 9k - 12 - 4k = 0 \;\therefore k = -\frac{4}{7}$
$\therefore$ Equation of plane is:
$x\left(1 - \frac{17}{7}\right) + y\left(2 + \frac{12}{7}\right) + z\left(3 - \frac{4}{7}\right) - \frac{12}{7} - 8 = 0$
$\therefore -5x + 26y + 17z + 68 = 0$

Task

1. *Find the equation of the plane which contains the line of intersection of the planes*
   $6x + 4y - 5z = 2$ *and* $x - 2y + 3z = 0$ and;
   a) *Perpendicular to the plane* $3x - 2y + z = 5$
   b) *Parallel to the line with direction cosines proportional to* $(1\,, 3\,, 2)$.
   Answer: $11x + 10y - 13z - 4 = 0, 2x - 20y + 29z + 2 = 0$
2. *Show that the lines* $\frac{1}{3}(x + 4) = \frac{1}{5}(y + 6) = -\frac{1}{2}(z - 1); 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ *are coplanar. Find also the coordinates of their point of intersection and the equation of the plane that contains their line.*
   Answer: $(2\,, 4\,, -3); 45x - 17y + 25z + 53 = 0$
3. *Show that the lines* $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{2}\,, 3x + 2y + z - 2 = 0 = x - 3y + 2z - 3$*are coplanar and find the equation of the plane in which they lie.*
   [$HINT$: $Required$: $(3x + 2y + z - 2) + \lambda(x - 3y + 2z - 3) = 0$]. This plane is parallel to the line if its normal is perpendicular to d.v. of the line. $i.e.\, 3(3 + \lambda) + (2 - 3\lambda) - 2(1 + 2\lambda) = 0 \;\therefore \lambda = \frac{9}{4} \Rightarrow (-5\,, -4\,, 7)\,, 21x - 19y + 22z - 125 = 0$

DISTANCE OF A POINT FROM A PLANE

Consider a point $P(a\,, b\,, c)$ and the plane $Ax + By + Cz + D = 0$.
Let $\overline{QP} = d$



Vector $\overrightarrow{QP}$ *is* // to $\boldsymbol{n}$

$\Rightarrow \overrightarrow{QP} = \lambda \boldsymbol{n}$

$But\ \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\Rightarrow a - x = \lambda A\ \ or\ x = a - \lambda A$

$\Rightarrow b - y = \lambda b\ \ or\ y = b - \lambda B$

$\Rightarrow c - z = \lambda A\ \ or\ z = c - \lambda C$

$\therefore Q(x, y, z) \equiv Q(a - \lambda A, b - \lambda B, c - \lambda C)$ lies on the plane $Ax + By + Cz + d = 0$

$\Rightarrow A(a - \lambda A) + B(b - \lambda B) + C(c - \lambda C) + d = 0$

$or\ Aa + Bb + Cc + d = \lambda(A^2 + B^2 + C^2)\ \therefore \lambda = \frac{Aa+Bb+Cc+d}{A^2+B^2+C^2}$ .... (1)

But $\overline{QP} = perpendicular\ distance = |\overrightarrow{QP}| = \lambda|\boldsymbol{n}|$

$\therefore \overline{QP} = \lambda \times \sqrt{A^2 + B^2 + C^2}$ ....
(2)

$subst. eqn. (1)\ in\ eqn. (2)$

$\Rightarrow Perpendicular\ distance = d = \frac{Aa+Bb+Cc+d}{A^2+B^2+C^2} \times \sqrt{A^2 + B^2 + C^2}$

$\therefore d = \frac{Aa+Bb+Cc+d}{\sqrt{A^2+B^2+C^2}}$ =Pperpendicular distance of point $P(a, b, c)$ from the plane $Ax + By + Cz + d = 0$

This distance can sometimes be negative, thus the absolute value (numerical value) is taken to ensure $+ve\ distance$.

Let $Ax + By + Cz + d = 0$ be the plane and $P(a, b, c)$ be a point. The distance d from the point is given by;

$$d = \left|\frac{Aa + Bb + Cc + d}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Rule to find perpendicular distance of a point from a plane with equation given in general form $Ax + By + Cz + d = 0\ (\boldsymbol{R.H.S} = 0)$

"Substitute the coordinates of the point in the equation of plane with $\boldsymbol{RHS} = 0$ and divide the result by $\sqrt{(coeff.of\ x)^2 + (coeff.of\ y)^2 + (coeff.of\ z)^2}$ "

Examples

1. *A plane passes through* $3-$*points* $A(0, 1, 3), B(1, 0, -4)$ *and* $C(1, 1, -1)$. *Find its equation and length of a perpendicular to it from* $(2, 4, -3)$.

Solution

Equation of plane is left for the student as an exercise. **Answer:** $\boldsymbol{4x - 3y + z = 0}$

Using the formula,

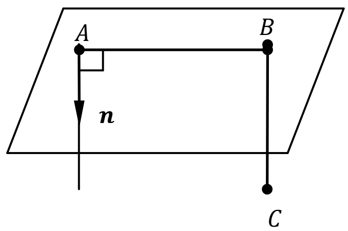
Perpendicular distance $= \frac{4(2)-3(4)-3}{\sqrt{4^2+(-3)^2+1^2}} = -\frac{7}{\sqrt{26}} = \frac{7}{\sqrt{26}} (numerically)$

2. *Show that the distance between the parallel planes* $2x - 2y + z + 3 = 0$ *and* $4x - 4y + 2z + 5 = 0$ *is* $\frac{1}{6}$.

Solution
Find any point on any plane.
Let $x = y = 0$, now finding z
For $2x - 2y + z + 3 = 0 \Rightarrow z = -3$
$\therefore (0\,,0\,,-3)$ lies on this plane.
Now perpendicular distance of $(0\,,0\,,-3)$ from $4x - 4y + 2z + 5 = 0$ is;
$d = \frac{4(0)-4(0)+2(-3)+5}{\sqrt{4^2+4^2+4}} = -\frac{1}{6} = \frac{1}{6}\,(Numerically)$

3. *Find the equations of the planes parallel to the plane* $x - 2y + 2z - 3 = 0$ *whose perpendicular distance from* $(1\,,2\,,3)$ *is* 1 *unit.*
Solution
Given plane: $x - 2y + 2z - 3 = 0$
Any plane parallel to it is: $x - 2y + 2z + k = 0$ .... (1)
Since perpendicular distance of eqn. (1) from $(1\,,2\,,3)$ $is = 1\,unit$
$\Rightarrow \pm\frac{1(1)-2(2)+2(3)+k}{\sqrt{1+4+4}} = 1$
[$\boldsymbol{use\ complete\ formul\ whenever\ perpendicular\ distance\ is\ given}$]
$\Rightarrow \pm\frac{3+k}{3} = 1\ or \pm(3+k) = 3\ \therefore k = 0\,,-6$
From eqn. (1), the required planes are;
$\therefore x - 2y + 2z = 0\,, x - 2y + 2z - 6 = 0$

4. *A plane contains the points* $A(-4\,,9\,,-9)$ *and* $B(5\,,-9\,,6)$ *and is perpendicular to the line which joins* $C(4\,,-6\,,k)$ *and B. Evaluates k and find the equation of the plane.*
Solution



$\overrightarrow{BC}\ //\boldsymbol{n}\ \Rightarrow \boldsymbol{n} = \lambda\overrightarrow{BC}$
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}4-5\\-6+9\\k-6\end{pmatrix} = \begin{pmatrix}-1\\3\\k-6\end{pmatrix}$
Also $\overrightarrow{AB}\ is\ perpendicular\ to\ \overrightarrow{BC}$
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}5+4\\-9-9\\6+9\end{pmatrix} = \begin{pmatrix}9\\-18\\15\end{pmatrix}$
$\Rightarrow \overrightarrow{AB}.\overrightarrow{BC} = 0\ \therefore -1(9) + 3(-18) + (k-6)(15) = 0$ .... $(\div 3)$
$\Rightarrow -51 + 5k = 0 \therefore k = \frac{51}{5}$
$Direction\ ratio\ of\ normal\ is -1 : 3 : k-6\,, k = \frac{51}{5}$
$\Rightarrow \boldsymbol{n}\ has\ direction\ ratio\ of -1 : 3 : \left(\frac{51}{5} - 6\right) = -5 : 15 : 21$
Using $\boldsymbol{r}.\boldsymbol{n} = d$
$\boldsymbol{r}.\begin{pmatrix}-5\\15\\21\end{pmatrix} = d$, this plane contains A and B.
For $A(-4\,,9\,,-9)\,, d = \begin{pmatrix}-4\\9\\-9\end{pmatrix}.\begin{pmatrix}-5\\15\\21\end{pmatrix} = 34$
$\therefore -5x + 15y + 21z - 34 = 0\ or\ \boldsymbol{r}.\begin{pmatrix}-5\\15\\21\end{pmatrix} = 34$

COMPLEX NUMBERS

Introduction:

A complex number is that can be expressed in the form:

$z = (Real\,part) + i(Imaginary\,part)\ .i.e.$

$z = a + ib\ ; a\,, b\ \in \mathbb{R}$

Examples

$z = 3 + 4i$, $z = 4i = 0 + 4i$, $z = 4 = 4 + 0i$ where $i = \sqrt{-1}$

Origin of symbol $i = \sqrt{-1}$

This originates from finding square root of negative numbers.

Consider

a) $\sqrt{4}$
b) $\sqrt{-4}$

Solution

a) $\sqrt{4} = \pm 2$

b) $\sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4}$ $\quad [By\ surds\colon \sqrt{ab} = \sqrt{a} \times \sqrt{b}]$

$= i \times \pm 2 = \pm 2i \quad [Let\ \sqrt{-1} = i\ (Mathematical\ notation)]$

Note:

There is no exact square root of a negative number that yields real numbers that are negative or positive.

Algebra of $i = \sqrt{-1}$

1) Simplify

a) i^2 b) i^3 c) i^5 d) i^4

Solution

a) $i^2 = [i]^2 = [\sqrt{-1}]^2 = \left[(-1)^{\frac{1}{2}}\right]^2 = -1$

$\therefore i^2 = -1$

b) $i^3 = i^2 \times i = -1 \times i = -i$

$\therefore i^3 = -i$

c) $i^5 = i^2 \times i^3 = -1 \times -i = i$

$\therefore i^5 = i$

d) $i^4 = (i^2)^2 = (-1)^2 = 1$

$\therefore i^4 = 1$

Note:

a. $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{-1 \times -1} = 1$

b. The result $i^2 = -1\,, i^3 = -i\ and\ i^4 = 1$ are important in the algebra of $i = \sqrt{-1}$.

2) Simplify

a) i^{20} b) i^{39} c) i^{40} d) i^{15}

Solution

a) $i^{20} = (i^2)^{10} = (-1)^{10} = 1$

$\therefore i^{20} = 1$

b) $i^{39} = (i^3)^{13} = (-i)^{13} = [-1 \times i]^{13} = -[i]^{13}$

$= -[i^4]^3 \times i = -[1]^3 \times i = -i$

$\therefore i^{39} = -i$

c) $i^{40} = [i^4]^{10} = [1]^{10} = 1$

$\therefore i^{40} = 1$

d) $i^{15} = [i^4]^3 \times i^3 = 1 \times -i = -i$
$\therefore i^{15} = -i$

Trial questions
Simplify
a) i^6 b) i^8 c) i^7 d) i^{17}
Answer:
a) $i^6 = (i^4)(i^2) = 1 \times -1 = -1$
b) $i^8 = [i^4]^2 = 1$
c) $i^7 = (i^4)(i^3) = -i$
d) $i^{17} = [i^4]^4 . i = i$

OPERATION IN COMPLEX NUMBERS
The following shall be considered:
i) Addition
ii) Subtraction
iii) Multiplication
iv) Division
Methods used
- Analytical method
- Graphical method

ADDITION OF COMPLEX NUMBERS
A. **Analytical method**

Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ *and* $z_3 = x_3 + iy_3$
Find: i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$
Solution
i. $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$
$= (x_1 + x_2) + i(y_1 + y_2)$
ii. $z_1 + z_2 + z_3 = (x_1 + iy_1) + (x_2 + iy_2) + (x_3 + iy_3)$
$= (x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)$
Deduction
When adding complex umbers, add the real parts and imaginary parts separately, and write the result in complex number form.
Examples
1) *Given that* $z_1 = 4 + i$, $z_2 = -3 - i$ *and* $z_3 = 3 + 2i$
Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$
Solution
i. $z_1 + z_2 = 4 + i + \ -3 - i$
$= (4 - 3) + i(1 - 1) = 1 + 0i = 1$
ii. $z_1 + z_2 + z_3 = 4 + i + \ -3 - i + 3 + 2i$
$= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i$
2) *Given that* $z_1 = 4$, $z_2 = 3i$ *and* $z_3 = 3 + 2i$
Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution
i. $z_1 + z_2 = 4 + 3i$

ii. $z_1 + z_2 + z_3 = 4 + 3i + 3 + 2i = 7 + 5i$

B. **Graphical method**

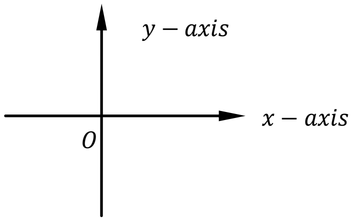
Before we look at this method, let us first look at Argand diagram.

Argand diagram

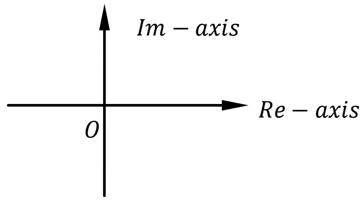
This is a graph having two axes.

i.e. Real (Re) axis as x − axis and Imaginary(Im) axis as y − axis

Cartesian graph



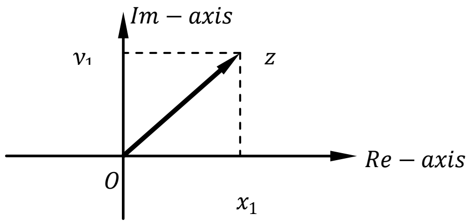
Argand diagram



Representation of a complex number on Argand diagram

Given that $z_1 = x_1 + iy_1$, z has a point (x_1 , y_1) in general form. Plotting (x_1 , y_1)

From the figure, a complex number is represented using an arrow from the origin to the point representing complex number.



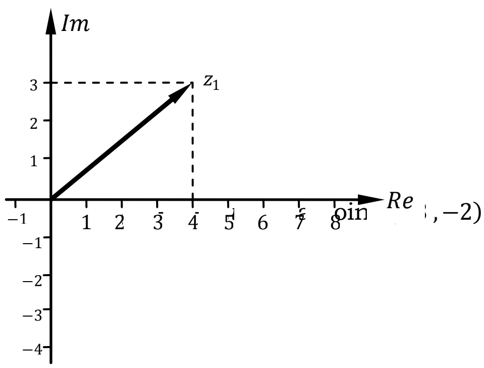
Example

1) *Given that* $z_1 = 4 + 3i$, $z_2 = -3 - 2i$, $z_3 = 5$ *and* $z_4 = 2i$

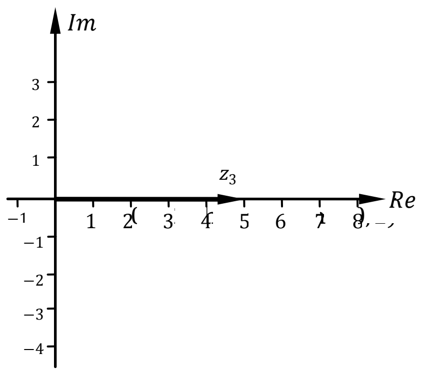
Represent the complex numbers on different Argand diagram.

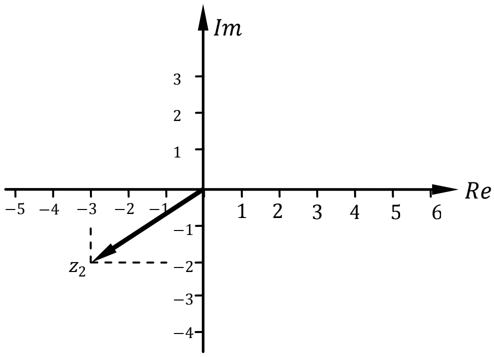
Solution

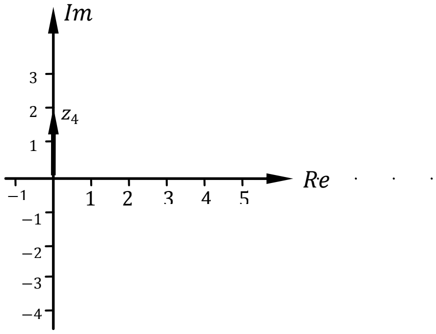
$z_1 = 4 + 3i$ has a point $(4 , 3)$



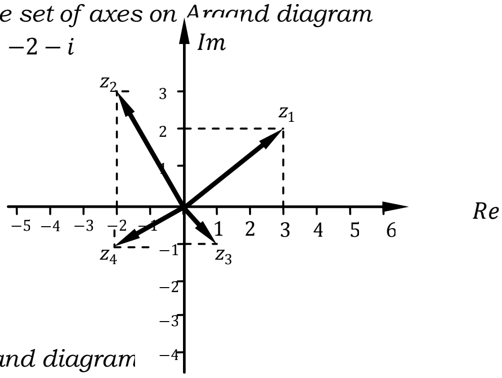
$z_3 = 5 + 0i$ has a point$(5 , 0)$







2) *Represent the complex numbers on the same set of axes on Argand diagram*
$z_1 = 3 + 2i$, $z_2 = -2 + 3i$, $z_3 = 1 - i$ *and* $z_4 = -2 - i$



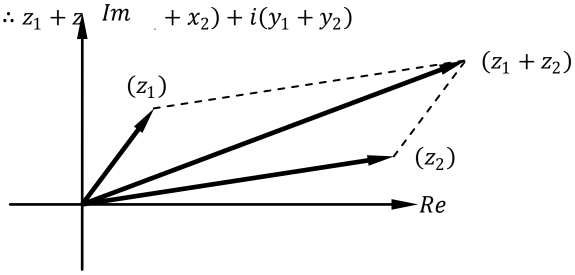
Trial questions

Represent the complex numbers on the Argand diagram
$z_1 = 3 - \frac{1}{2}i$, $z_2 = \frac{3}{2} + \frac{1}{2}i$, $z_3 = 4 - 3i$

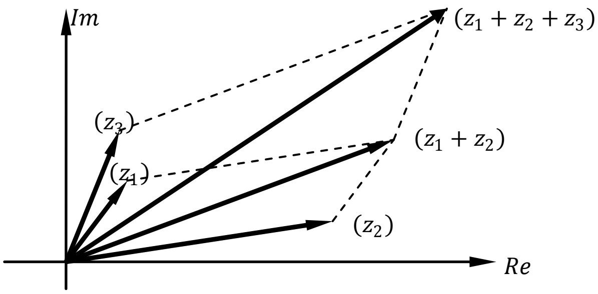
Now adding complex numbers by Graphical method

Two complex numbers are added using a parallelogram. Each complex number is represented on the Argand diagram, and the two complex numbers form the adjacent sides of a parallelogram. The addition result is given by the diagonal of a parallelogram.

Consider $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$
$\therefore z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$



Addition of complex numbers by graphical method



Three complex numbers are added as a continuous chain $.i.e. z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$. First add any two $.i.e.$ $z_1 + z_2$ to get the result and this result with z_3 give a diagonal of a parallelogram which is the required result.

Example

Given that $z_1 = 4 + i$, $z_2 = -2 + i$, $z_3 = -3 - 2i$ *and* $z_4 = 5 + i$

Find:

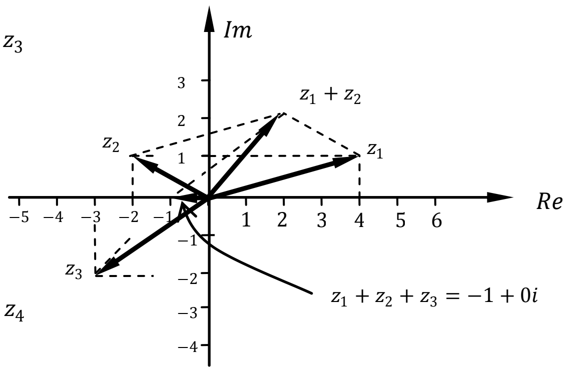
a) $z_1 + z_2 + z_3$

b) $z_2 + z_3 + z_4$

Solution

a) $z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$

Adding $z_1 + z_2$

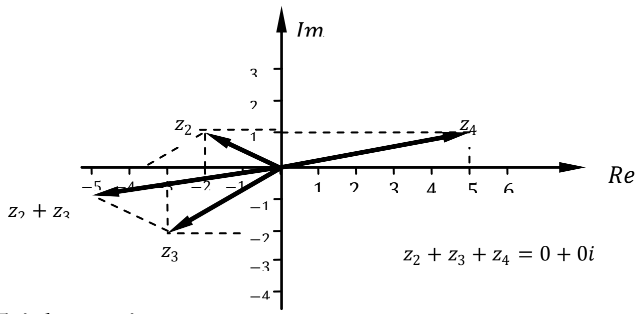


b) $z_2 + z_3 + z_4 = (z_2 + z_3) + z_4$

Adding $z_2 + z_3$

NOTICE:

The diagonal for a parallelogram from z_4 and $z_2 + z_3$ is just a straight line whose midpoint is $(0,0). i.e.$ Parallelogram cannot be drawn.



Trial questions:

1) *Given that* $z_1 = 3 + 4i$, $z_2 = -1 + 2i$ *and* $z_3 = -3i + 2$. *Represent the complex numbers clearly on the Argand diagram.*

   a) $z_1 + z_2$

   b) $z_2 + z_3$

   c) $z_1 + z_2 + z_3$

2) *Given that* $z_1 = 4 + 3i$, $z_2 = -4 - 3i$, $z_3 = 5$ *and* $z_4 = 2i$. *using Argand diagram find;*

   a) $z_1 + z_2$

   b) $z_2 + z_3$

   c) $z_1 + z_2 + z_3$

Note: $z_3 = 5 = 5 + 0i , z_4 = 2i - 0 + 2i$

SUBTRACTION OF COMPLEX NUMBERS

A. **Analytical method**

Given that $z_1 = x_1 + iy_1\ , z_2 = x_2 + iy_2$ *and* $z_3 = x_3 + iy_3$

Find: i) $z_1 - z_2$ *ii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 + z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$= (x_1 - x_2) + i(y_1 - y_2)$ $(collecting\ like\ terms)$

ii. $z_1 - z_2 - z_3 = (x_1 + iy_1) - (x_2 + iy_2) - (x_3 + iy_3)$

$= (x_1 - x_2 - x_3) + i(y_1 - y_2 - y_3)$ $(collecting\ like\ terms)$

Deduction

When subtracting complex umbers, subtract the real parts and imaginary parts separately, and write the result in complex number form.

Examples

1) *Given that* $z_1 = 3 + 2i, z_2 = 4 - 3i$ *and* $z_3 = 4 + 3i$

Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution

i. $z_1 + z_2 = 4 + i + \ -3 - i$

$= (4 - 3) + i(1 - 1) = 1 + 0i = 1$

ii. $z_1 + z_2 + z_3 = 4 + i + \ -3 - i + 3 + 2i$

$= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i$

2) *Given that* $z_1 = 3 + 2i$, $z_2 = 4 - 3i$ *and* $z_3 = 4 + 3i$

Find i) $z_1 - z_2$ *ii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 - z_2 = (3 + 2i) - (4 - 3i)$

$= (3 - 4) + i\big(2 - -3\big)$ $(collecting\ like\ terms) = -1 + 5i$

ii. $z_1 - z_2 - z_3 = (3 + 2i) - (4 - 3i) - (4 + 3i)$

$= (3 - 4 - 4) + i\big(2 - -3 - 3\big) = -5 + 2i$

Task

Given that $z_1 = 4 + 3i\ , z_2 = 3 - i$ *and* $z_3 = -4 - 3i$. *Find:*

a) $z_1 - z_2$ $\boldsymbol{Answer}: \mathbf{1 + 4}\boldsymbol{i}$

b) $z_2 - z_3$ $\boldsymbol{Answer}: \mathbf{7 + 2}\boldsymbol{i}$

c) $z_1 - z_2 - z_3$ $\boldsymbol{Answer}: \mathbf{5 + 7}\boldsymbol{i}$

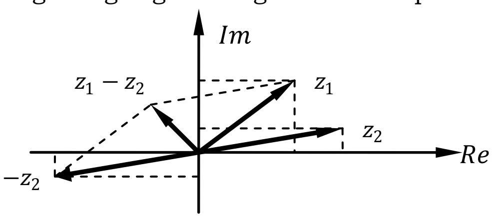
Graphical method

This is done using the Argand diagram just like in addition.

Consider two complex numbers $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$. To find $z_1 - z_2$, proceed as follows.

$z_1 - z_2 = z_1 + (-z_2)$ $[\boldsymbol{Note\ this}]$

Adding using Argand diagram the complex numbers z_1 *and* $-z_2$

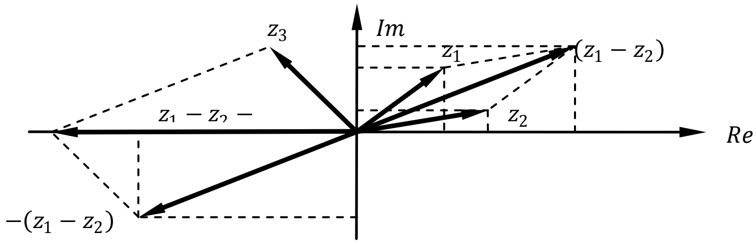


If $z_3 = x_3 + iy_3$, find $z_1 - z_2 - z_3$ if $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

Solution

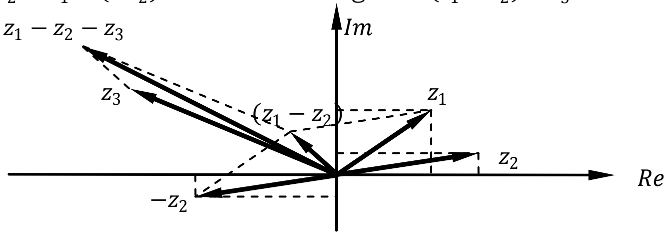
$z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$ *or* $z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$

Considering $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$



Considering second alternative $z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$

Dealing with $z_1 - z_2 = z_1 + (-z_2)$ and then dealing with $(z_1 - z_2) - z_3 = (z_1 - z_2) + (-z_3)$



Note:

Two complex numbers are subtracted by addition of the negative of the second complex number being subtracted from the first. *i.e.* $z_1 - z_2 = z_1 + (-z_2)$. A parallelogram diagonal gives the result, and the answer must be expressed in complex number form.

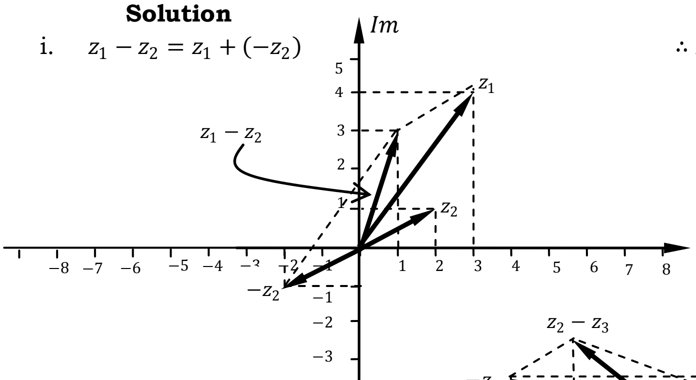
Example

Using the Argand diagram simplify given that $z_1 = 3 + 4i$, $z_2 = 2 + i$ *and* $z_3 = 6 - 2i$

i) $z_1 - z_2$ *ii)* $z_2 - z_3$ *iii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 - z_2 = z_1 + (-z_2)$ $\therefore z_1 - z_2 = 1 + 3i$

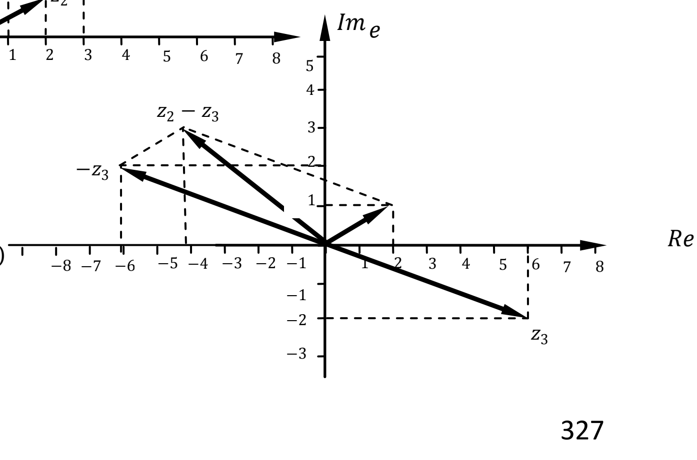


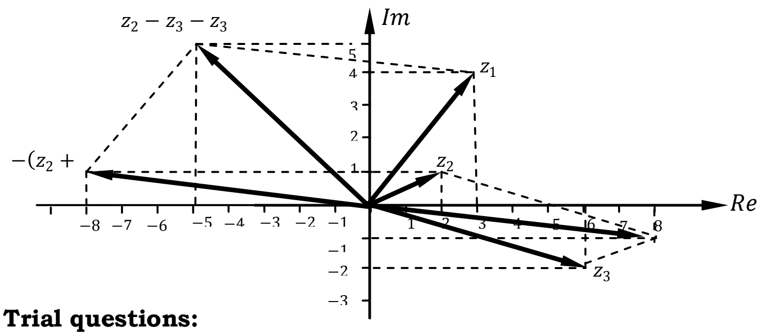
ii. $z_2 - z_3 = z_2 + (-z_3)$

$\therefore z_2 - z_3 = -4 - 3i$

iii. $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$

$\therefore z_1 - z_2 - z_3 = -5 + 5i$





Trial questions:

Given that $z_1 = 3 - 2i$, $z_2 = 4$ *and* $z_3 = 2i$. *Using the Argand diagram find;*

i) $z_1 - z_2$ *ii)* $z_2 - z_3$ *iii)* $z_1 + z_2 - z_3$ *iv)* $z_1 - z_2 - z_3$

Answers**:** $\boldsymbol{i)}\ \mathbf{-1-2}\boldsymbol{i}$ $(\boldsymbol{ii})\ \mathbf{4-2}\boldsymbol{i}$ $(\boldsymbol{iii})\ \mathbf{7-4}\boldsymbol{i}$ $(\boldsymbol{iv})\ \mathbf{-1-4}\boldsymbol{i}$

MULTIPLICATION OF COMPLEX NUMBERS

A. **Analytical method**

Given that $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

To find $z_1.z_2$ proceed as follows:

$z_1.z_2 = (x_1 + iy_1).(x_2 + iy_2)$

$= x_1.x_2 + ix_1.y_2 + iy_1.x_2 + i^2y_1.y_2$

$= (x_1.x_2 - y_1.y_2) + i(x_1.y_2 + y_1.x_2)$ $[\because i^2 = -1]$

Deduction

When multiplying complex numbers, open the brackets just like in expansion of binomials, simplify the expressions and write the result in complex number form.

Examples

Given that $z_1 = 3 + 2i$, $z_2 = -1 + 2i$ $z_3 = 4 - 3i$ *and* $z_4 = -3 - 4i$. *Find:*

i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$ *iv)* $z_2.z_3.z_4$

Solution

i. $z_1.z_2 = (3 + 2i).(-1 + 2i)$

$= 3(-1 + 2i) + 2i(-1 + 2i)$

$= -3 + 6i - 2i + 4i^2 = -3 - 4 + 4i$ $[\because i^2 = -1]$

$= -7 + 4i$

ii. $z_2.z_3 = (-1 + 2i).(4 - 3i)$

$= -1(4 - 3i) + 2i(4 - 3i)$

$= -4 + 3i + 8i - 6i^2 = -4 + 6 + 11i$ $[\because i^2 = -1]$

$= 2 + 11i$

iii. $z_3.z_4 = (4 - 3i).(-3 - 4i)$

$= 4(-3 - 4i) - 3i(-3 - 4i)$

$= -12 - 16i + 9i + 12i^2 = -12 - 12 - 7i$ $[\because i^2 = -1]$

$= -24 - 7i$

iv. $z_1.z_2.z_3 = [z_1.z_2].z_3$. But from (i) above, $z_1.z_2 = -7 + 14i$

$= [-7 + 14i].(4 - 3i)$

$= -7(4 - 3i) + 4i(4 - 3i)$

$= -28 + 21i + 16i - 12i^2$

$= -28 + 12 + 37i$

$= -16 + 37i$

v. $z_2.z_3.z_4 = [z_2.z_3].z_4$. But $z_2.z_3 = 2 + 11i$

$$= [2 + 11i].(-3 - 4i)$$
$$= 2(-3 - 4i) + 11i(-3 - 4i)$$
$$= -6 - 8i - 33i - 44i^2$$
$$= -6 + 44 - 41i$$
$$= 38 - 41i$$

Note:

Graphical multiplication of complex numbers requires Demoivre's theorem which has not been tackled. Graphical multiplication of complex numbers shall be handled later.

Trial questions

1. *Given that* $z_1 = 2 + i$, $z_2 = -2 + i$ *and* $z_3 = 3 - 2i$. *Find:*
   i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$
2. *Given that* $z_1 = 1 + i\sqrt{3}$, $z_2 = \sqrt{3} + i$ *and* $z_3 = 1 + \sqrt{2}i$. *Find:*
   i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$

Answers:

DIVISION OF COMPLEX NUMBERS

Before we look at this, let us first look at a conjugate of a complex number.

Conjugate of a complex number

If $z = a + bi$, then conjugate of z is $\bar{z} = a - ib$

If $z = a - bi$, then conjugate of z is $\bar{z} = a + ib$

Example

Find $\bar{z}$ *if* *i)* $z = 2 + 3i$

ii) $z = -3 - 2i$

Solution

i) $z = 2 + 3i;\ \bar{z} = 2 - 3i$

ii) $z = -3 - 2i;\ \bar{z} = -3 + 2i$

Thus when two complex numbers are to be divided, multiply the numerator and denominator of the quotient being divided by the conjugate of the denominator. In simple terms *rationalize* the denominator.

Consider $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \ldots\ldots\ldots\ldots\ldots\ldots \times \frac{(x_2-iy_2)}{(x_2-iy_2)}$$
$$= \frac{(x_1+iy_1).(x_2-iy_2)}{(x_2+iy_2).(x_2-iy_2)}$$
$$= \frac{x_1.x_2+ix_1.y_2+iy_1.x_2+i^2y_1.y_2}{x_1.x_2-ix_1.y_2+iy_1.x_2-i^2y_1.y_2}$$
$$= \frac{(x_1.x_2-y_1.y_2)+i(x_1.y_2+y_1.x_2)}{x_2{}^2+y_2{}^2}$$

Note:

$\frac{z_1}{z_2}$, has now a *real* denominator

Example

1. Given that $z_1 = 2 + 3i$, $z_2 = 4 - 3i$ *and* $z_3 = 1 + i$

   Find i) $\frac{z_1}{z_2}$ ii) $\frac{z_2}{z_3}$

Solution

i. $\frac{z_1}{z_2} = \frac{(2+3i)}{(4-3i)} \ldots\ldots\ldots\ldots\ldots\ldots \times \frac{(4+3i)}{(4+3i)}$

$= \frac{(2+3i).(4+3i)}{(4-3i).(4+3i)} = \frac{8+6i+12i+9i^2}{16+12i-12i-9i^2}$

$= \frac{8-9+18i}{16+9} = \frac{-1+18i}{25}$

$\therefore \frac{z_1}{z_2} = -\frac{1}{25} + \frac{18}{25}i$

ii. $\frac{z_2}{z_3} = \frac{(4-3i)}{(1+i)}$ .... $\times \frac{(1-i)}{(1-i)}$

$= \frac{(4-3i).(1-i)}{(1+i).(1-i)} = \frac{4-4i-3i+3i^2}{1+i-i-i^2}$

$= \frac{4-3-7i}{1+1} = \frac{1-7i}{2}$

$\therefore \frac{z_2}{z_3} = \frac{1}{2} - \frac{7}{2}i$

2. *Given that* $z_1 = 1 + i\sqrt{3}$, $z_2 = 1 - 2\sqrt{3}i$ *and* $z_3 = -1 + \sqrt{3}i$. *Find:*

i) $\frac{z_1}{z_2}$ *ii)* $\frac{z_2}{z_3}$ *iii)* $\frac{z_3}{z_1z_2}$ *iv)* $\frac{z_2z_3}{z_1}$

Solution

i. $\frac{z_1}{z_2} = \frac{(1+i\sqrt{3})}{(1-2\sqrt{3}i)}$ .... $\times \frac{(1+2\sqrt{3}i)}{(1+2\sqrt{3}i)}$

$= \frac{(1+i\sqrt{3}).(1+2\sqrt{3}i)}{(1-2\sqrt{3}i).(1+2\sqrt{3}i)} = \frac{1+2\sqrt{3}i+\sqrt{3}i+2\sqrt{9}i^2}{1+2\sqrt{3}i-2\sqrt{3}i-4\sqrt{9}i^2}$

$= \frac{1+3\sqrt{3}i-2(3)}{1+4(3)} = \frac{1-6+3\sqrt{3}i}{1+12}$

$\therefore \frac{z_1}{z_2} = -\frac{5}{13} + \frac{3\sqrt{3}}{13}i$

ii. $\frac{z_2}{z_3} = \frac{(1-2\sqrt{3}i)}{(-1+\sqrt{3}i)}$ .... $\times \frac{(-1-\sqrt{3}i)}{(-1-\sqrt{3}i)}$

$= \frac{(1-2\sqrt{3}i).(-1-\sqrt{3}i)}{(-1+\sqrt{3}i).(-1-\sqrt{3}i)} = \frac{-1-\sqrt{3}i+2\sqrt{3}i+2\sqrt{9}i^2}{1+\sqrt{3}i-\sqrt{3}i-\sqrt{9}i^2}$

$= \frac{-1-2(3)i+\sqrt{3}i}{1+3} = \frac{-7+\sqrt{3}i}{4}$

$\therefore \frac{z_2}{z_3} = -\frac{7}{4} + \frac{\sqrt{3}}{4}i$

iii. $\frac{z_3}{z_1z_2} = \frac{(-1+\sqrt{3}i)}{(1+i\sqrt{3}).(1-2\sqrt{3}i)} = \frac{(-1+\sqrt{3}i)}{1+i\sqrt{3}-2\sqrt{3}i-2\sqrt{9}i^2}$

$= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)}$ .Now rationalizing the denominator

$= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)}$ .... $\times \frac{(7+\sqrt{3}i)}{(7+\sqrt{3}i)}$

$= \frac{(-1+\sqrt{3}i).(7+\sqrt{3}i)}{(7-\sqrt{3}i).(7+\sqrt{3}i)} = \frac{-7-\sqrt{3}i+7\sqrt{3}i+\sqrt{9}i^2}{49+7\sqrt{3}i-7\sqrt{3}i-\sqrt{9}i^2}$

$= \frac{-7-3+6\sqrt{3}i}{49+3} = \frac{-10+6\sqrt{3}i}{52}$

$= -\frac{10}{52} + \frac{6\sqrt{3}}{52}i = -\frac{5}{26} + \frac{3\sqrt{3}}{26}i$

$\therefore \frac{z_3}{z_1z_2} = -\frac{5}{26} + \frac{3\sqrt{3}}{26}i$

vi. $\frac{z_2z_3}{z_1} = \frac{(1-2\sqrt{3}i).(-1+\sqrt{3}i)}{(1+i\sqrt{3})} = \frac{-1+\sqrt{3}i+2\sqrt{3}i-2\sqrt{9}i^2}{(1+i\sqrt{3})}$

$= \frac{-1+2(3)+3\sqrt{3}i}{(1+i\sqrt{3})} = \frac{5+3\sqrt{3}i}{(1+i\sqrt{3})}$

Now rationalizing the denominator

$\frac{z_2z_3}{z_1} = \frac{(5+3\sqrt{3}i).(1-i\sqrt{3})}{(1+i\sqrt{3}).(1-i\sqrt{3})} = \frac{5-5\sqrt{3}i+3\sqrt{3}i-3\sqrt{9}i^2}{1-i\sqrt{3}+i\sqrt{3}-\sqrt{9}i^2}$

$= \frac{5+3(3)-i2\sqrt{3}}{1+3} = \frac{14-i2\sqrt{3}}{4}$

$= \frac{14}{4} - \frac{2\sqrt{3}}{4} i = \frac{7}{2} - \frac{\sqrt{3}}{2} i \quad \therefore \frac{z_2 z_3}{z_1} = \frac{7}{2} - \frac{\sqrt{3}}{2} i$

3. *Given that* $z_1 = 1 + i$, $z_2 = 2 - i$ *and* $z_3 = 3 + 2i$. *Find:*

i) $\frac{z_1+z_2}{z_3}$ *ii)* $\frac{{z_1}^2}{{z_2}^2 . z_3}$ *iii)* $\frac{z_2+z_3}{z_1+z_2}$ *iv)* $\frac{z_1+z_2+z_3}{z_2-z_3}$ *v)* $\frac{{z_1}^3 . z_2}{z_1+z_3}$

Solution

i. $\frac{z_1+z_2}{z_3}$; $z_1 + z_2 = 1 + i + 2 - i = 3$

$\therefore \frac{z_1+z_2}{z_3} = \frac{3}{3+2i}$; rationalizing the denominator

$\frac{z_1+z_2}{z_3} = \frac{3}{3+2i} = \frac{3(3-2i)}{(3+2i).(3-2i)}$

$= \frac{9-6i}{9-6i+6i-4i^2} = \frac{9-6i}{9+4} \quad \therefore \frac{z_1+z_2}{z_3} = \frac{9}{13} - \frac{6}{13} i$

ii. $\frac{{z_1}^2}{{z_2}^2 . z_3}$;

${z_1}^2 = z_1 . z_1 = (1 + i).(1 + i) = 1 + i + i + i^2 = 2i$

${z_2}^2 = z_2 . z_2 = (2 - i).(2 - i) = 4 - 2i - 2i + i^2 = 5 - 4i$

$\therefore {z_2}^2 . z_3 = (5 - 4i).(3 + 2i) = 15 + 10i - 12i - 8i^2 = 23 - 2i$

$\therefore \frac{{z_1}^2}{{z_2}^2 . z_3} = \frac{2i}{(23-2i)} = \frac{2i(23+2i)}{(23-2i).(23+2i)}$

$= \frac{46i+4i^2}{529+46i-46i-4i^2} = \frac{-4+46i}{529+4} = \frac{-4+46i}{533}$

$\therefore \frac{{z_1}^2}{{z_2}^2 . z_3} = -\frac{4}{533} + \frac{46}{533} i$

iii. $\frac{z_2+z_3}{z_1+z_2} = \frac{2-i+3+2i}{1+i+2-i} = \frac{5+i}{3} \quad \therefore \frac{z_2+z_3}{z_1+z_2} = \frac{5}{3} + \frac{1}{3} i$

iv. $\frac{z_1+z_2+z_3}{z_2-z_3} = \frac{1+i+2-i+3+2i}{2-i-(3+2i)} = \frac{6+2i}{-1-3i}$; rationalizing denominator

$= \frac{(6+2i).(-1+3i)}{(-1-3i).(-1+3i)} = \frac{-6+18i-2i+6i^2}{1-3i+3i-9i^2} = \frac{-12+16i}{1+9} = -\frac{12}{10} + \frac{16}{10} i$

$\therefore \frac{z_1+z_2+z_3}{z_2-z_3} = -\frac{6}{5} + \frac{8}{5} i$

v. $\frac{{z_1}^3 . z_2}{z_1+z_3}$; ${z_1}^3 = {z_1}^2 . z_1 = 2i.(1 + i) = -2 + 2i$

$\therefore {z_1}^3 . z_2 = (-2 - 2i).(2 - i) = -4 + 2i - 4i + 2i^2 = -2 - 2i$

$z_1 + z_3 = 1 + i + 3 + 2i = 4 + 3i$

$\therefore \frac{{z_1}^3 . z_2}{z_1+z_3} = \frac{(-2-2i)}{(4+3i)} = \frac{(-2-2i).(4-3i)}{(4+3i).(4-3i)}$

$= \frac{-8+6i-8i+6i^2}{16-12i-12i-9i^2} = \frac{-14-2i}{16+9} = -\frac{14}{25} - \frac{2}{9} i$

$\therefore \frac{{z_1}^3 . z_2}{z_1+z_3} = -\frac{14}{25} - \frac{2}{9} i$

Trial questions

1) *Given that* $z = \sqrt{3} + i$; *find i)* z^2 *ii)* $\frac{1}{z}$

2) *Given that* $z = 3 + 4i$, *find the value of the expression* $z + \frac{25}{z}$

$\left[\boldsymbol{Hint}: 1^{st}\ find \frac{25}{z}\ and\ add\ the\ result\ to\ z\right]$

3) *The total impedance z in an electric circuit with branches* z_1 *and* z_2 *is given by;* $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. *Given that* $z_1 = 3 + i4$ *and* $z_2 = 5 + i5$, *where* $i = \sqrt{-1}$, *calculate the total impedance in z in the form* $a + ib$.

$\left[\begin{array}{c}\boldsymbol{Hint}: \frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1+z_2}{z_1 . z_2}. Now\ proceed\ to\ find\ z_1 + z_2\ and\ z_1 . z_2\ then\ z. \\ Or\ find\ \frac{1}{z_1} and \frac{1}{z_2}\ by\ rationalisation\ and\ then\ z\end{array}\right]$

4) *Given that* $z_1 = 3 - i$, $z_2 = 3 + i$. *find*

a) $\frac{{z_1}^2 . {z_2}^3}{z_1 + z_2}$ b) $\frac{z_2}{z_1}$ c) $\frac{z_2 . z_1}{z_2 - z_1}$

Note:

Graphical division of complex numbers requires Demoivre's theorem. This shall be tackled later.

DIFFERENT FORMS OF COMPLEX NUMBERS

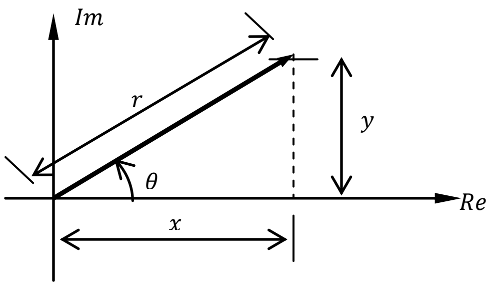
A complex number can mainly be expressed in three forms:

- Standard form . $i.e.z = x + iy$
- Polar form . $i.e.z = r(cos\theta + isin\theta)$
- Exponential form . $i.e.z = re^{\pm i\theta}$

However the most commonly examined at A level are: standard and polar form. Before we look at these in detail, let us first look at modulus and argument of a complex number.

MODULUS AND ARGUMENT OF A COMPLEX NUMBER

Consider the complex number $z = x + iy$ in standard form. Representing z on the Argand diagram



Argument , θ

Argument of a complex number is the angle the complex number makes with *positive Real axis*

Denotion

Argument of z is denoted as $arg(z)$

From the triangle above,

$tan\theta = \frac{y}{x}$ $\therefore \theta = tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\therefore arg(z) = arg(x + iy) = \theta = tan^{-1}\left(\frac{y}{x}\right)}$$

Note:

i. When finding $tan^{-1}\left(\frac{y}{x}\right)$, there is no i on y from $z = x + iy$ which is the commonest mistake students make . $i.e.tan^{-1}\left(\frac{iy}{x}\right)$, this is not correct.

ii. The range of θ is $- 180^0 < \theta \leq 180^0$ or $-\pi < \pi \leq \pi$ in radians . $i.e.$ *principal argument*

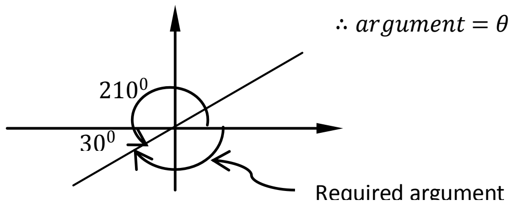
iii. When θ is being measured, it is done with respect to the $+ve$ *real axis* and θ is $+ve$ if measurement is in *anticlockwise direction* where as $-ve$ if measured in a *clockwise direction*

iv. *Measurement of* $\theta = arg(z)$ requires the knowledge of trigonometry about tangents of angles in the four quadrants.

If θ is not in the interval $-\pi < \pi \leq \pi$ then expressing the angle in terms of acute angles must be done to bring the angle in the required interval.

$e.g.$ If $\theta = 210^0$ then $tan210^0 = +tan30^0$

$\therefore argument = \theta = -150^0 = -\frac{5}{6}\pi$



Modulus /Magnitude /length of a complex number

From the right angled triangle above, applying Pythagoras Theorem,

$x^2 + y^2 = r^2 \therefore r = \sqrt{x^2 + y^2}$

Denotion

Modulus of z is denoted as $|z| = r$

$$\boxed{\therefore |z| = r = \sqrt{x^2 + y^2}}$$

Note:

In the formula, there is no use of i *on* y $.i.e.\therefore |z| = r \neq \sqrt{x^2 + (iy)^2}$

Examples

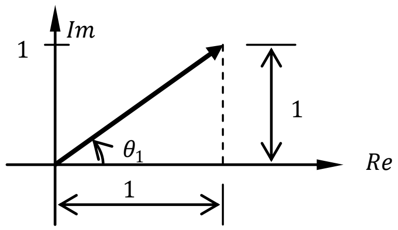
1. *Given that* $z_1 = 1 + i$, $z_2 = -\sqrt{3} + i$, $z_3 = -5 - 12i$ *and* $z_4 = \sqrt{3} - i$

Find i) $arg(z_1)$ ii) $arg(z_2)$ iii) $arg(z_3)$ iv) $arg(z_4)$

v) $|z_1|$ vi) $|z_2|$ vii) $|z_3|$ viii) $|z_4|$

Solution

i. $z_1 = 1 + i$



From $|z| = \sqrt{x^2 + y^2}$

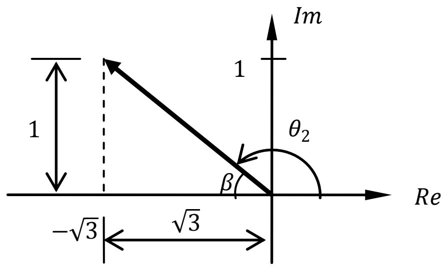
$\therefore |z_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\, units$

From $arg(z) = tan^{-1}\left(\frac{y}{x}\right)$

$\therefore arg(z_1) = tan^{-1}\left(\frac{1}{1}\right) = 45^0 = \frac{\pi}{4}$

$\therefore |z_1| = \sqrt{2}\, units , arg(z_1) = 45^0 = \frac{\pi}{4}$

ii. $z_2 = -\sqrt{3} + i$



From $|z| = \sqrt{x^2 + y^2}$

$\therefore |z_2| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2\, units$

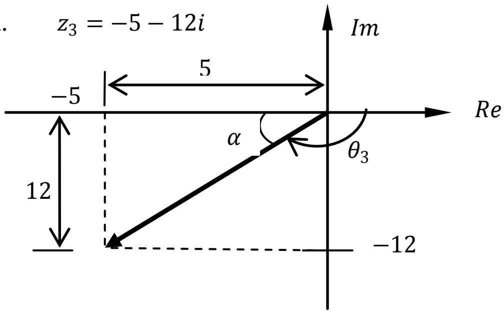
$arg(z_2) = 180^0 - tan^{-1}\left(\frac{y}{x}\right)$

From Δ, $\beta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0$

$\therefore arg(z_2) = \theta_2 = 180^0 - 30^0 = 150^0 = \frac{5\pi}{6}$

$\therefore |z_2| = 2\, units , arg(z_2) = 150^0 = \frac{\pi}{6}$

iii. $z_3 = -5 - 12i$



From $|z| = \sqrt{x^2 + y^2}$

$\therefore |z_2| = \sqrt{(5)^2 + (12)^2} = 13\, units$

$arg(z_3) = \theta_3 = -(180^0 - \alpha)$

From Δ, $\alpha = tan^{-1}\left(\frac{12}{5}\right) = 67.4^0$

$\therefore arg(z_3) = \theta_3 = -(180^0 - 67.4^0\ ) = -112.6^0$

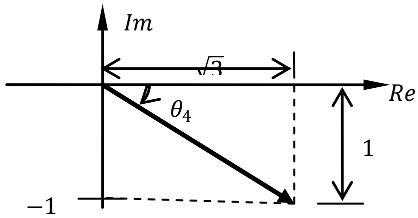
$\therefore |z_3| = 13\, units\, , arg(z_3) = -112.6^0$

iv. $z_4 = \sqrt{3} - i$, From $|z| = \sqrt{x^2 + y^2}$

$\therefore |z_4| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2\, units$

$arg(z_4) = \theta_4 = -\,tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0 = -\frac{\pi}{6}$

$\therefore |z_2| = 2\, units\, , arg(z_2) = 30^0 = -\frac{\pi}{6}$



2. Given that $z_1 = 1 + i\, , z_2 = -1 - i, z_3 = 2 + 3i.$ *Find the modulus and argument of:*

i) $z_1 . z_2$ *ii)* $\frac{z_2 . z_3}{z_1}$ *iii)* $\frac{{z_1}^2}{z_2 . z_3}$ *iv)* $\frac{{z_2}^3}{z_1 . z_3}$ *v)* ${z_1}^2$

Solution

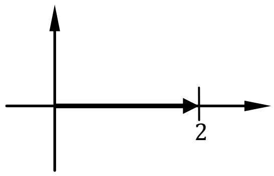
i. $z_1 . z_2 = (1 + i).(1 + i) = 1 - i + i - i^2$

$= 2 + 0i$

$\therefore |z_1 . z_2| = \sqrt{(2)^2 + 0^2} = 2\, units$

$arg(z_1 . z_2) = tan^{-1}\left(\frac{0}{2}\right) = 0^0$

$\therefore |z_1 . z_2| = 2\, units\, ;\ arg(z_1 . z_2) =\ 0^0$



ii. $\frac{z_2 . z_3}{z_1} = \frac{(1-i).(2+3i)}{(1+i)} = \frac{2+3i-2i-i^2.3}{1+i}$

$= \frac{2+3+i}{1+i} = \frac{5+i}{1+i}$, rationalizing denominator

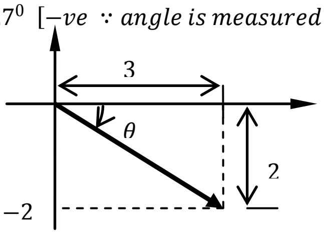
$\frac{(5+i).(1-i)}{(1+i).(1-i)} = \frac{5-5i+i-i^2}{1+i-i-i^2} = \frac{5+1-4i}{1+1} = \frac{6-4i}{2} = 3 - 2i$

$\therefore \frac{z_1 . z_3}{z_1} = 3 - 2i$

$\Rightarrow \left|\frac{z_1 . z_3}{z_1}\right| = \sqrt{3^2 + 2^2} - \sqrt{13}\, units$

From the figure, $tan\theta = \frac{2}{3}\ \therefore \theta = 33.7^0$

$\therefore arg\left(\frac{z_1 . z_3}{z_1}\right) = -33.7^0$ $[-ve\ \because angle\ is\ measured\ in\ clockwise\ direction]$



iii. $\frac{{z_1}^2}{z_2 . z_3} = \frac{(1+i)^2}{(1-i).(2+3i)} = \frac{1+i+i-i^2}{2+3i-2i-i^2.3} = \frac{2i}{5+i}$

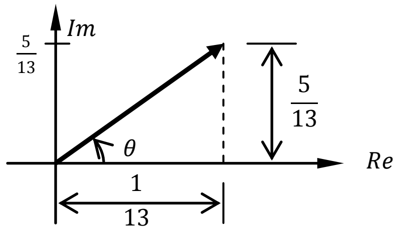
$\frac{2i}{5+i} = \frac{2i\,.(5-i)}{(5+i).(5-i)} = \frac{10i-2i^2}{25-5i+5i-i^2} = \frac{2+10i}{25+1} = \frac{2+10i}{26}$

$\therefore \frac{z_1^2}{z_2.z_3} = \frac{1}{13} + \frac{5}{13}i$

$\Rightarrow \left|\frac{z_1^2}{z_2.z_3}\right| = \sqrt{\left(\frac{1}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2 [1 + 25]} = \frac{\sqrt{26}}{13} units$

$arg\left(\frac{z_1^2}{z_2.z_3}\right) = tan^{-1}\left(\frac{5/13}{1/13}\right) = tan^{-1}(5) = 78.7^0$

$\therefore \left|\frac{z_1^2}{z_2.z_3}\right| = \frac{\sqrt{26}}{13} units\,;\ arg\left(\frac{z_1^2}{z_2.z_3}\right) = 78.7^0$



iv. $\frac{z_2^3}{z_1.z_3} = \frac{(1-i)^3}{(1+i).(2+3i)} = \frac{(1-i)^2.(1-i)}{2+3i+2i+3i^2} = \frac{(1-2i+i^2).(1-i)}{2-3+5i}$

$= \frac{(-2i).(1-i)}{-1+5i} = \frac{(-2i+2i^2)}{(-1+5i)} = \frac{(-2-2i)}{(-1+5i)}$; rationalizing denominator

$\frac{(-2-2i)}{(-1+5i)} = \frac{(-2-2i).(-1-5i)}{(-1+5i).(-1-5i)} = \frac{2+10i+2i+10i^2}{1+5i-5i-25i^2}$

$\therefore \frac{z_2^3}{z_1.z_3} = \frac{2-10+12i}{1+25} = \frac{-8+12i}{26} = -\frac{4}{13} + \frac{6}{13}i$

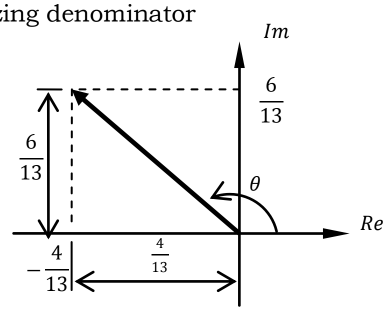
$\therefore \left|\frac{z_2^3}{z_1.z_3}\right| = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2 [16 + 36]}$

$= \frac{\sqrt{52}}{13} = \frac{\sqrt{4\times13}}{13} = \frac{2}{\sqrt{13}} units$

$arg\left(\frac{z_2^3}{z_1.z_3}\right) = 180^0 - tan^{-1}\left(\frac{\frac{6}{13}}{\frac{4}{13}}\right)$

$= 180^0 - tan^{-1}\left(\frac{6}{4}\right) = 180^0 - 56.3^0 = 123.7^0$

$\therefore \left|\frac{z_2^3}{z_1.z_3}\right| = \frac{2}{\sqrt{13}} units\,;\ \ arg\left(\frac{z_2^3}{z_1.z_3}\right) = 123.7^0$



STANDARD FORM OF A COMPLEX NUMBER

A complex number of the form $z = x + iy\ where\ x, y \epsilon \mathbb{R}$ is said to be in standard form.

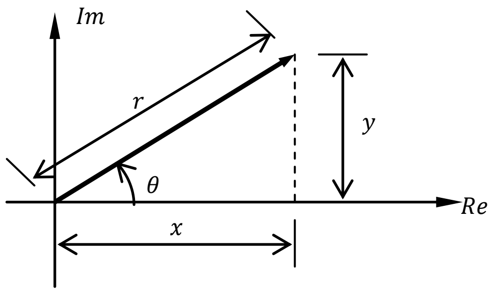
$.e.g. z = 2 + 3i\,, z = 4 - 3i$

POLAR/MODULUS-ARGUMENT FORM OF A COMPLEX NUMBER

This originates from the standard form. A complex number in the form $z = r(cos\theta + isin\theta)$ is in polar form.

Derivation

Consider $z = x + iy$



$|z| = \sqrt{x^2 + y^2}$

$arg(z) = tan^{-1}\left(\frac{y}{x}\right)$

Now $cos\theta = \frac{x}{r} \therefore x = rcos\theta$

$sin\theta = \frac{y}{r} \therefore y = rsin\theta$

From $z = x + iy$

$\Rightarrow z = rcos\theta + rsin\theta$

$\therefore z = r(cos\theta + isin\theta)$

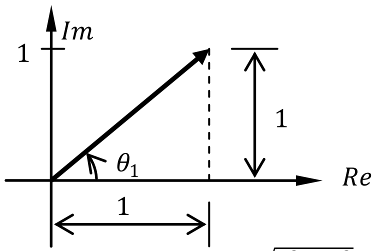
Note:
Expressing a complex number in polar form requires the knowledge of finding modulus and argument of a complex number.

Examples

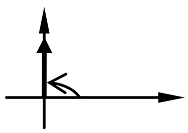
1) *Given that* $z_1 = 1 + i$, $z_2 = -3i$, $z_3 = -\sqrt{3} - i$ *and* $z_4 = \sqrt{3} - i$. *Express the complex numbers in polar form.*

Solution

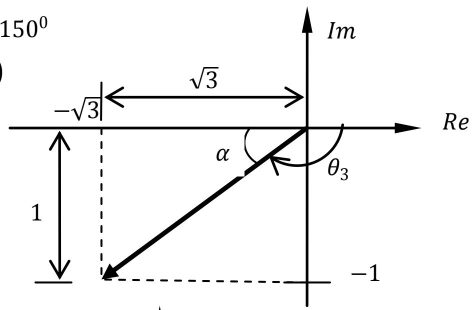
$z_1 = 1 + i$; $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}\, units$; $arg(z_1) = tan^{-1}(1) = 45^0$
$\therefore z_1 = \sqrt{2}\big(cos(45^0) + isin(45^0)\big)$



$z_2 = -3i = 0 + 3i$; $|z_2| = \sqrt{0^2 + 3^2} = 3\, units$; $arg(z_2) = 90^0$
$\therefore z_2 = 3\big(cos(90^0) + isin(90^0)\big)$

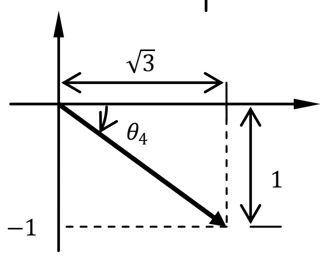


$z_3 = -\sqrt{3} - i$; $|z_1| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2\, units$;
$arg(z_3) = -\left[180^0 - tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -150^0$
$\therefore\ z_3 = 2\big(cos(-150^0) + isin(-150^0)\big)$



$z_4 = \sqrt{3} - i$; $|z_3| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2\, units$;
$arg(z_3) = -\,tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^0$

$\therefore z_4 = 2\big(cos(-30^0) + isin(-30^0)\big)$



2) *Given that* $z_1 = 2 + 3i$, $z_2 = 1 + i$, *and* $z_3 = 1 - i$. *Express the complex numbers in modulus-argument form*

i) $\frac{z_1.z_2}{z_3}$ *ii)* $\frac{z_2.z_3}{z_1}$ *iii)* $z_2 + \frac{z_1.z_2}{z_3}$ *iv)* $\frac{z_1+z_2}{z_2.z_3}$ *v)* $\frac{z_1+z_2}{z_2-z_3}$

Solution

i. $\frac{z_1.z_2}{z_3} = \frac{(2+3i).(1+i)}{1-i} = \frac{2+2i+3i+3i^2}{1-i} = \frac{2-3+5i}{1-i}$

$\frac{z_1.z_2}{z_3} = \frac{-1+5i}{1-i}$; rationalizing denominator

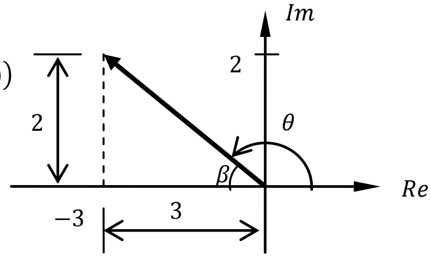
$\frac{z_1.z_2}{z_3} = \frac{(-1+5i)(1+i)}{(1-i)(1+i)} = \frac{-1-i+5i+5i^2}{(1)^2-(i)^2}$ $\left[\because (a+b)(a-b) = a^2 - b^2 \text{ in the denominator}\right]$

$= \frac{-1-5+6i}{1+1} = \frac{-6+4i}{2} = -3+2i$

$\therefore \left|\frac{z_1.z_2}{z_3}\right| = \sqrt{3^2+2^2} = \sqrt{13}$

$arg\left(\frac{z_1.z_2}{z_3}\right) = 180^0 - tan^{-1}\left(\frac{2}{3}\right) = 146.3^0$

$\therefore \frac{z_1.z_2}{z_3} = \sqrt{13}\left(cos(146.3^0) + isin(146.3^0)\right)$



ii. $\frac{z_2.z_3}{z_1} = \frac{(1+i).(1-i)}{2+3i} = \frac{1^2-i^2}{2+3i} = \frac{1+1}{2+3i} = \frac{2}{(2+3i)}$

$\frac{z_2.z_3}{z_1} = \frac{2}{(2+3i)}$; rationalizing denominator

$\frac{z_2.z_3}{z_1} = \frac{2(2-3i)}{(2+3i)(2-3i)} = \frac{4-6i}{(2)^2-(3i)^2}$ $\left[\because (a+b)(a-b) = a^2 - b^2 \text{ in the denominator}\right]$

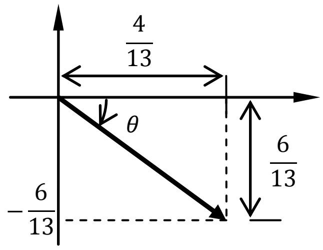
$= \frac{4-6i}{4+9} = \frac{4}{13} - \frac{6}{13}i$

$\therefore \left|\frac{z_2.z_3}{z_1}\right| = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \frac{2}{\sqrt{13}}$ *units*

$arg\left(\frac{z_2.z_3}{z_1}\right) = -tan^{-1}\left(\frac{6/13}{4/13}\right) = -56.3^0$

$\therefore \frac{z_2.z_3}{z_1} = \frac{2}{\sqrt{13}}\left(cos(-56.3^0) + isin(-56.3^0)\right)$

$= \frac{2}{\sqrt{13}}\left(cos(56.3^0) - isin(56.3^0)\right)$



iii. $z_2 + \frac{z_1.z_2}{z_3} = (1+i) + \frac{(2+3i).(1+i)}{1-i} = 1+i+\frac{2+2i+3i+3i^2}{1-i}$

$= 1+i+\frac{2-3+5i}{1-i} = 1+i+\frac{-1+5i}{1-i}$

$= 1+i+\frac{(-1+5i)(1+i)}{(1-i)(1+i)} = 1+i+\frac{-1-i+5i+5i^2}{(1)^2-(i)^2}$

$= 1+i+\frac{-1-5+6i}{1+1} = 1+i+\frac{-6+4i}{2}$

$= 1+i+-3+2i$

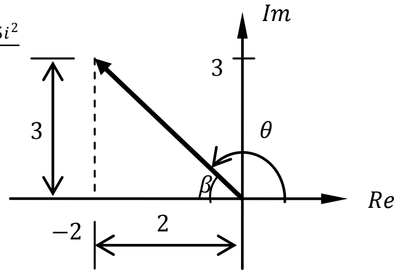
$\therefore z_2 + \frac{z_1.z_2}{z_3} = -2+3i$

$\left|z_2 + \frac{z_1.z_2}{z_3}\right| = \sqrt{2^2+3^2} = \sqrt{13}$ *units*

$arg\left(z_2 + \frac{z_1.z_2}{z_3}\right) = 180^0 - tan^{-1}\left(\frac{3}{2}\right)$

$= 123.7^0$

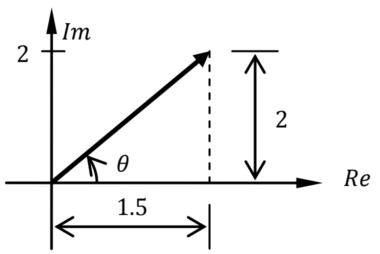
$\therefore z_2 + \frac{z_1.z_2}{z_3} = \sqrt{13}\left(cos(123.7^0) + isin(123.7^0)\right)$



iv. $\frac{z_1+z_2}{z_2.z_3} = \frac{2+3i+1+i}{(1+i).(1-i)} = \frac{3+4i}{(1)^2-(i)^2} = \frac{3+4i}{2} = \frac{3}{2} + 2i$

$\left|\frac{z_1+z_2}{z_2.z_3}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \frac{5}{2}$ *units*

$arg\left(\frac{z_1+z_2}{z_2.z_3}\right) = tan^{-1}\left(\frac{2}{1.5}\right) = 53.1^0$



$\therefore \frac{z_1+z_2}{z_2.z_3} = \frac{5}{2}\left(cos(53.1^0) + isin(53.1^0)\right)$

v. $\frac{z_1+z_2}{z_2-z_3} = \frac{2+3i+1+i}{(1+i)-(1-i)} = \frac{3+4i}{2i} = \frac{3+4i}{0+2i} = \frac{(3+4i)(-2i)}{(2i)(-2i)}$

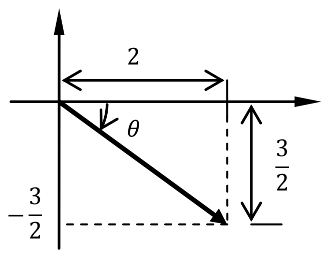
$= \frac{-6i-8i^2}{-4i^2} = \frac{8-6i}{4} = 2 - \frac{3}{2}i$

$\therefore \left|\frac{z_1+z_2}{z_2-z_3}\right| = \sqrt{2^2 + \left(\frac{3}{2}\right)^2} = \frac{5}{2}\ units$

$arg\left(\frac{z_1+z_2}{z_2-z_3}\right) = -\ tan^{-1}\left(\frac{1.5}{2}\right) = -36.9^0$

$\therefore \frac{z_1+z_2}{z_2-z_3} = \frac{5}{2}\left(cos(-36.9^0) + isin(-36.9^0)\right)$

$= \frac{5}{2}\left(cos(36.9^0) - isin(36.9^0)\right)$



Trial questions

Given that $z_1 = 4 + 3i$, $z_2 = -4 - 3i$, *and* $z_3 = 1 - i$. *Express the complex numbers in polar form.*

i) $z_1.z_2$ *ii)* $z_1.z_2.z_3$ *iii)* $z_3 + \frac{z_1.z_2}{z_3}$ *iv)* $\frac{z_1.{z_2}^3}{{z_3}^2}$ *v)* $\frac{z_1.z_3}{z_2} + \frac{z_2}{z_2.z_3}$

$\boldsymbol{ANSWER}$: ***i)*** $\boldsymbol{r = 25\ ,\theta = -106.3^0}$ ***ii)*** $\boldsymbol{r = 35.4\ ,\theta = -151.3^0}$ ***iii)*** $\boldsymbol{r = 19\ ,\theta = -60.1^0}$ ***iv)*** $\boldsymbol{r = 312.5\ ,\theta = 57.5^0}$ ***v)*** $\boldsymbol{r = 1.581\ ,\theta = 108.4^{\ 0}}$

MULTIPLICATION OF COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ which can be expressed in the polar form.$i.e.$

$z_1 = r_1(cos\theta_1 + isin\theta_1)$ *and* $z_2 = r_2(cos\theta_2 + isin\theta_2)$ respectively.

To find $z_1.z_2$, proceed as follows;

$z_1.z_2 = [r_1(cos\theta_1 + isin\theta_1)].[r_2(cos\theta_2 + isin\theta_2)]$

$= r_1.r_2[(cos\theta_1 + isin\theta_1).(cos\theta_2 + isin\theta_2)]$

$= r_1.r_2[cos\theta_1 cos\theta_2 + icos\theta_1 sin\theta_2 + isin\theta_1 cos\theta_2 + i^2 sin\theta_1 sin\theta_2]$

$= r_1.r_2[(cos\theta_1 cos\theta_2 + i^2 sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$

$= r_1.r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$

$= r_1.r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$

In general when multiplying complex numbers in polar form, first multiply their moduli, add their arguments and write the result in polar form.$i.e.$

If $z_1 = r_1(cos\ \theta_1 + isin\ \theta_1)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$z_n = r_n(cos\ \theta_n + isin\ \theta_n)$

$\Rightarrow z_1.z_2 \ldots.. z_n = r_1.r_2 \ldots\ldots r_n[cos(\theta_1 + \theta_2 + \cdots + \theta_n) + isin(\theta_1 + \theta_2 + \cdots + \theta_n)]$

Deduction

When multiplying complex numbers in polar form,

i. $arg(z_1.z_2 \ldots.. z_n) = arg(z_1) + arg(z_2) + \ldots.. + arg(z_n)$

ii. $|z_1.z_2 \ldots.. z_n| = |z_1||z_2| \ldots\ldots\ldots\ldots\ldots |z_n|$

Note

The sense of measurement of the argument *must* be the same. $i.e.$ clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

1. *Given that* $z_1 = 1 + i\sqrt{3}$, $z_2 = \sqrt{3} - i$, $z_3 = -1 - i$ *and* $z_4 = -1 + i$. *Express the complex numbers in polar form.* $z_1.z_2$, $z_3.z_4$ *and* $z_1.z_2.z_3$

Solution

$z_1 = 1 + i\sqrt{3}$; $|z_1| = \sqrt{1^2 + (\sqrt{3})^2} = 2\ units$

$arg(z_1) = tan^{-1}(\sqrt{3}) = 60^0$

$\therefore z_1 = 2(cos60^0 + isin60^0)$

$z_2 = \sqrt{3} - i$; $|z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = 2\ units$

$arg(z_2) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^0$ [$\boldsymbol{principal\,argument}$]

$\therefore z_2 = 2\big(cos(-30^0) + isin(-30^0)\big)$

$z_3 = -1 - i$; $|z_3| = \sqrt{1^2 + 1^2} = \sqrt{2}\ units$

$arg(z_3) = -\left[180^0 - tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -135^0$ [$\boldsymbol{principal\,argument}$]

$\therefore z_3 = \sqrt{2}\big(cos(-135^0) + isin(-135^0)\big)$

$z_4 = -1 + i$; $|z_3| = \sqrt{1^2 + 1^2} = \sqrt{2}\ units$

$arg(z_3) = 180^0 - tan^{-1}(1) = 135^0$ [$\boldsymbol{principal\,argument}$]

$\therefore z_4 = \sqrt{2}\big(cos(135^0) + isin(135^0)\big)$

$z_1.z_2 = [2(cos60^0 + isin60^0)].\big[2\big(cos(-30^0) + isin(-30^0)\big)\big]$

$= 4[cos(60^0 + -30^0) + isin(60^0 + -30^0)]$

$= 4\big(cos(30^0) + isin(30^0)\big)$

$z_3.z_4 = \big[\sqrt{2}\big(cos(-135^0) + isin(-135^0)\big)\big].\big[\sqrt{2}\big(cos(135^0) + isin(135^0)\big)\big]$

$= 2[cos(-135^0 + 135^0) + isin(-135^0 + 135^0)]$

$= 2[cos0^0 + isin0^0]$

$z_1.z_2.z_3 = [2(cos60^0 + isin60^0)].\big[2\big(cos(-30^0) + isin(-30^0)\big)\big].\big[\sqrt{2}\big(cos(-135^0) + isin(-135^0)\big)\big]$

$= 4\sqrt{2}[cos(60^0 + -30^0 + -135^0) + isin(60^0 + -30^0 + -135^0)]$

$= 4\sqrt{2}\big[\big(cos(-105^0) + isin(-105^0)\big)\big]$

2. *Given that* $z_1 = 2 + i$, $z_2 = -2 + i$ *and* $z_3 = 2 - 3i$. *Express the complex numbers in polar form.* $z_1.z_2$, $z_3.z_4$ *and* $z_1.z_2.z_3$

Solution

$|z_1| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $arg(z_1) = 26.6^0$

$|z_2| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $arg(z_2) = 153.4^0$

$|z_3| = \sqrt{2^2 + 3^2} = \sqrt{13}$; $arg(z_3) = -56.3^0$

$\therefore z_1.z_2 = \Big[\sqrt{5}\big[\big(cos(26.6^0) + isin(26.6^0)\big)\big]\Big].\sqrt{5}\big[\big(cos(153.4^0) + isin(153.4^0)\big)\big]$

$= 5[cos(26.6^0 + 153.4^0) + isin(26.6^0 + 153.4^0)]$

$= 5\big(cos(180^0) + isin(180^0)\big)$

$z_3.z_4 = \Big[\sqrt{5}\big[\big(cos(153.4^0) + isin(153.4^0)\big)\big]\Big].\sqrt{13}\big[\big(cos(-56.3^0) + isin(-56.3^0)\big)\big]$

$= \sqrt{65}[cos(-56.3^0 + 153.4^0) + isin(-56.3^0 + 153.4^0)]$

$= \sqrt{65}\left(cos(97.1^0) + isin(97.1^0)\right)$

$z_1.z_2.z_3 =$

$\left[\sqrt{5}\left[\left(cos(26.6^0) + isin(26.6^0)\right)\right]\right].\sqrt{5}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right].\sqrt{13}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right]$

$= 5\sqrt{13}[cos(26.6^0 + 153.4^0 - 56.3^0) + isin(26.6^0 + 153.4^0 - 56.3^0)]$

$= 5\sqrt{13}\left[\left(cos(123.7^0) + isin(123.7^0)\right)\right]$

Trial questions

Given that $z_1 = 2$, $z_2 = -1 + i$ *and* $z_3 = 2i$.

Express z_1, z_2, z_3, $z_1.z_2$, $z_2.z_3$ *and* $z_1.z_2.z_3$ *in polar form.*

ANSWER: $\boldsymbol{r = 2, \theta = 0; r = \sqrt{2}, \theta = 135^0; r = 2, \theta = -90^0; r = 2\sqrt{2}, \theta = 135^0; r = 4\sqrt{2}, \theta = 45^0}$

DIVIDING COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ which can be expressed in the polar form.*i.e.*

$z_1 = r_1(cos\theta_1 + isin\theta_1)$ *and* $z_2 = r_2(cos\theta_2 + isin\theta_2)$ respectively.

To find $\frac{z_1}{z_2}$, proceed as follows;

$\frac{z_1}{z_2} = \frac{r_1(cos\,\theta_1 + isin\,\theta_1)}{r_2(cos\,\theta_2 + isin\,\theta_2)} = \frac{r_1}{r_2}.\frac{(cos\,\theta_1 + isin\,\theta_1)}{(cos\,\theta_2 + isin\,\theta_2)}$; rationalizing the denominator

$\frac{z_1}{z_2} = \frac{r_1}{r_2}.\frac{(cos\,\theta_1 + isin\,\theta_1).(cos\,\theta_2 - isin\,\theta_2)}{(cos\,\theta_2 + isin\,\theta_2).(cos\,\theta_2 - isin\,\theta_2)}$

$= \frac{r_1}{r_2}.\frac{cos\,\theta_1 cos\,\theta_2 - icos\,\theta_1 sin\,\theta_2 + isin\,\theta_1 cos\,\theta_2 - i^2 sin\,\theta_1 sin\,\theta_2}{(cos\,\theta_2)^2 - (isin\,\theta_2)^2}$ $\left[\begin{matrix} \because (a+b)(a-b) = a^2 - b^2 \\ in\ the\ denominator \end{matrix}\right]$

$= \frac{r_1}{r_2}.\frac{(cos\,\theta_1 cos\,\theta_2 + sin\,\theta_1 sin\,\theta_2) + i(sin\,\theta_1 cos\,\theta_2 - cos\,\theta_1 sin\,\theta_2)}{cos^2\theta_2 + sin^2\theta_2}$

$= \frac{r_1}{r_2}.\frac{cos\,(\theta_1 - \theta_2) + isin\,(\theta_1 - \theta_2)}{1}$

$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}.[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$

In general when dividing complex numbers in polar form, first divide their moduli, subtract their arguments and write the result in polar form.

Deduction

When dividing complex numbers in polar form

i. $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

ii. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Note

The sense of measurement of the argument *must* be the same.*i.e.* clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

1. *Given that* $z_1 = 1 + i$, $z_2 = -1 + i$, $z_3 = -i$ *and* $z_4 = 2$. *Express in polar form;*

i) $\frac{z_1}{z_2}$ *ii)* $\frac{z_2}{z_3}$ *iii)* $\frac{z_4}{z_2}$

Solution

$z_1 = 1 + i = \sqrt{2}\left(cos(45^0) + isin(45^0)\right)$

$z_2 = -1 + i = \sqrt{2}\left(cos(135^0) + isin(135^0)\right)$

$z_3 = -i = 0 - i = 1\left(cos(-90^0) + isin(-90^0)\right)$

$z_4 = 2 = 2 + 0i = 2\big(cos(0^0) + isin(0^0)\big)$

$[Expressing\, complex\, numbers\, in\, polar\, form\, is\, left\, as\, an\, exercise]$

i. $\frac{z_1}{z_2} = \frac{\sqrt{2}\left(cos\,(45^0)+isin\,(45^0)\right)}{\sqrt{2}\left(cos\,(135^0)+isin\,(135^0)\right)} = cos(45^0 - 135^0) + isin(45^0 - 135^0)$

$\therefore \frac{z_1}{z_2} = [cos(-90^0) + isin(-90^0)]$

ii. $\frac{z_2}{z_3} = \frac{\sqrt{2}\left(cos\,(135^0)+isin\,(135^0)\right)}{1\left(cos\,(-90^0)+isin\,(-90^0)\right)} = \sqrt{2}[cos(135^0 - -90^0) + isin(135^0 - -90^0)]$

$= \sqrt{2}[cos(225^0) + isin(225^0)]$

Expressing angles trigonometric ratios in terms *principal arguments*

Angle 225^0 lies in the 3^{rd} *quadrant*. Thus the angle nearest to $+ve\, real\, axis$ $is\, 135^0$. Hence principal argument is-135^0

$\therefore \frac{z_2}{z_3} = \sqrt{2}[cos(-135^0\,) + isin(-135^0\,)]$

iii. $\frac{z_4}{z_2} = \frac{2\left(cos\,(0^0)+isin\,(0^0)\right)}{\sqrt{2}\left(cos\,(135^0)+isin\,(135^0)\right)} = \frac{2}{\sqrt{2}}[cos(0^0 - 135) + isin(0^0 - 135)]$

$\therefore \frac{z_4}{z_2} = \frac{2}{\sqrt{2}}[cos(-135^0\,) + isin(-135^0\,)]$

Summary of rules in complex numbers

Complex number	Modulus rules	Argument rules
$z_1 . z_2$	$\lvert z_1\rvert . \lvert z_2\rvert$	$arg(z_1 . z_2) = arg(z_1) + arg(z_2)$
$\frac{z_1}{z_2}$	$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$	$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Note:

i. $arg(z_1) \times arg(z_2) \neq arg(z_1 + z_2) \neq arg(z_1) + arg(z_2)$

ii. $\frac{arg\,(z_1)}{arg\,(z_2)} \neq arg(z_1) - arg(z_2)$

2. *Given that* $z_1 = 2 + 3i\,, z_2 = 1 - i\,, z_3 = -\sqrt{3} + i$ *and* $z_4 = -1 - i\sqrt{3}$. *Express the complex numbers in polar form. Hence express the following in polar form:*

a) $\frac{z_1 . z_2}{z_3}$

b) $\frac{z_1}{z_2 . z_3}$ c) $\frac{z_1 . z_2}{z_3 . z_4}$

Solution

$z_1 = 2 + 3i = \sqrt{13}[cos(56.3^0\,) + isin(56.3^0\,)]$

$z_2 = 1 - i = \sqrt{2}[cos(-45^0\,) + isin(-45^0\,)]$

$z_3 = -\sqrt{3} + i = 2[cos(150^0\,) + isin(150^0\,)]$

$z_4 = -1 - i\sqrt{3} = 2[cos(-120^0\,) + isin(-120^0\,)]$

$[To\, express\, the\, complex\, numbers\, in\, polar\, form\, see\, notes\, on\, polar\, form\, of\, a\, complex\, number]$

a) $\frac{z_1 . z_2}{z_3} = r(cos\theta + isin\theta)$

$r = \left\lvert\frac{z_1 . z_2}{z_3}\right\rvert = \frac{\lvert z_1 . z_2\rvert}{z_3} = \frac{\lvert z_1\rvert . \lvert z_2\rvert}{\lvert z_3\rvert} = \frac{\sqrt{13}\times\sqrt{2}}{2} = \frac{\sqrt{26}}{2}$

$\theta = arg\left(\frac{z_1 . z_2}{z_3}\right) = arg(z_1 . z_2) - arg(z_3) = arg(z_1) + arg(z_2) - arg(z_3)$

$= 56.3^0 + -45^0 - 150^0 = -138.7^0$

$\therefore \frac{z_1 . z_2}{z_3} = \frac{\sqrt{26}}{2}[cos(-138.7^0\,) + isin(-138.7^0\,)]$

b) $\frac{z_1}{z_2 . z_3} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1}{z_2.z_3}\right| = \frac{|z_1|}{|z_2|.|z_3|} = \frac{\sqrt{13}}{\sqrt{2}\times 2} = \frac{1}{2}\sqrt{\left(\frac{13}{2}\right)}$

$\theta = arg\left(\frac{z_1}{z_2.z_3}\right) = arg(z_1) - arg(z_2.z_3) = arg(z_1) - [arg(z_2) + arg(z_3)]$

$= 56.3^0 - [-45^0 + 150^0] = -48.7^0$

$\therefore \frac{z_1}{z_2.z_3} = \frac{1}{2}\sqrt{\left(\frac{13}{2}\right)}[cos(-48.7^0) + isin(-48.7^0)]$

c) $\frac{z_1.z_2}{z_3.z_4} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1.z_2}{z_3.z_4}\right| = \frac{|z_1|.|z_2|}{|z_3|.|z_4|} = \frac{\sqrt{13}\times\sqrt{2}}{2\times 2} = \frac{\sqrt{26}}{4}$

$\theta = arg\left(\frac{z_1.z_2}{z_3.z_4}\right) = arg(z_1) + arg(z_2) - [arg(z_3) + arg(z_4)]$

$= 56.3^0 + -45^0 - [150^0 + -120^0] = -18.7^0$

$\therefore \frac{z_1.z_2}{z_3.z_4} = \frac{\sqrt{26}}{4}[cos(-18.7^0) + isin(-18.7^0)]$

3. *Given that* $z_1 = 1 + i$, $z_2 = 1 - i$, $z_3 = 2 + 3i$. *Express the complex numbers in polar form. Hence express the following in polar form:*

i) $\frac{z_2.z_3}{z_1}$ *ii)* $\frac{z_1.z_2}{z_3}$ *iii)* $z_1.z_2 + \frac{1}{z_3}$ *iv)* $\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}$

Solution

$z_1 = 1 + i = \sqrt{2}[cos(45^0) + isin(45^0)]$

$z_2 = 1 - i = \sqrt{2}[cos(-45^0) + isin(-45^0)]$

$z_3 = 2 + 3i = \sqrt{13}[cos(56.3^0) + isin(56.3^0)]$

i. $\frac{z_2.z_3}{z_1} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_2.z_3}{z_1}\right| = \frac{|z_2|.|z_3|}{|z_1|} = \frac{\sqrt{2}\times\sqrt{13}}{\sqrt{2}} = \sqrt{13}$

$\theta = arg\left(\frac{z_2.z_3}{z_1}\right) = arg(z_2) + arg(z_3) - arg(z_1)$

$= -45^0 + 56.3^0 - 45^0 = -33.7^0$

$\therefore \frac{z_1.z_2}{z_3.z_4} = \sqrt{13}[cos(-33.7^0) + isin(-33.7^0)]$

ii. $\frac{z_1.z_2}{z_3} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1.z_2}{z_3}\right| = \frac{|z_1|.|z_2|}{|z_3|} = \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{13}} = \frac{2}{\sqrt{13}}$

$\theta = arg\left(\frac{z_1.z_2}{z_3}\right) = arg(z_1) + arg(z_2) - arg(z_3)$

$= 45^0 + -45^0 - 56.3^0 = -56.3^0$

$\therefore \frac{z_1.z_2}{z_3} = \frac{2}{\sqrt{13}}[cos(-56.3^0) + isin(-56.3^0)]$

iii. $z_1.z_2 + \frac{1}{z_3} = r(cos\theta + isin\theta)$

Dealing with $z_1.z_2$

$\Rightarrow z_1.z_2 = (1 + i)(1 - i) = 1^2 - i^2 = 2$

$\Rightarrow \frac{1}{z_3} = \frac{1}{2+3i} = \frac{1.(2-3i)}{(2+3i).(2-3i)} = \frac{(2-3i)}{(2)^2-(3i)^2} = \frac{2}{13} - \frac{3}{13}i$

$\therefore z_1.z_2 + \frac{1}{z_3} = 2 + \frac{2}{13} - \frac{3}{13}i = \frac{28}{13} - \frac{3}{13}i$

$r = \left|z_1.z_2 + \frac{1}{z_3}\right| = \sqrt{\left(\frac{28}{13}\right)^2 + \left(\frac{3}{13}\right)^2} = \frac{\sqrt{793}}{13}$

$\theta = arg\left(z_1.z_2 + \frac{1}{z_3}\right) = -tan^{-1}\left(\frac{3/13}{2/13}\right) = -6.1^0$

$\therefore z_1.z_2 + \frac{1}{z_3} = \frac{\sqrt{793}}{13}[cos(-6.1^0\ ) + isin(-6.1^0\ )]$

iv. $\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = r(cos\theta + isin\theta)$

$\frac{z_1}{z_2} = \frac{(1+i)}{(1-i)} = \frac{(1+i).(1+i)}{(1-i).(1+i)} = \frac{1+2i+i^2}{1^2-i^2} = \frac{2i}{2} = i$

$\frac{z_2.z_3}{z_1} = \frac{(1-i).(2+3i)}{(1+i)} = \frac{2+3i-2i-3i^2}{(1+i)} = \frac{2+i+3}{(1+i)} = \frac{(5+i)}{(1+i)}$

$\frac{(5+i)}{(1+i)} = \frac{(5+i).(1-i)}{(1+i).(1-i)} = \frac{5-5i+i-i^2}{1^2-i^2} = \frac{5-4i+1}{2} = \frac{6-4i}{2}$

$\therefore \frac{z_2.z_3}{z_1} = 3 - 2i$

$\therefore \frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = i + 3 - 2i = 3 - i$

$\Rightarrow r = \left|\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\Rightarrow \theta = arg\left(\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}\right) = -\,tan^{-1}\left(\frac{1}{3}\right) = -18.4^0$

$\therefore \frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = \sqrt{10}[cos(-18.4^0\ ) + isin(-18.4^0\ )]$

Task

Given that $z_1 = 1 + i\ , z_2 = 2i\ , z_3 = -1 + i$. *Express the complex numbers in polar form. Hence express the following in polar form:*

i) $\frac{1}{z_1}$ *ii)* $\frac{1}{z_2}$ *iii)* $\frac{1}{z_3}$ *iv)* $\frac{z_1}{z_2} + z_3$ *iv)* $\frac{z_1.z_2}{z_3} + z_3$ *v)* $\frac{z_1.z_2}{z_3} + \frac{z_3}{z_1}$

Answer: $\boldsymbol{i)\ r = \frac{1}{\sqrt{2}}\ , \theta = -45^0\ \ ii)\ r = \frac{1}{2}\ , \theta = -90^0\ \ iii)\ r = \frac{1}{\sqrt{2}}\ , \theta = -135^0}$

$\boldsymbol{iv)\ r = \frac{1}{\sqrt{2}}\ , \theta = -135^0\ \ v)\ r = \sqrt{2}\ , \theta = 45^0\ \ vi)\ r = 2.236\ , \theta = 26.5^0}$

DEMOIVRE'S THEOREM

This is derived from multiplication of complex numbers in polar form.

Consider $z_1 = r_1(cos\theta_1 + isin\theta_1)$

$z_2 = r_2(cos\theta_2 + isin\theta_2)$

$z_3 = r_3(cos\theta_3 + isin\theta_3)$

$\Rightarrow z_1.z_2.z_3 = r_1.r_2.r_2[cos(\theta_1 + \theta_2 + \theta_3) + isin(\theta_1 + \theta_2 + \theta_3)]$

If $z = z_1 = z_2 = z_3 = r(cos\theta + isin\theta)$

$\Rightarrow z.z.z = r.r.r[cos(\theta + \theta + \theta) + isin(\theta + \theta + \theta)]$

$\therefore z^3 = r^3[cos3\theta + isin3\theta]$

$$\therefore z^3 = r^3[cos3\theta + \ i\ sin3\theta]$$

$$\therefore z^n = r^n[cosn\theta + i\ sinn\theta]$$

$\Rightarrow [r(cos\theta + isin\theta)]^n = r^n[cosn\theta + i\ sinn\theta]$

$\Rightarrow r^n(cos\theta + isin\theta)^n = r^n[cosn\theta + i\ sinn\theta]$

$$\boxed{\therefore (\mathbf{cos\theta + isin\theta})^{\mathbf{n}} = \mathbf{cos(n)\theta + i\ sin(n)\theta}}$$

This is Demoivre's theorem.

Proof

The proof is based on induction.

$(cos\theta + isin\theta)^n = cos(n)\theta + i\ sin(n)\theta$

For $n = 1$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^1 = cos\theta + i\ sin\theta$

$\boldsymbol{R.H.S} = cos(1)\theta + i\ sin(1)\theta = cos\theta + i\ sin\theta$

Since $\boldsymbol{L.H.S} = \boldsymbol{R.H.S}$, the proof holds for $n = 1$

For $n = k$

$\Rightarrow (cos\theta + isin\theta)^k = cos(k)\theta + i\ sin(k)\theta$ …………………… (1)

Let the proof hold for $n = k$

Now for $n = k + 1$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^{k+1} = (cos\theta + isin\theta)^k.(cos\theta + i\ sin\theta)^1$ ……. (2)

$eqn.(1) in\ to\ eqn.(2)$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^{k+1} = [cos(k)\theta + i\ sin(k)\theta].(cos\theta + i\ sin\theta)^1$

$= [cos(k)\theta + i\ sin(k)\theta].(cos\theta + i\ sin\theta)$

$= cos(k)\theta cos\theta + icos(k)\theta sin\theta + i\ sin(k)\theta cos\theta + i^2 sin(k)\theta sin\theta$

$= [cos(k)\theta cos\theta - sin(k)\theta sin\theta] + i[cos(k)\theta sin\theta + sin(k)\theta cos\theta]$

$= cos[k\theta + \theta] + i\ sin[k\theta + \theta]$

$= cos(k + 1)\theta + i\ sin(k + 1)\theta$

$\boldsymbol{R.H.S} = cos(k + 1)\theta + i\ sin(k + 1)\theta$

Since $\boldsymbol{L.H.S} = \boldsymbol{R.H.S}$, the proof holds for $n = k + 1$

Since the proof holds for $n = 1, 2, 3 \ldots\ldots\ldots..k, k + 1$, then the proof holds for all values of n.

Deduction from Demoivre's theorem

If $z = r(cos\theta + isin\theta)$, then $z^n = r^n(cos\theta + isin\theta)^n = r^n[cosn\theta + i\ sinn\theta]$

i. $Arg(z^n) = nArg(z)$

ii. $|z^n| = |z|^n$

Properties of Argument

$Arg(\lambda z) = Arg(z)$, where $\lambda \geq 0\ and\ \lambda \epsilon \mathbb{R}$

Application of Demoivre's Theorem

- Simplifying powers of complex numbers in polar form
- Proof of trigonometric identities
- Simplifying complex number expressions
- Expansions

SIMPLIFYING EXPRESSIONS

1) *Simplify*

a) $\frac{cos\,3\theta + isin\,3\theta}{cos\theta\ + isin\theta}$

b) $\frac{\cos\frac{3}{2}\theta + isin\frac{3}{2}\theta}{\cos\frac{1}{2}\theta + isin\frac{1}{2}\theta}$

c) $\frac{cos\,2\theta + isin\,2\theta}{cos\theta\ - isin\theta}$

d) $\frac{cos\,3\theta - isin\,3\theta}{cosn\theta\ + isinn\theta}$

Solution

a) $\frac{cos\,3\theta + isin\,3\theta}{cos\theta\ + isin\theta} = \frac{(cos\theta\ + isin\theta\ )^3}{(cos\theta\ + isin\theta\ )^1} = (cos\theta + isin\theta)^2 = cos2\theta + isin2\theta$

b) $\frac{\cos\frac{3}{2}\theta+isin\frac{3}{2}\theta}{\cos\frac{1}{2}\theta+isin\frac{1}{2}\theta}=\frac{(cos\theta+isin\theta)^{\frac{3}{2}}}{(cos\theta+isin\theta)^{\frac{1}{2}}}=(cos\theta+isin\theta)^{\frac{3}{2}-\frac{1}{2}}=(cos\theta+isin\theta)^{1}=cos\theta+isin\theta$

c) $\frac{cos\,2\theta+isin\,2\theta}{cos\theta-isin\theta}=\frac{cos\,2\theta+isin\,2\theta}{cos\,(-1)\theta+isin\,(-1)\theta}=\frac{(cos\theta+isin\theta)^{2}}{(cos\theta+isin\theta)^{-1}}=(cos\theta+isin\theta)^{3}=cos3\theta+isin3\theta$

d) $\frac{cos\,3\theta-isin\,3\theta}{cosn\theta+isinn\theta}=\frac{cos\,(-3)\theta+isin\,(-3)\theta}{cos\,(n)\theta+isin\,(n)\theta}=\frac{(cos\theta+isin\theta)^{-3}}{(cos\theta+isin\theta)^{n}}=(cos\theta+isin\theta)^{-3-n}$

$=cos(3+n)\theta-isin(3+n)\theta$

2) *Express in the form* $a+ib$

a) $\sqrt{\left(\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right)}$

b) $\frac{\left(\cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)^{2}}{\left(\cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)^{3}(\cos\pi-isin\pi)^{\frac{1}{3}}}$

c) $\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}-isin\frac{\pi}{3}\right)^{2}}{(\cos\,4\pi-isin\,4\pi)}$

Solution

a) $\sqrt{\left(\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right)}=\left[\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right]^{\frac{1}{2}}=\frac{(cos\theta+isin\theta)^{\frac{1}{2}}}{[(cos\theta+isin\theta)^{-2}]^{\frac{1}{2}}}=\frac{(cos\theta+isin\theta)^{\frac{1}{2}}}{(cos\theta+isin\theta)^{-1}}$

$=(cos\theta+isin\theta)^{\frac{1}{2}-\,-1}=\cos\frac{3}{2}\theta+isin\frac{3}{2}\theta$

b) $\frac{\left(\cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)^{2}}{\left(\cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)^{3}(\cos\pi-isin\pi)^{\frac{1}{3}}}=\frac{\left[(cos\pi+isin\pi)^{\frac{1}{4}}\right]^{2}}{\left[(cos\pi+isin\pi)^{\frac{1}{6}}\right]^{3}[(cos\pi+isin\pi)^{-1}]^{\frac{1}{3}}}$

$=\frac{(cos\pi+isin\pi)^{\frac{1}{2}}}{(cos\pi+isin\pi)^{\frac{1}{2}}.(cos\pi+isin\pi)^{-\frac{1}{3}}}=(cos\pi+isin\pi)^{\frac{1}{3}}=\cos\frac{\pi}{3}+isin\frac{\pi}{3}$

c) $\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}-isin\frac{\pi}{3}\right)^{2}}{(\cos\,4\pi-isin\,4\pi)}=\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\left(-\frac{\pi}{3}\right)+isin\left(-\frac{\pi}{3}\right)\right)^{2}}{(\cos\,(-4)\pi+isin\,(-4)\pi)}=\frac{(\cos\pi+isin\pi)^{\frac{1}{3}}(\cos\pi+isin\pi)^{-\frac{2}{3}}}{(\cos\pi+isin\pi)^{-4}}$

$(\cos\pi+isin\pi)^{\frac{1}{3}-\frac{2}{3}-\,-4}=(\cos\pi+isin\pi)^{\frac{11}{3}}=\cos\frac{11}{3}\theta+isin\frac{11}{3}\theta$

Task

Prove that

a) $\frac{(cos\,3\theta+i\,sin\,3\theta)^{4}(cos\,4\theta-i\,sin\,4\theta)^{5}}{(cos\,4\theta+i\,sin\,4\theta)^{3}(cos\,5\theta+i\,sin\,5\theta)^{-4}}=1$

b) $\frac{(cos\,5\theta-i\,sin\,5\theta)^{2}(cos\,7\theta+i\,sin\,7\theta)^{-3}}{(cos\,4\theta-i\,sin\,4\theta)^{9}(cos\theta+i\,sin\theta)^{5}}=1$

c) $\frac{(cos\alpha+i\,sin\alpha)^{2}}{(sin\beta+i\,cos\beta)^{5}}=sin(4\alpha+5\beta)-i\,cos(4\alpha+5\beta)$

d) $\left(\frac{cos\theta+i\,sin\theta}{sin\theta+i\,cos\theta}\right)^{4}=cos8\theta+i\,sin8\theta$

EXPANSION OF $\boldsymbol{cos\,n\theta}$ $\boldsymbol{and\,sin\,n\theta}$

From Demoivre's theorem $(cos\theta+isin\theta)^{n}=cosn\theta+i\,sinn\theta$, to expand $\boldsymbol{cos\,n\theta\ and\,sin\,n\theta}$ the **L.H.S** $(cos\theta+isin\theta)^{n}$ is expanded.

Examples

1. *Express* $cos3\theta$ *,* $sin3\theta$ *in terms of* $cos\theta$ *and* $sin\theta$ *respectively. Hence find* $tan3\theta$

Solution

For $n=3$ in the Demoivre's theorem

$\Rightarrow\ (cos\theta+isin\theta)^{3}=cos3\theta+i\,sin3\theta$

$(cos\theta + isin\theta)^3 = \mathbf{1}(cos\theta)^3(isin\theta)^0 + \mathbf{3}(cos\theta)^2(isin\theta)^1 + \mathbf{3}(cos\theta)^1(isin\theta)^2 + (isin\theta)^3$

$= cos^3\theta + i3cos^2\theta sin\theta - 3cos\theta sin^2\theta - isin^3\theta$

$= (cos^3\theta - 3cos\theta sin^2\theta) + i(3cos^2\theta sin\theta - sin^3\theta)$

$\therefore cos3\theta = cos^3\theta - 3cos\theta sin^2\theta = cos^3\theta - 3cos\theta(1 - cos^2\theta)$

$= cos^3\theta - 3cos\theta + 3cos^3\theta = 4cos^3\theta - 3cos\theta$

$\therefore cos3\theta = 4cos^3\theta - 3cos\theta$

$\therefore sin3\theta = 3cos^2\theta sin\theta - sin^3\theta = 3(1 - sin^2\theta)sin\theta - sin^3\theta$

$= 3sin\theta - 3sin^3\theta - sin^3\theta = 3sin\theta - 4sin^3\theta$

$\therefore sin3\theta = 3sin\theta - 4sin^3\theta$

$\therefore tan3\theta = \frac{sin3\theta}{cos3\theta} = \frac{3cos^2\theta sin\theta - sin^3\theta}{cos^3\theta - 3cos\theta\, sin^2\theta}$, dividing Numerator and Denominator by $cos^3\theta$

$\Rightarrow tan3\theta = \frac{\frac{3cos^2\theta sin\theta}{cos^3\theta} - \frac{sin^3\theta}{cos^3\theta}}{\frac{cos^3\theta}{cos^3\theta} - \frac{3cos\theta\, sin^2\theta}{cos^3\theta}} = \frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}$

$\therefore tan3\theta = \frac{3tan\theta - 4tan^3\theta}{1 - 3tan^2\theta}$

2. *Show that;*

   a) $tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

   b) $tan5\theta = \frac{5tan\theta - 10tan^3\theta + tan^5\theta}{1 - 6tan^2\theta + 5tan^4\theta}$

Solution

a) $tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

$\boldsymbol{L.H.S}: tan4\theta = \frac{sin4\theta}{cos4\theta}$

For $n = 4$ in the Demoivre's theorem

$\Rightarrow (cos\theta + isin\theta)^4 = cos4\theta + i\ sin4\theta$

$(cos\theta + isin\theta)^3 = \mathbf{1}(cos\theta)^4(isin\theta)^0 + \mathbf{4}(cos\theta)^3(isin\theta)^1 + \mathbf{6}(cos\theta)^2(isin\theta)^2 + \mathbf{4}(cos\theta)^1(isin\theta)^3 + \mathbf{1}(isin\theta)^4$

$= cos^4\theta + i4cos^3\theta sin\theta - 6cos^2\theta sin^2\theta - i4cos\theta sin^3\theta + sin^4\theta$

$= (cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta) + i(4cos^3\theta sin\theta - 4cos\theta sin^3\theta)$

$\therefore cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$\therefore sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

$\therefore tan4\theta = \frac{sin4\theta}{cos4\theta} = \frac{4cos^3\theta sin\theta - 4cos\theta\, sin^3\theta}{cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta}$, dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow tan4\theta = \frac{\frac{4cos^3\theta sin\theta}{cos^4\theta} - \frac{4cos\theta\, sin^3\theta}{cos^4\theta}}{\frac{cos^4\theta}{cos^4\theta} - 6\frac{cos^2 sin^2\theta}{cos^4\theta} + \frac{sin^4\theta}{cos^4\theta}} = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

$\therefore tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

b) $Left\ as\ an\ exercise\ for\ the\ students$

3. *Prove that;*

   a) $cos4\theta = \frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$

   b) $sin4\theta = \frac{4tan\theta - 4tan^3\theta}{tan^4\theta + 2tan^2\theta + 1}$

Solution

From example (2) above,

$cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

a) $cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$= \frac{cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta}{1}$

Dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow cos4\theta = \frac{\frac{cos^4\theta}{cos^4\theta} - \frac{6cos^2 sin^2\theta}{cos^4\theta} + \frac{sin^4\theta}{cos^4\theta}}{\frac{1}{cos^4\theta}} = \frac{1-6tan^2\theta + tan^4\theta}{sec^4\theta}$

$= \frac{1-6tan^2\theta + tan^4\theta}{(sec^2\theta)^2} = \frac{1-6tan^2\theta + tan^4\theta}{(1+tan^2\theta)^2}$

$\therefore cos4\theta = \frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$ *As required*

b) $sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

$= \frac{sin4\theta = 4cos^3\theta sin\theta - 4cos\theta\, sin^3\theta}{1}$

Dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow sin4\theta = \frac{\frac{4cos^3\theta sin\theta}{cos^4\theta} - \frac{4cos\theta\, sin^3\theta}{cos^4\theta}}{\frac{1}{cos^4\theta}} = \frac{4tan\theta - 4tan^3\theta}{sec^4\theta}$

$= \frac{4tan\theta - 4tan^3\theta}{(sec^2\theta)^2} = \frac{4tan\theta - 4tan^3\theta}{(1+tan^2\theta)^2}$

$\therefore sin4\theta = \frac{4tan\theta - 4tan^3\theta}{tan^4\theta + 2tan^2\theta + 1}$

Task

Use Demoivre's theorem to show that;

i. $cos2\theta = \frac{1-t^2}{1+t^2}$; $t = tan\theta$

ii. $sin2\theta = \frac{2t}{1+t^2}$; $t = tan\theta$

EXPANSION OF $cos^n\theta$ and $sin^n\theta$

This is based on $z^n + \frac{1}{z^n}$ *and* $z^n - \frac{1}{z^n}$ results.

Consider $z = cos\theta + i\, sin\theta$

$\Rightarrow z^n = cos\, n\,\theta + i\, sin\, n\,\theta$ .... (1)

$z^{-n} = \frac{1}{z^n} = cos(-n)\,\theta + i\, sin(-n)\,\theta = cos\, n\,\theta - i\, sin\, n\,\theta$

$\Rightarrow \frac{1}{z^n} = cos\, n\,\theta - i\, sin\, n\,\theta$ .... (2)

Adding the eqns. (1) *and* (2)

$z^n + \frac{1}{z^n} = 2\, cos\, n\theta$ .... (3)

Subtracting the eqns. (1) *and* (2)

$z^n - \frac{1}{z^n} = i2\, sin\, n\theta$ .... (4)

From eqns. (3) and (4), when $n = 1$

$\Rightarrow z + \frac{1}{z} = 2cos\theta$ .... (5)

$\Rightarrow z - \frac{1}{z} = i2sin\theta$ .... (6)

The above equations are important in this kind of expansions.

Example

1. *Express*

a) $cos^3\theta$ in terms of $cos\theta$ and $cos3\theta$

b) $sin^3\theta$ in terms of $sin\theta$ and $sin3\theta$

c) $cos^4\theta$ in terms of $cos4\theta$ and $cos2\theta$

d) $sin^4\theta$ in terms of $sin4\theta$ and $sin2\theta$

Solution

a) To expand $cos^3\theta, z+\frac{1}{z}=2cos\theta$ is used.

Cubing the equation to create $cos^3\theta$

$\Rightarrow \left(z+\frac{1}{z}\right)^3=(2cos\theta)^3=8cos^3\theta$

$z^3+3.z^2.\frac{1}{z}+3z.\frac{1}{z^2}+\frac{1}{z^3}=8cos^3\theta$

$\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)=8cos^3\theta$

But from $z^n+\frac{1}{z^n}=2\cos n\theta, for\ n=3,$

$z^3+\frac{1}{z^3}=2cos3\theta$

$\therefore 8cos^3\theta=2cos3\theta+3(2cos\theta)$

$=2cos3\theta+6cos\theta$

$\therefore cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$

b) To expand $sin^3\theta, z-\frac{1}{z}=i2sin\theta$ is used.

Cubing the equation to create $cos^3\theta$

$\Rightarrow \left(z-\frac{1}{z}\right)^3=(i2sin\theta)^3=-i8sin^3\theta$

$z^3+3.z^2.\left(-\frac{1}{z}\right)+3z.\left(-\frac{1}{z}\right)^2+\left(-\frac{1}{z}\right)^3=-i8sin^3\theta$

$\left(z^3-\frac{1}{z^3}\right)-3\left(z-\frac{1}{z}\right)=-i8sin^3\theta$

But from $z^n-\frac{1}{z^n}=i2\sin n\theta, for\ n=3,$

$z^3-\frac{1}{z^3}=i2sin3\theta$

$\therefore -i8sin^3\theta=i2sin3\theta-3(i2sin\theta)$

$=i2sin3\theta-i6sin\theta$, dividing through by $-i8$

$\therefore sin^3\theta=-\frac{1}{4}sin3\theta+\frac{3}{4}sin\theta$

c) To expand $cos^4\theta, z+\frac{1}{z}=2cos\theta$ is used.

Quarting both sides

$\Rightarrow \left(z+\frac{1}{z}\right)^4=(2cos\theta)^4=16cos^4\theta$

$z^4+4.z^3.\frac{1}{z}+6z^2.\frac{1}{z^2}+4z.\frac{1}{z^3}+\frac{1}{z^4}=16cos^4\theta$

$\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6=16cos^4\theta$

But from $z^n+\frac{1}{z^n}=2\cos n\theta, for\ n=4\ and\ 2,$

$z^4+\frac{1}{z^4}=2cos4\theta$; $z^2+\frac{1}{z^2}=2cos2\theta$

$\therefore 16cos^4\theta\ =2cos4\theta+4(2cos2\theta)+6$

$=2cos4\theta+8cos2\theta+6$, dividing through by16

$\therefore cos^4\theta=\frac{1}{8}cos3\theta+\frac{1}{2}cos2\theta+\frac{3}{8}$

d) $Left\ as\ an\ exercise\ for\ students$

Task

a) Prove that $sin^5\theta=\frac{1}{16}(sin5\theta-5sin3\theta+10sin\theta)$

b) Prove that $cos^6\theta=\frac{1}{32}(cos6\theta+6cos\theta+15cos\theta+10)$

FINDING THE N^{TH} ROOT OF A COMPLEX NUMBER

Consider $z = r(cos\theta + i\,sin\theta)$. To find $\sqrt[n]{z} = z^{\frac{1}{n}}$, proceed as follows.

$z^{\frac{1}{n}} = [r(cos\theta + i\,sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}}$

Since the result must also be in polar form, then let

$r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$

Recall: From trigonometry,

$cos(\theta) = cos(\theta + 360^0k)$

$sin(\theta) = sin(\theta + 360^0k)$, where $k = 0,1,2,3 \ldots \ldots$

$\therefore r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$, becomes;

$r^{\frac{1}{n}}\left(cos(\theta + 360^0k) + i\,sin(\theta + 360^0k)\right)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$

$\Rightarrow r^{\frac{1}{n}}\left[cos\left(\frac{\theta+360^0k}{n}\right) + i\,sin\left(\frac{\theta+360^0k}{n}\right)\right] = R(cos\varphi + i\,sin\varphi)$

Comparing

$\Rightarrow R = r^{\frac{1}{n}}$ ………………………………. (1)

$\Rightarrow \varphi = \frac{\theta+360^0k}{n}$ ……………………… (2)

$where\ k = 0,1,2,3 \ldots \ldots \ldots \ldots \ldots\ (n-1)$

Taking values of k in to equation (2) yields the argument of different roots to $\sqrt[n]{z}$.

Example

1. *Given that* $z_1 = 1 + \sqrt{3}i\,, z_2 = 2 + i\,, z_3 = 1 - i,$ *find;*

a)	$\sqrt{z_1}$	d)	$\sqrt[3]{z_1}$
b)	$\sqrt{z_2}$	e)	$\sqrt[3]{z_3}$
c)	$\sqrt{z_3}$	f)	$\sqrt[4]{z_1}$

Solution

Expressing complex numbers in polar form first,

a) $z_1 = 1 + \sqrt{3}i = 2\left(cos(60^0) + i\,sin(60^0)\right)$

$\sqrt{z_1} = \left[2\left(cos(60^0) + i\,sin(60^0)\right)\right]^{\frac{1}{2}}$

$= \sqrt{2}\left[\left(cos(60^0) + i\,sin(60^0)\right)\right]^{\frac{1}{2}}$

$= \sqrt{2}\left[cos\left(\frac{60^0+360^0k}{2}\right) + i\,sin\left(\frac{60^0+360^0k}{2}\right)\right]$

Now $k = 0,1$ [$\because n = 2\ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt{2}\left[cos\left(\frac{60^0+360^0\times 0}{2}\right) + i\,sin\left(\frac{60^0+360^0\times 0}{2}\right)\right]$

$= 1.225 + 0.707i$

For $k = 1$

$Root\ 2 = \sqrt{2}\left[cos\left(\frac{60^0+360^0\times 1}{2}\right) + i\,sin\left(\frac{60^0+360^0\times 1}{2}\right)\right]$

$= -1.225 - 0.707i$

$\therefore \sqrt{z_1} = \pm(1.225 + 0.707i)$

b) $z_2 = 2 + i = \sqrt{5}\left(cos(26.6^0) + i\,sin(26.6^0)\right)$

$\sqrt{z_2} = \left[\sqrt{5}\left(cos(26.6^0) + i\,sin(26.6^0)\right)\right]^{\frac{1}{2}}$

$= \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0k}{2}\right) + i\,sin\left(\frac{26.6^0+360^0k}{2}\right)\right]$

Now $k = 0,1$ [$\because n = 2\ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0\times0}{2}\right) + i\ sin\left(\frac{26.6^0+360^0\times0}{2}\right)\right]$

$= 1.455 + 0.344i$

For $k = 1$

$Root\ 2 = \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0\times1}{2}\right) + i\ sin\left(\frac{26.6^0+360^0\times1}{2}\right)\right]$

$= -1.455 - 0.344i$

$\therefore \sqrt{z_1} = \pm(1.455 + 0.344i)$

c) $z_3 = 1 - i = \sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)$

$\sqrt{z_3} = {z_3}^{\frac{1}{2}} = \left[\sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)\right]^{\frac{1}{2}}$

$= \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0k}{2}\right) + i\ sin\left(\frac{-45^0+360^0k}{2}\right)\right]$

Now $k = 0{,}1$ [$\because n = 2\ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0\times0}{2}\right) + i\ sin\left(\frac{-45^0+360^0\times0}{2}\right)\right]$

$= 1.099 - 0.455i$

For $k = 1$

$Root\ 2 = \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0\times1}{2}\right) + i\ sin\left(\frac{-45^0+360^0\times1}{2}\right)\right]$

$= -1.099 + 0.455i$

$\therefore \sqrt{z_3} = \pm(1.099 - 0.455i)$

d) $\sqrt[3]{z_1} = {z_1}^{\frac{1}{3}} = \left[2\left(cos(60^0) + i\ sin(60^0)\right)\right]^{\frac{1}{3}}$

$= \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0k}{3}\right) + i\ sin\left(\frac{60^0+360^0k}{3}\right)\right]$

Now $k = 0{,}1{,}2$ [$\because n = 3\ \therefore k\ takes\ on\ 0\ ,1\ and\ 2$]

For $k = 0$

$Root\ 1 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times0}{3}\right) + i\ sin\left(\frac{60^0+360^0\times0}{3}\right)\right]$

$= 1.184 + 0.431i$

For $k = 1$

$Root\ 2 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times1}{3}\right) + i\ sin\left(\frac{60^0+360^0\times1}{3}\right)\right]$

$= -0.965 + 0.810i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times2}{3}\right)\right]$

$= -0.219 - 1.241i$

$\therefore \sqrt[3]{z_1} = 1.184 + 0.431i\ ; -0.965 + 0.810i\ ; -0.219 - 1.241i$

e) $\sqrt[3]{z_3} = {z_3}^{\frac{1}{3}} = \left[\sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)\right]^{\frac{1}{3}}$

$= \sqrt[6]{2}\left[cos\left(\frac{-45^0+360^0k}{3}\right) + i\ sin\left(\frac{-45^0+360^0k}{3}\right)\right]$

Now $k = 0{,}1{,}2$ [$\because n = 3\ \therefore k\ takes\ on\ 0\ ,1\ and\ 2$]

For $k = 0$

$Root\ 1 = \sqrt[6]{2}\left[cos\left(\frac{-45^0+360^0\times0}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times0}{3}\right)\right]$

$= 1.084 - 0.291i$

For $k = 1$

$Root\ 2 = \sqrt[3]{2}\left[cos\left(\frac{-45^0+360^0\times1}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times1}{3}\right)\right]$

$= -0.291 + 1.084i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{-45^0+360^0\times2}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times2}{3}\right)\right]$

$= -0.794 - 0.794i$

$\therefore \sqrt[3]{z_1} = 1.084 - 0.291i\ ; -0.291 + 1.084i\ ; -0.794 - 0.794i$

f) $\sqrt[4]{z_1} = \sqrt[4]{z_1} = z_1^{\frac{1}{4}} = \left[2\left(cos(60^0) + i\ sin(60^0)\right)\right]^{\frac{1}{4}}$

$= \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0k}{4}\right) + i\ sin\left(\frac{60^0+360^0k}{4}\right)\right]$

Now $k = 0,1,2\ ,3\ [\because n = 4\ \therefore k\ takes\ on\ 0\ ,1\ ,2\ and\ 3]$

For $k = 0$

$Root\ 1 = \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0\times0}{4}\right) + i\ sin\left(\frac{60^0+360^0\times0}{4}\right)\right]$

$= 1.149 + 0.308i$

For $k = 1$

$Root\ 2 = \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0\times1}{3}\right) + i\ sin\left(\frac{60^0+360^0\times1}{3}\right)\right]$

$= -0.308 + 1.149i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times2}{3}\right)\right]$

$= -1.149 - 0.308i$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times2}{3}\right)\right]$

$= 0.308 - 1.149i$

$\therefore \sqrt[3]{z_1} = 1.149 + 0.308i\ ; -0.308 + 1.149i\ ;\ -1.149 - 0.308i; 0.308 - 1.149i$

Task

Find the fourth root of:

a. $5 + 12i$

b. $-4 + 3i$

c. $4 - 3i$

ANSWER:

EXPRESSING POWERED COMPLEX NMBER IN POLAR FORM

The rules of modulus and argument of complex numbers are employed.

Example

1) *Express the following in polar form:*

a. $\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}$ c. $\frac{1+2i}{1-(1-i)^2}$

b. $\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}$ d. $\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}$

Solution

a. let $z = \frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3} = r(cos\theta + i\ sin\theta)$

$r = |z| = \left|\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}\right| = \frac{|(1+i)^2||1+i\sqrt{3}|}{\left|(1-i\sqrt{3})^3\right|}$

$= \frac{|1+i|^2|1+i\sqrt{3}|}{|1-i\sqrt{3}|^3} = \frac{(\sqrt{1+1})^2(\sqrt{1+3})}{(\sqrt{1+3})^3} = \frac{2\times2}{2^3} = \frac{1}{2}$

$\theta = arg(z) = arg\left(\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}\right)$

$= arg(1+i)^2.(1+i\sqrt{3}) - arg(1-i\sqrt{3})^3$

$\left[\because arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)\right]$

$= 2arg(1+i) + arg(1+i\sqrt{3}) - 3arg(1-i\sqrt{3})$

$[\because Arg(z^n) = nArg(z)]$

$= 2 \times 45^0 + 60^0 - 3 \times (-60^0) = 330^0\ or - 30^0$ [$\boldsymbol{Principal\, angle}$]

$\therefore z = \frac{1}{2}\left(cos(-30^0) + i\, sin(-30^0)\right)$

b. let $z = \frac{(1-i)^2(3+4i)^3}{(-5+12i)^2} = r(cos\theta + i\, sin\theta)$

$r = |z| = \left|\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}\right| = \frac{|(1-i)^2||(3+4i)^3|}{|(-5+12i)^2|}$

$= \frac{|1-i|^2|3+4i|^3}{|-5+12i|^2} = \frac{(\sqrt{1+1})^2(\sqrt{9+16})^3}{(\sqrt{25+144})^2} = \frac{2\times5^3}{13^2} = \frac{250}{169}$

$\theta = arg(z) = arg\left(\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}\right)$

$= arg(1-i)^2.(3+4i)^3 - arg(-5+12i)^2$

$= 2arg(1-i) + 3arg(3+4i) - 2arg(-5+12i)$

$= 2 \times -45^0 + 3 \times 53.1^0 - 2 \times 112.6^0 = -155.9^0$ [$\boldsymbol{Principal\, angle}$]

$\therefore z = \frac{250}{169}\left(cos(-155.9^0) + i\, sin(-155.9^0)\right)$

c. let $z = \frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-2i+i^2)} = \frac{1+2i}{1+2i} = 1 + 0i = r(cos\theta + i\, sin\theta)$

$r = |z| = |1+0i| = \sqrt{1+0} = 1$

$\theta = arg(z) = 0$

$\therefore z = 1(cos0 + i\, sin0)$

d. let $z = \frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}} = r(cos\theta + i\, sin\theta)$

$r = |z| = \left|\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}\right| = \frac{|(1+i\sqrt{3})^{12}|}{|(\sqrt{3}-i)^{11}|} = \frac{|1+i\sqrt{3}|^{12}}{|\sqrt{3}-i|^{11}}$

$= \frac{(\sqrt{1+3})^{12}}{(\sqrt{3+1})^{11}} = \frac{2^{12}}{2^{11}} = 2$

$\theta = arg(z) = arg\left(\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}\right)$

$= arg(1+i\sqrt{3})^{12} - arg(\sqrt{3}-i)^{11}$

$= 12arg(1+i\sqrt{3}) - 11arg(\sqrt{3}-i)$

$= 12 \times 60^0 - 11 \times -30^0 = 1050^0 or - 30^0$ [$\boldsymbol{Principal\, angle}$]

$\therefore z = 2\left(cos(-30^0) + i\, sin(-30^0)\right)$

STATING THE PRINCIPAL ANGLE OF A COMPLEX NUMBER

The interval of a principal angle is $-\pi < \theta \leq \pi\ or - 180^0 < \theta \leq 180^0$

The knowledge of trigonometry of angles in quadrants is required.

Definition

The *principal angle* is the angle whose complex number (number) is nearest to the *positive Real axis.*

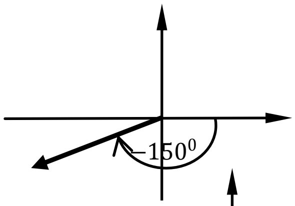
Example

1) *Given that* $z_1 = \sqrt{3} - i$, $z_2 = -1 + i$ *and* $z_3 = 1 - i\sqrt{3}$. *Find the principal argument of;*

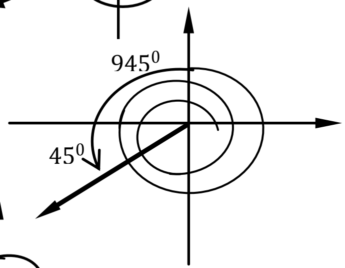
   a) $z_1{}^5$

b) $z_2{}^7$ c) $z_3{}^8$

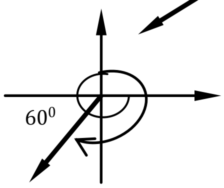
Solution



a) $arg(z_1{}^5) = 5arg(z_1)$
$= 5 \times -30^0 = -150^0$



b) $arg(z_2{}^7) = 7arg(Z_2)$
$= 7 \times 135^0 = 945^0$
But $tan945^0 = -tan135^0$
$\therefore arg(z_2{}^7) = -135^0$

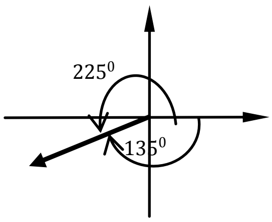


c) $arg(z_3{}^8) = 8arg(Z_3)$
But $arg(z_3) = -60^0$
$\therefore arg(z_3{}^8) = -60^0 \times 8 = -480^0$
$\therefore arg(z_3{}^8) = -120^0$

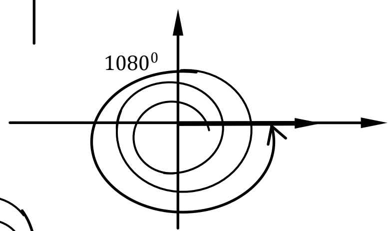
2) *Given that* $z_1 = 1 + i$, $z_2 = -1 + i$ *and* $z_3 = -1 - i$. *Find the principal argument of;*

c) $z_1{}^5$
d) $z_2{}^8$ c) $z_3{}^6$

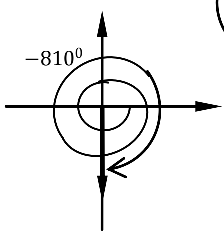
Solution



a) $arg(z_1{}^5) = 5arg(z_1)$
$= 5 \times 45^0 = 225^0$
$\therefore arg(z_1{}^5) = -135^0$



b) $arg(z_2{}^8) = 8arg(Z_3)$
But $arg(z_3) = 135^0$
$\therefore arg(z_3{}^8) = 135^0 \times 8 = 1080^0$
$\therefore arg(z_3{}^8) = 0^0$



c) $arg(z_3{}^6) = 6arg(Z_3)$
But $arg(z_3) = -135^0$
$\therefore arg(z_3{}^8) = -135^0 \times 6 = -810^0$
$\therefore arg(z_3{}^8) = -90^0$

Task

Given that $z_1 = 5 + 12i$, $z_2 = -2 + i$ *and* $z_3 = 8 - i4$. *Find the principal argument of;*

a) $z_1{}^3$
b) $z_2{}^5$ c) $z_3{}^8$

Answer: $(\boldsymbol{a})\ -\mathbf{157.9^0}$ $(\boldsymbol{b})\ -\mathbf{47.2^0}$ $(\boldsymbol{c})\ -\mathbf{65^0}$

FURTHER EXAMPLES ON PROOFS BY DEMOIVRE'S THEOREM

The knowledge of trigonometry is applicable.

Example

1. *If* $z_1 = cos\alpha + i\ sin\alpha$, $z_2 = cos\beta + i\ sin\beta$, *show that* $\frac{1}{2i}\left[\frac{z_1}{z_2} - \frac{z_2}{z_1}\right] = sin(\alpha - \beta)$

Solution

$\frac{z_1}{z_2} = \frac{cos\alpha\ + i\ sin\,\alpha}{cos\beta\ + i\ sin\beta} = cos(\alpha - \beta) + i\ sin(\alpha - \beta)$
$\frac{z_2}{z_1} = \frac{cos\beta\ + i\ sin\beta}{cos\alpha\ + i\ sin\alpha} = cos(\beta - \alpha) + i\ sin(\beta - \alpha)$
$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = cos(\alpha - \beta) + i\ sin(\alpha - \beta) - [cos(\beta - \alpha) + i\ sin(\beta - \alpha)\ ]$
$= cos(\alpha - \beta) + i\ sin(\alpha - \beta) - cos(\beta - \alpha) - i\ sin(\beta - \alpha)$
$= cos(\alpha - \beta) + i\ sin(\alpha - \beta) - cos[-(\alpha - \beta)] - i\ sin[-(\alpha - \beta)]$
$= cos(\alpha - \beta) + i\ sin(\alpha - \beta) - cos(\alpha - \beta) + i\ sin(\alpha - \beta)$
$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = 2i\ sin(\alpha - \beta)$
$\therefore \frac{1}{2i}\left[\frac{z_1}{z_2} - \frac{z_2}{z_1}\right] = \frac{1}{2i} \times 2i\ sin(\alpha - \beta) = sin(\alpha - \beta)$

2. *If* $z = cos\theta + i\ sin\theta$ *, prove that* $\frac{2}{1+z} = 1 - i\ tan\frac{\theta}{2}$ *and* $\frac{1+z}{1-z} = i\ cot\frac{\theta}{2}$

Solution

$\frac{2}{1+z} = \frac{2}{1+cos\theta\ + i\ sin\theta} = \frac{2}{2cos^2\frac{\theta}{2} + i\ 2sin\frac{\theta}{2}cos\frac{\theta}{2}}$
$= \frac{1}{cos\frac{\theta}{2}\left(cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right)} = \frac{1}{cos\frac{\theta}{2}}.\left(cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right)^{-1}$
$= \frac{1}{cos\frac{\theta}{2}}.\ cos\left(-\frac{\theta}{2}\right) + i\ sin\left(-\frac{\theta}{2}\right) = \frac{1}{cos\frac{\theta}{2}}.\left[cos\frac{\theta}{2} - i\ sin\frac{\theta}{2}\right]$
$\therefore \frac{2}{1+z} = 1 - i\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} = 1 - i\ tan\frac{\theta}{2}$
$\frac{1+z}{1-z} = \frac{1+cos\theta\ + i\ sin\theta}{1-cos\theta\ - i\ sin\theta} = \frac{(1+cos\theta\ ) + i\ sin\theta}{(1-cos\theta\ ) - i\ sin\theta}$
$= \frac{2cos^2\frac{\theta}{2} + i\ 2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2} - i\ 2sin\frac{\theta}{2}cos\frac{\theta}{2}} = \frac{2cos\frac{\theta}{2}\left[cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right]}{2sin\frac{\theta}{2}\left[sin\frac{\theta}{2} - i\ cos\frac{\theta}{2}\right]}$

Writing $sin\frac{\theta}{2} - i\ cos\frac{\theta}{2}$ in the form $cos\varphi + i\ sin\varphi$

Recall: $cos\left(90^0 - \frac{\theta}{2}\right) = sin\frac{\theta}{2}$ $and\ sin\left(90^0 - \frac{\theta}{2}\right) = cos\frac{\theta}{2}$

$= cot\frac{\theta}{2}.\frac{\left[cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right]}{\left[cos\left(90^0 - \frac{\theta}{2}\right) - i\ sin\left(90^0 - \frac{\theta}{2}\right)\right]}$
$= cot\frac{\theta}{2}.\frac{\left[cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right]}{\left[cos\left\{-\left(90^0 - \frac{\theta}{2}\right)\right\} - i\ sin\left\{-\left(90^0 - \frac{\theta}{2}\right)\right\}\right]}$
$= cot\frac{\theta}{2}.\left[cos\frac{\theta}{2} + i\ sin\frac{\theta}{2}\right].\left[cos\left(\frac{\theta}{2} - 90^0\right) + i\ sin\left(\frac{\theta}{2} - 90^0\right)\right]^{-1}$
$= cot\frac{\theta}{2}.\left[cos\left(\frac{\theta}{2}\right) + i\ sin\left(\frac{\theta}{2}\right)\right].\left[cos\left\{-\left(\frac{\theta}{2} - 90^0\right)\right\} + i\ sin\left\{-\left(\frac{\theta}{2} - 90^0\right)\right\}\right]$
$= cot\frac{\theta}{2}.\left[cos\left(\frac{\theta}{2}\right) + i\ sin\left(\frac{\theta}{2}\right)\right].\left[cos\left(90^0 - \frac{\theta}{2}\right) + i\ sin\left(90^0 - \frac{\theta}{2}\right)\right]$
$= cot\frac{\theta}{2}.\left[cos\left(\frac{\theta}{2} + 90^0 - \frac{\theta}{2}\right) + i\ sin\left(\frac{\theta}{2} + 90^0 - \frac{\theta}{2}\right)\right]$
$= cot\frac{\theta}{2}.[cos(90^0) + i\ sin(90^0)]$
$\therefore \frac{1+z}{1-z} = i\ cot\frac{\theta}{2}$

3. *Prove that* $(sin\theta + i\ cos\theta)^n = cos\left(\frac{n\pi}{2} - n\theta\right) + i\ sin\left(\frac{n\pi}{2} - n\theta\right)$

Solution

$(sin\theta + i\ cos\theta)^n = ?$
$sin\theta = cos\left(\frac{\pi}{2} - \theta\right)$ and $cos\theta = sin\left(\frac{\pi}{2} - \theta\right)$
$\Rightarrow sin\theta + i\ cos\theta = cos\left(\frac{\pi}{2} - \theta\right) + i\ sin\left(\frac{\pi}{2} - \theta\right)$

$\therefore (sin\theta + i\,cos\theta)^n = \left[cos\left(\frac{\pi}{2} - \theta\right) + i\,sin\left(\frac{\pi}{2} - \theta\right)\right]^n$

$= cos\left\{n\left(\frac{\pi}{2} - \theta\right)\right\} + i\,sin\left\{n\left(\frac{\pi}{2} - \theta\right)\right\}$

$\therefore (sin\theta + i\,cos\theta)^n = cos\left(\frac{n\pi}{2} - n\theta\right) + i\,sin\left(\frac{n\pi}{2} - n\theta\right)$

4. (a) *Find the modulus and argument of* $(a + ib)^2$ *where* a *and* b *are real numbers. Hence show that* $tan^{-1}\left(\frac{2ab}{a^2-b^2}\right) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

(b) *Two complex numbers* z_1 *and* z_2 *are such that* $|z_1 - z_2| = |z_1 + z_2|$.
Show that $|\theta_1 - \theta_2| = \frac{\pi}{2}$, *where* θ_1 *and* θ_2 *are arguments of* z_1 *and* z_2 *respectively.*

Solution

a) let $z = (a + ib)^2$

$|z| = |a + ib|^2 = \left[\sqrt{a^2 + b^2}\right]^2$

$\therefore |z| = a^2 + b^2$

$arg(z) = arg(a + ib)^2 = 2arg(a + i\,b) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

Also $z = (a + ib)^2 = a^2 + i2ab + i^2b^2$

$= a^2 - b^2 + i2ab$

$\Rightarrow arg(z) = tan^{-1}\left(\frac{2ab}{a^2-b^2}\right) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

b) $|\theta_1 - \theta_2| = \frac{\pi}{2}$; $|z_1 - z_2| = |z_1 + z_2|$

Let $z_1 = r_1(cos\theta_1 + i\,sin\theta_1)$, $z_2 = r_2(cos\theta_2 + i\,sin\theta_2)$

$z_1 - z_2 = r_1(cos\theta_1 + i\,sin\theta_1) - r_2(cos\theta_2 + i\,sin\theta_2)$

$= r_1cos\theta_1 + i\,r_1sin\theta_1 - r_2cos\theta_2 - r_2i\,sin\theta_2$

$= (r_1cos\theta_1 - r_2cos\theta_2) + i\,(r_1sin\theta_1 - r_2sin\theta_2)$

$z_1 + z_2 = r_1(cos\theta_1 + i\,sin\theta_1) + r_2(cos\theta_2 + i\,sin\theta_2)$

$= r_1cos\theta_1 + i\,r_1sin\theta_1 + r_2cos\theta_2 + r_2i\,sin\theta_2$

$= (r_1cos\theta_1 + r_2cos\theta_2) + i\,(r_1sin\theta_1 + r_2sin\theta_2)$

$|z_1 - z_2| = \sqrt{[(r_1cos\theta_1 - r_2cos\theta_2)]^2 + [(r_1sin\theta_1 - r_2sin\theta_2)]^2}$

$= \sqrt{\begin{matrix} r_1{}^2cos^2\theta_1 - r_1cos\theta_1r_2cos\theta_2 + r_2{}^2cos^2\theta_2 \\ +r_1{}^2sin^2\theta_1 - r_1sin\theta_1r_2sin\theta_2 + r_2{}^2sin^2\theta_2 \end{matrix}}$

$= \sqrt{\begin{matrix} r_1{}^2(cos^2\theta_1 + sin^2\theta_1) - 2r_1r_2(cos\theta_1cos\theta_2 + sin\theta_1sin\theta_2) \\ +r_2{}^2(cos^2\theta_2 + sin^2\theta_2) \end{matrix}}$

$\therefore |z_1 - z_2| = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2cos(\theta_1 - \theta_2)}$

$|z_1 + z_2| = \sqrt{[(r_1cos\theta_1 + r_2cos\theta_2)]^2 + [(r_1sin\theta_1 + r_2sin\theta_2)]^2}$

$= \sqrt{\begin{matrix} r_1{}^2cos^2\theta_1 + 2r_1cos\theta_1r_2cos\theta_2 + r_2{}^2cos^2\theta_2 \\ +r_1{}^2sin^2\theta_1 + 2r_1sin\theta_1r_2sin\theta_2 + r_2{}^2sin^2\theta_2 \end{matrix}}$

$= \sqrt{\begin{matrix} r_1{}^2(cos^2\theta_1 + sin^2\theta_1) + 2r_1r_2(cos\theta_1cos\theta_2 + sin\theta_1sin\theta_2) \\ +r_2{}^2(cos^2\theta_2 + sin^2\theta_2) \end{matrix}}$

$\therefore |z_1 + z_2| = \sqrt{r_1{}^2 + r_2{}^2 + 2r_1r_2cos(\theta_1 - \theta_2)}$

Now $|z_1 - z_2| = |z_1 + z_2|$; squaring both sides;

$r_1{}^2 + r_2{}^2 - 2r_1r_2cos(\theta_1 - \theta_2) = r_1{}^2 + r_2{}^2 + 2r_1r_2cos(\theta_1 - \theta_2)$

Simplifying

$-cos(\theta_1 - \theta_2) = cos(\theta_1 - \theta_2)$ or $2cos(\theta_1 - \theta_2) = 0$
$cos(\theta_1 - \theta_2) = 0 \Rightarrow \theta_1 - \theta_2 = cos^{-1}(0) = \frac{\pi}{2}$
$\therefore \theta_1 - \theta_2 = \frac{\pi}{2}$ or $|\theta_1 - \theta_2| = \frac{\pi}{2}$

Task

1. $If \quad z = 1 + cos2\theta + i\, sin2\theta \quad , \quad where -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$ $Prove \quad that |z| = 2cos\theta \quad and$ $arg(z) = \theta$
2. $If \quad x = cos\alpha + i\, sin\alpha \,, y = cos\beta + i\, sin\beta \,, z = cos\lambda + i\, sin\lambda \, and\, x + y + z = 0,$ $prove$ $that \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$
3. $If\, arg\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \frac{\pi}{2},$ $show\ that\ |z_1| = |z_2|$

SOLVING EQUATIONS IN COMPLEX NUMBERS

a) **Equations involving equality of complex numbers**

Equality of complex numbers occurs if:

i. If $a + i\,b = 0, then\; a = 0\,,\; b = 0$
ii. If $a + i\,b = c + i\,d, then\; a = c\,,\; b = d$

Example

1) *Solve the simultaneous equations;*
$z_1 + z_2 = 8\,, 4z_1 - i3\,z_2 = 26 + 8\,i$

Solution

Let $z_1 = x_1 + iy_1\,,\; z_2 = x_2 + iy_2$
$\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = 8 + 0i$
$\therefore x_1 + x_2 = 8$ .... (1)
$y_1 + y_2 = 0$ .... (2)
$4z_1 - i3\,z_2 = 26 + 8\,i$
$\Rightarrow 4(x_1 + iy_1) - i3(x_2 + iy_2) = 26 + i\,8$
$4x_1 + i4y_1 - i3x_2 - i^2 3y_2 = 26 + i\,8$
$(4x_1 + 3y_2) + i(4y_1 - 3x_2) = 26 + i\,8$
$4x_1 + 3y_2 = 26$ .... (3)
$4y_1 - 3x_2 = 8$ .... (4)
From eqn. (1)
$x_2 = 8 - x_1$ .... (5)
$eqn. (5) in\ to\ (4)$
$4y_1 - 3(8 - x_1) = 8 \Rightarrow 4y_1 + 3x_1 - 24 = 8$
$4y_1 + 3x_1 = 32$ .... (6)
From eqn. (2)
$y_2 = -y_1$ .... (7)
$eqn. (7) in\ to\ (3)$
$4x_1 - 3y_1 = 26$ .... (8)
$Solving\ eqn. (6)\ and\ (8)$
$x_1 = 8\,, y_1 = 2$
From eqn. (2) and eqn. (5)
$y_2 = -2\,, x_2 = 0$
$\therefore z_1 = 8 + 2\,i\,, z_2 = 0 - 2\,i$

Alternatively

$z_1 + z_2 = 8$ .... (1)

$4z_1 - i3\,z_2 = 26 + 8\,i$ .... (2)

$eqn.(1) \times 4 - eqn.(2)$

$$\begin{array}{rl} & 4z_1 + 4\,z_2 = 32 + 0\,i \\ - & \underline{4z_1 - i3\,z_2 = 26 + 8\,i} \\ & (4 + 3\,i)z_2 = 6 - 8\,i \end{array}$$

$\therefore z_2 = \frac{6-8\,i}{4+3\,i} = \frac{(6-8\,i)(4-3\,i)}{(4+3\,i)(4-3\,i)}$

$= \frac{24-18i-32i+i^2 24}{16-(3i)^2} = \frac{0-50\,i}{16+9}$

$\therefore z_2 = 0 - 2\,i$

From eqn. (1)

$z_1 = 8 - z_2 = 8 - \; -2i = 8 + 2i$

$\therefore z_1 = 8 + 2\,i$

2) *Given that* $(3 + 2z^*)z = 5 + 2z$, *where* z^* *is the conjugate of z. find z.*

Solution

$3z + 2zz^* = 5 + 2z$

$\Rightarrow 2zz^* = 5 - z$

Let $z = x + i\,y \quad \Rightarrow z^* = x - i\,y$

$\therefore 2(x + i\,y)(x - i\,y) = 5 - (x + i\,y)$

$\Rightarrow 2(x^2 - (i\,y)^2) = (5 - x) - i\,y$

$\Rightarrow 2x^2 + 2y^2 = (5 - x) - i\,y$

Equating Real parts and imaginary parts

$\therefore 2x^2 + 2y^2 = 5 - x$ .... (1)

$\therefore 0 = -y \Rightarrow y = 0$

$For\ y = o, x = ?$

$\Rightarrow 2x^2 = 5 - x \ \ or\ 2x^2 + x - 5 = 0$

$\therefore x = 1.351\,, -1.851$

$\therefore z = 1.351 + 0\,i\,;\ -1.851 + 0\,i$

3) *Given that z is a complex number such that* $z = \frac{p}{2-i} + \frac{q}{1+3i}$, *where p and q are real.*

Given that $arg(z) = \frac{\pi}{2}$ *and* $|z| = 7$, *find p and q.*

Solution

$z = \frac{p}{2-i} + \frac{q}{1+3i} = \ z = \frac{p(2+i)}{(2-i)(2+i)} + \frac{q(1-3i)}{(1+3i)(1-3i)}$

$= \frac{2p+2i}{4-i^2} + \frac{q-3qi}{1-i^2 3^2} = \frac{2p}{5} + \frac{2}{5}i + \frac{q}{10} - \frac{3q}{10}i$

$\therefore z = \left(\frac{2p}{5} + \frac{q}{10}\right) + i\left(\frac{2}{5} - \frac{3q}{10}\right)$

$arg(z) = tan^{-1}\left(\frac{\left(\frac{2}{5}-\frac{3q}{10}\right)}{\left(\frac{2p}{5}+\frac{q}{10}\right)}\right) = tan^{-1}\left(\frac{2p-3q}{4p+q}\right) = \frac{\pi}{2}$

$\therefore \frac{2p-3q}{4p+q} = \tan\frac{\pi}{2} = \infty$

$\Rightarrow 4p + q = 0$ .... (1)

$|z| = 7 = \sqrt{\left(\frac{2p}{5} + \frac{q}{10}\right)^2 + \left(\frac{2}{5} - \frac{3q}{10}\right)^2}$

Squaring both sides

$\Rightarrow 49 = \frac{(4p+q)^2+(2p-3q)^2}{100}$

$\therefore 4900 = (4p+q)^2 + (2p-3q)^2$ …………………… (2)

$From\ eqn. (1)$

$q = -4p$ ………………………………………… (3)

$eqn. (3) in\ to\ (2)$

$\Rightarrow 4900 = (4p-4p)^2 + \left(2p-3(-4p)\right)^2$

$\Rightarrow 4900 = (14p)^2$

Finding square root on both sides

$\pm 70 = 14p \therefore p = \pm 5$

$For\ p = \pm 5, q = ?$

From eqn. (1)

$q = -4p = \pm 5 \times (-4) = \pm 20$

$\therefore p = 5. q = -20; p = -5, q = 20$

4) *Given that* $u = -10 + i\,9$ *as a complex number,*
Find the complex number v *which satisfies the equation* $uv = -11 + i\,28$.
Hence verify that $|u+v| = 8\sqrt{2}$.

Solution

$uv = -11 + i\,28 \Rightarrow v = \frac{-11+i\,28}{u} = \frac{-11+i\,28}{-10+i\,9}$

$= \frac{(-11+i\,28)(-10-i\,9)}{(-10+i\,9)(-10-i\,9)}$

$= \frac{110+i\,99-i\,280-i^2 252}{(-10)^2-(i9)^2} = \frac{110-i\,181+252}{100+81}$

$= \frac{362}{181} - i\frac{181}{181} = 2 - i \therefore v = 2 - i$

$|u+v| = |-1 + i\,9 + 2 - i| = 8\sqrt{2}$

$= |-8 + 8i| = \sqrt{8^2 + 8^2} = 8\sqrt{2}$

5) *Find* z *if* $arg(z+1) = \frac{\pi}{4}$ *and* $arg(z-1) = \frac{2\pi}{3}$

Solution

Let $z = x + i\,y$, $z + 1 = x + 1 + i\,y$

$\Rightarrow arg(z+1) = tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$

$\Rightarrow tan\frac{\pi}{4} = \frac{y}{x+1} = 1$

$\therefore y = x + 1$ ……………………………… (1)

$z - 1 = x - 1 + i\,y$

$\Rightarrow arg(z-1) = tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$

$\Rightarrow tan\frac{2\pi}{3} = \frac{y}{x-1} = -\sqrt{3}$

$\therefore -\sqrt{3}x + \sqrt{3} = y$ …………………… (2)

$eqn. (1) = eqn. (2)$

$x + 1 = -\sqrt{3}x + \sqrt{3}$

$x\left(1+\sqrt{3}\right) + 1 - \sqrt{3} = 0 \therefore x = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$x = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$= \frac{3-2\sqrt{3}+1}{3-1} = 1 - \sqrt{3} \therefore x = 1 - \sqrt{3}$

From eqn. (1)

$y = x + 1 = 1 - \sqrt{3} + 1 = 2 - \sqrt{3}$

$\therefore z = \left(1-\sqrt{3}\right) + i\left(2-\sqrt{3}\right)$

6) *If* $(1+i3)z_1 = 5(1+i)$, *find* z_1

Solution

$(1+i3)z_1 = 5(1+i) \Rightarrow z_1 = \frac{5(1+i)}{(1+i3)}$

$z_1 = \frac{(5+5i)}{(1+i3)} = \frac{(5+5i)(1-i3)}{(1+i3)(1-i3)} = \frac{5-i\,15+i\,5-i^2\,15}{1^2-(i3)^2}$

$= \frac{5+15-10i}{1+9} = \frac{20-i\,10}{10} = 2-i$

$\therefore\ z_1 = 2-i$

7) *Given that x and y are real, find the values of x and y which satisfy the equation*

$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$

Solution

$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$

$\frac{2y}{2x+y} + \frac{4}{2x+y}\,i - \frac{y(x+i)}{(x-i)(x+i)} = 0$

$\frac{2y}{2x+y} + \frac{4}{2x+y}\,i - \frac{xy}{x^2+1} - \frac{y}{x^2+1}\,i = 0$

For equality of complex numbers

For real part: $\frac{2y}{2x+y} - \frac{xy}{x^2+1} = 0$ ………………………. (1)

For imaginary part: $\frac{4}{2x+y} - \frac{y}{x^2+1} = 0$ ………………………. (2)

From eqn. (2)

$\frac{4}{2x+y} = \frac{y}{x^2+1}$, substituting in eqn. (1)

$\Rightarrow \frac{2y}{2x+y} - \frac{4}{2x+y}.x = 0$

$\therefore \frac{2y-4x}{2x+y} = 0 \ \Rightarrow 2y-4x = 0$

$\therefore y = 2x$ …………………………………………. (3)

Eqn. (3) in to eqn. (1)

$\Rightarrow \frac{2(2x)}{2x+2x} - \frac{x(2x)}{x^2+1} = 0 \ \therefore 1 - \frac{2x^2}{x^2+1} = 0$

$\Rightarrow x^2 - 2x^2 + 1 = 0 \ or \ x^2 = 1$

$\therefore x = \pm 1$

From eqn. (3)

$y = 2(\pm 1) = \pm 2$

$\therefore x = 1\,, y = 2\,; x = -1\,, y = -2$

8) Find the values of x and y in the equation

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$ $(\boldsymbol{Uneb\,2006})$

Solution

Expressing complex numbers in the form $a+ib$

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$

$\frac{x(2-i\,3)}{(2+i\,3)(2-i\,3)} - \frac{y(3+i\,2)}{(3-i\,2)(3+i\,2)} = \frac{(6+2\,i)(1-i\,8)}{(1+i\,8)(1-i\,8)}$

$\frac{2x-i\,3x}{4+9} - \frac{3y+i\,2y}{9+4} = \frac{6-i\,48+i\,2-i^2\,16}{1+64}$

$\frac{2x-i\,3x}{13} - \frac{3y+i\,2y}{13} = \frac{6+16-i\,46}{65}$

$\Rightarrow \frac{2x}{13} - i\frac{3x}{13} - \frac{3y}{13} - i\frac{2y}{13} = \frac{22}{65} - i\frac{46}{65}$

Equating real parts and imaginary parts

For real part: $\Rightarrow \frac{2x}{13} - \frac{3y}{13} = \frac{22}{65} \ldots\ldots\ldots\ldots\ldots\ldots.\ldots. \times 65$

$\therefore 10x - 15y = 22$ ……………………………………………………. (1)

For imaginary part: $-\frac{3x}{13} - \frac{2y}{13} = -\frac{46}{65} \ldots\ldots\ldots\ldots.. \times 65$

$\therefore 15x + 10y = 46$ ………………………………………………………. (2)

Solving eqn. (1) and eqn. (2) simultaneously

$x = 2.8, y = 0.4$

Alternatively

Making the equation linear

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{x(3-i\,2)-y(2+i\,3)}{(2+i\,3)(3-i\,2)} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{3x-i\,2x-2y-i\,3y}{6-i\,4+i\,9-i^2\,6} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{(3x-2y)+i\,(-2x-3y)}{12+i\,5} = \frac{(6+2\,i)}{(1+i\,8)} \ldots\ldots\ldots\ldots \times (12+i\,5)$

$\Rightarrow (3x-2y) + i\,(-2x-3y) = \frac{(6+2\,i)(12+i\,5)}{(1+i\,8)}$

$= \frac{72+i\,30+i\,24+i^2\,10}{1+i8}$

$\frac{72+i\,30+i\,24+i^2\,10}{1+i8} = \frac{62+i\,54}{1+8\,i} = \frac{(62+i\,54)(1-i\,8)}{(1+8\,i)(1-i\,8)}$

$= \frac{62+i\,54-i\,496-i^2\,432}{1+64}$

$= \frac{494}{65} - i\frac{442}{65} = \frac{38}{5} - i\frac{34}{5}$

$\therefore (3x-2y) = \frac{38}{5}$ $or\ 15x - 10y = 38$ …………………… (1)

$(-2x-3y) = -\frac{38}{5}$ $or\ 10x + 15y = 34$ ……………….. (2)

Solving eqn. (1) and eqn. (2) simultaneously

$\therefore x = 2.8, y = 0.4$

Task

Given that x and y are real, find the values of x and y which satisfy the equation

a) $\frac{x+4\,i}{2x+y} - \frac{x+i}{y} = 0$

b) $\frac{3z}{1-i} + \frac{3z}{i} = \frac{4}{3-i}$

$\boldsymbol{Answer}: (\boldsymbol{b})\ \boldsymbol{x = 0.27, y = 0.53}$

ROOTS OF A GIVEN EQUATION

It can be proved that if $A + iB = x$ is a root of a quadratic equation $ax^2 + bx + c = 0$, then its conjugate is also a root.

Formulation of a quadratic equation with two given roots

The general formula used is:

$x^2 - (sum\ of\ the\ roots)x + product\ of\ roots = 0$

Example

1) *Find the equation whose one root is;*

(i) $2 - 3i$ (ii) $1 + i$ (iii) $3 + 4i$

Solution

i) Since $x = 2 - 3i$ is a root, then the other root $\bar{x} = 2 + 3i$ is also a root.

Sum of the roots: $2-3i+2+3i=4$
Product of roots: $(2-3i)(2+3i)=(2)^2-(i3)^2=4+9=13$
$\therefore x^2-4x+13=0$

ii) $x=1+i \Rightarrow x=1-i$ is also a root.
Sum of the roots: $1+i+1-i=2$
Product of roots: $(1+i)(1-i)=(1)^2-(i)^2=1+1=2$
$\therefore x^2-2x+2=0$

iii) $x=3+4i \Rightarrow x=3-4i$ is also a root.
Sum of the roots: $3+4i+3-4i=6$
Product of roots: $(3+4i)(3-4i)=(3)^2-(i4)^2=9+16=25$
$\therefore x^2-6x+25=0$

2) *Show that* $z=1$ *is the root of the equation* $z^3-5z^2+7z-5=0$. *Hence solve the equation.*(***Uneb* 2003**)

Solution

Let $f(z)=z^3-5z^2+7z-5=0$
For $z=1$
$\Rightarrow f(1)=(1)^3-5(1)^2+7(1)-5=0$. Hence $z=1$ is a root. Thus $(z-1)$ is a root of $f(z)$. Factorizing $f(z)$

$$\begin{array}{r|l}
 & z^2-4z+5 \\
\hline
z-1 & z^3-5z^2+7z-5 \\
 & z^3-z^2 \\
\hline
 & \quad -4z^2+9z-5 \\
 & \quad -4z^2+4z \\
\hline
 & \qquad\quad 5z-5 \\
 & \qquad\quad 5z-5 \\
\hline
\end{array}$$

$\therefore f(z)=(z-1)(z^2-4z+5)=0$
$z^2-4z+5=0$
$z=\frac{4\pm\sqrt{16-4(5)}}{2}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i$
$\therefore z=2\pm i$

3) *Show that* $z=2+3i$ *is a root of the equation* $z^4-5z^3+18z^2-17z+13=0$

Solution

$z=2+3i$
$z^2=(2+3i)^2=4+12i+i^2 9=-5+12i$
$z^3=(-5+12i)(2+3i)=-10-15i+24i+i^2 36$
$\quad=-10+9i-36=-46+9i$
$z^4=(-5+12i)^2=25-120i+i^2 144=-119-120i$
Let $f(z)=z^4-5z^3+18z^2-17z+13=0$
$f(z)=-119-120i-5(-46+9i)+18(-5+12i)-17(2+3i)+13$
$\quad=(-119+230-90-34+13)+i(-120-45+216-51)=0$
Hence $z=2+3i$ is a root of the equation.
Since $2+3i$ is a root of the equation, also $2-3i$ is a root.
Quadratic factor in $f(z)=z^4-5z^3+18z^2-17z+13$:
Sum of roots: $2+3i+2-3i=4$

Product of roots: $(2+3i)(2-3i) = (2)^2 - (3i)^2 = 4 + 9 = 13$

$\therefore z^2 - 4z + 13$ is a quadratic factor of $f(z)$

Factorizing $f(z)$

$$
\begin{array}{r|l}
 & z^2 - z + 1 \\
\hline
z^2 - 4z - 13 & z^4 - 5z^3 + 18z^2 - 17z + 13 \\
 & z^4 - 4z^3 + 13z^2 \\
\hline
 & \quad -z^3 + 5z^2 - 17z + 13 \\
 & \quad -z^3 + 4z^2 - 3z \\
\hline
 & \qquad z^2 - 4z + 13 \\
 & \qquad z^2 - 4z + 13 \\
\hline
\end{array}
$$

$\therefore f(z) = (z^2 - 4z + 13)(z^2 - z + 1) = 0$

$z^2 - z + 1 = 0$

$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} \quad \therefore z = \frac{1 \pm i\sqrt{3}}{2}$

4) *Show that* $z - 1 - i$ *is a factor of the expression* $z^4 - 6z^3 + 23z^2 - 34z + 26$. *Hence solve the equation* $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$

Solution

If $z - 1 - i$ is a factor then the root of the equation $f(z) = 0$ must be $z - 1 - i = 0$ *or*

$z = 1 + i$. Thus showing that $z = 1 + i$ is a root of $f(z) = 0$.

Let $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$

$z = 1 + i$

$z^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$

$z^3 = (2i)(1+i) = 2i + i^2 2 = -2 + 2i$

$z^4 = (2i)^2 = -4$

From $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$

$f(z) = -4 - 6(-2+2i) + 23(2i) - 34(1+i) + 26$

$\qquad = (-4 + 12 - 34 + 26) + i(-12 + 46 - 34) = 0$

Hence $z = 1 + i$ is a root of the equation hence $z - 1 - i$ is a factor of the expression

$z^4 - 6z^3 + 23z^2 - 34z + 26$

Since $1 + i$ is a root of the equation, also $1 - i$ is a root.

Quadratic factor in $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$:

Sum of roots: $1 + i + 1 - i = 2$

Product of roots: $(1+i)(1-i) = (1)^2 - (i)^2 = 1 + 1 = 2$

$\therefore z^2 - 2z + 2$ is a quadratic factor of $f(z)$

Factorizing $f(z)$

$$
\begin{array}{r|l}
 & z^2 - 4z + 13 \\
\hline
z^2 - 2z + 2 & z^4 - 6z^3 + 23z^2 - 34z + 26 \\
 & z^4 - 2z^3 + 2z^2 \\
\hline
 & \quad -4z^3 + 21z^2 - 34z + 26 \\
 & \quad -4z^3 + 8z^2 - 8z \\
\hline
 & \qquad 13z^2 - 26z + 26 \\
 & \qquad 13z^2 - 26z + 26 \\
\hline
\end{array}
$$

$\therefore f(z) = (z^2 - 2z + 2)(z^2 - 4z + 13) = 0$
$z^2 - 4z + 13 = 0$
$z = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm i6}{2} = 2 \pm 3i$
$\therefore z = 2 \pm 3i$

5) *Solve the equation* $z^4 - 6z^2 + 25 = 0$

Solution

$z^4 - 6z^2 + 25 = 0$ is an equation reducable to quadratic of the form $\boldsymbol{au^2 + bu + c = 0}$

Let $z^2 = u$
$\Rightarrow u^2 - 6u + 25 = 0$
$\therefore u = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$
But $z^2 = u \;\therefore z = \sqrt{u}$
For $u = 3 \pm 4i$; let $z = a + bi$
$\Rightarrow (a + bi)^2 = 3 + 4i$
$\Rightarrow a^2 + i2ab - b^2 = 3 \pm 4i$
$\therefore a^2 - b^2 = 3$ ………………………………. (1)
$\Rightarrow 2ab = \pm 4 \; or \; ab = \pm 2$
$\therefore a = \pm \frac{2}{b}$ ……………………………………………. (2)
$eqn. (2) \; in \; to \; eqn. (1)$
$\Rightarrow \left(\frac{2}{b}\right)^2 - b^2 = 3 \Rightarrow \frac{4}{b^2} - b^2 = 3$
$Let \; m = b^2$
$\Rightarrow \frac{4}{m} - m = 3 \;\therefore m^2 + 3m - 4 = 0$
Solving the equation yields $m = -4, 1$
$\Rightarrow b = \sqrt{m}$. Now for $m = -4, b = \sqrt{-4} = \pm 2i$
For $m = 1, b = \sqrt{1} = \pm 1$
From eqn. (3)
$a = \pm \frac{2}{b} \Rightarrow a = \frac{2}{b}$. For $b = \pm 2i \Rightarrow a = \pm \frac{2}{\pm 2i} = \pm i$
Also for $b = \pm 1 \Rightarrow a = \pm \frac{2}{\pm 1} = \pm 2$
$\therefore b = \pm 1, a = \pm 2 ; b = \pm 2i, a = \pm i$
$\Rightarrow z = \pm(2 + i)$

Task

1. *Show that* $z = 1$ *is a root of the equation* $z^4 - 3z^3 + 4z^2 - 3z + 1 = 0$. *Hence find the other roots.* ***Answer***: $\mathbf{1, 1, \frac{1}{2}(1 \pm i\sqrt{3})}$
2. *Given that that the complex number z and its conjugate* $\bar{z}$ *satisfy the equation:* $z\bar{z} + 3\bar{z} = 34 - 12i$. *Find the values of z.* (***Uneb*** **2010**)
3. *Given the complex number such that* $z = \frac{p}{2-i} + \frac{q}{1+3i}$ *where p and q are real.* *If* $arg(z) = \frac{\pi}{2}$ *and* $|z| = 7$, *find the values of p and q*
4. *Find the real values of a and b such that* $(a + ib)^2 = i$, *hence or otherwise solve the equation* $z^2 + 2z + 1 - i = 0$, *where* $z = a + ib$.
5. *Solve for z in the form* $a + ib$ if $\frac{2z-3}{1-4i} = \frac{3}{1+i}$

Equations involving n^{th} root of a complex number
Example
Solve the following equations
i) $z^3 + 1 = 0$
ii) $z^3 - 8 = 0$
iii) $z^4 - 8 - i\,8\sqrt{3} = 0$
iv) $z^3 - 5 + i\,12 = 0$
v) $z^3 - \left(\frac{2-i}{2+i}\right) = 0$

Solution

i) $z^3 + 1 = 0 \Rightarrow z^3 = -1 + 0\,i \;\therefore z = \sqrt[3]{-1 + 0\,i}$
Expressing z in polar form
$\Rightarrow z = 1\left(cos(-180^0) + i\,sin(-180^0)\right)$
$\sqrt[3]{z} = [cos(-180^0) + i\,sin(-180^0)]^{\frac{1}{3}}$
$= cos\left(\frac{-180^0+360^0k}{3}\right) + i\,sin\left(\frac{-180^0+360^0k}{3}\right)$, where $k = 0, 1\; and\; 2$
For $k = 0$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times 0}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times 0}{3}\right)$
$= \frac{1}{2} - \frac{i\sqrt{3}}{2}$
For $k = 1$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times 1}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times 1}{3}\right)$
$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$
For $k = 2$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times 2}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times 2}{3}\right)$
$= -1 + 0\,i$
$\therefore \sqrt[3]{z} = \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}\;, -1$

ii) $z^3 - 8 = 0 \Rightarrow z^3 = 8 + 0\,i \;\therefore z = \sqrt[3]{8 + 0\,i}$
Expressing z in polar form
$\Rightarrow z = 8\left(cos(0^0) + i\,sin(0^0)\right)$
$\sqrt[3]{z} = \left[8\left(cos(0^0) + i\,sin(0^0)\right)\right]^{\frac{1}{3}}$
$= 2\left[cos\left(\frac{0^0+360^0k}{3}\right) + i\,sin\left(\frac{0^0+360^0k}{3}\right)\right]$, where $k = 0, 1\; and\; 2$
For $k = 0$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times 0}{3}\right) + i\,sin\left(\frac{0^0+360^0\times 0}{3}\right)\right]$
$= 2 + 0\,i$
For $k = 1$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times 1}{3}\right) + i\,sin\left(\frac{0^0+360^0\times 1}{3}\right)\right]$
$= -1 + i\,\sqrt{3}$
For $k = 2$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times 2}{3}\right) + i\,sin\left(\frac{0^0+360^0\times 2}{3}\right)\right]$
$= -1 - i\,\sqrt{3}$
$\therefore \sqrt[3]{z} = 2 + 0\,i,\;, -1 \pm i\,\sqrt{3}$

iii) $z^4 - 8 - i\,8\sqrt{3} = 0 \Rightarrow z^4 = -8 + i\,8\sqrt{3} = z_1\;(say)$
$\therefore z = \sqrt[4]{z_1}$
Expressing z_1 in polar form
$|z_1| = \sqrt{(8)^2 + \left(8\sqrt{3}\right)^2} = 8\sqrt{1+3} = 16$
$arg(z_1) = arg\left[8\left(-1 + i\sqrt{3}\right)\right]$

$= arg\left(-1 + i\sqrt{3}\right)\ [\because arg(\lambda z) = arg(z)\ for\ \lambda > 0]$

$\therefore arg(z_1) = 120^0$

$\Rightarrow z_1 = 16\left(cos(120^0) + i\ sin(120^0)\right)$

$\Rightarrow z = \sqrt[4]{z_1} = \sqrt[4]{\left[16\left(cos(120^0) + i\ sin(120^0)\right)\right]}$

$= 2[cos(120^0) + i\ sin(120^0)]^{\frac{1}{4}}$

$= 2\left[cos\left(\frac{120^0+360^0k}{4}\right) + i\ sin\left(\frac{120^0+360^0k}{4}\right)\right]$, where $k = 0, 1, 2\ and\ 3$

For $k = 0, \sqrt[4]{z_1} = 2\left[cos\left(\frac{120^0+360^0\times 0}{4}\right) + i\ sin\left(\frac{120^0+360^0\times 0}{4}\right)\right]$

$= \sqrt{3} + i$

For $k = 1, \sqrt[3]{z} = 2\left[cos\left(\frac{120^0+360^0\times 1}{4}\right) + i\ sin\left(\frac{120^0+360^0\times 1}{4}\right)\right]$

$= -1 + i\sqrt{3}$

For $k = 2, \sqrt[3]{z} = 2\left[cos\left(\frac{120^0+360^0\times 2}{4}\right) + i\ sin\left(\frac{120^0+360^0\times 2}{4}\right)\right]$

$= -\sqrt{3} - i$

For $k = 3,\ 2\left[cos\left(\frac{120^0+360^0\times 3}{4}\right) + i\ sin\left(\frac{120^0+360^0\times 3}{4}\right)\right]$

$= 1 - i\sqrt{3}$

$\therefore z = \pm\left(\sqrt{3} + i\right), \pm\left(1 - i\sqrt{3}\right)$

iv) *Left as an exercise* ***Answer***: $\mathbf{1\lfloor 102^0 18'}, \mathbf{1\lfloor 222^0 18'}, \mathbf{1\lfloor 342^0 18'}, \mathbf{0.953 - i\ 0.304}$

LOCUS PROBLEMS IN COMPLEX NUMBERS

Consider a complex number $z = x + i\ y$. If z varies when subjected to some given condition, the corresponding set of points in the Argand diagram is called the *locus* of a point $P(x, y)$ representing z.

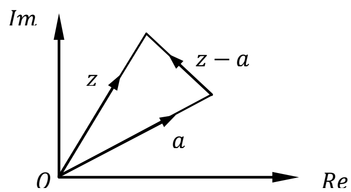
Loci are defined by:

- Distances
- Arguments/ angles

In general, equations of loci in Argand diagram involve moduli and arguments of complex variables.

Locus from Distances:

Consider the figure below.



$z = x + i\ y, P(x, y)$

$a = a_1 + i\ a_2$

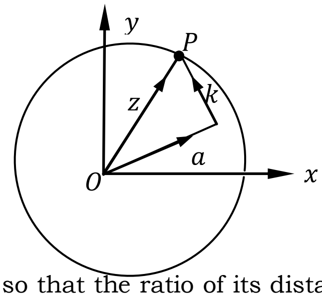
$z - a = \overrightarrow{AP}$

Finding magnitude of $\overrightarrow{AP}$

$\Rightarrow \left|\overrightarrow{AP}\right| = |z - a| = k\ (+ve\ constant)$

Thus Distance of P from A is a constant. Hence

$\lvert z - a\rvert = k$ is the equation of a circle with centre a at the point A and radius k



NOTE

If a point $P(x\,,y)$ move so that the ratio of its distance from two fixed points A and B is constant, then the locus is a circle. This locus is referred to as *Apollonius* circle and is represented in the Argand diagram by the equation of the form

$$\boxed{|z-a| = k|z-b| \;\; or \;\; \left|\frac{z-a}{z-b}\right| = k \; for \; k \neq 1}$$

Example

1) *Sketch the loci defined by the equations:*

i) $|z| = 2$

ii) $|z-3+2i| = 5$

iii) $|z-4+3i| = 4$

iv) $|z+i| = 3$

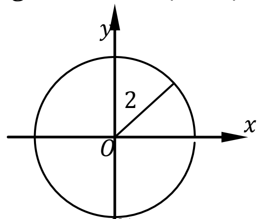
Solution

These are straight forward equations. Re-writing in the form $|z-a| = k$

i) $|z| = 2 \Rightarrow |z-(0+0i)| = 2$

$\therefore Centre\ of\ circle: (0\,,0)\,; Radius: 2$

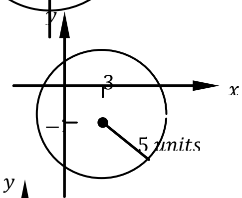
$|x+iy| = 2 \Rightarrow x^2+y^2 = 2^2$



ii) $|z-3+2i| = 5 \Rightarrow |z-(3-2i)| = 5$

Centre $(3\,,-2)$: $Radius$: 5

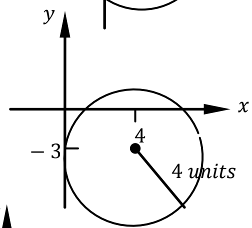
$|(x-3)+i(y+2)| = 5 \Rightarrow (x-3)^2+(y+2)^2 = 5^2$



iii) $|z-4+3i| = 4 \Rightarrow |z-(4-3i)| = 4$

Centre $(4\,,-3)$: $Radius$: 4

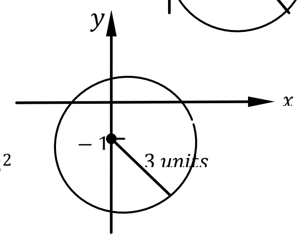
$|(x-4)+i(y+3)| = 5 \Rightarrow (x-4)^2+(y+3)^2 = 4^2$



iv) $|z+i| = 3 \Rightarrow |z-(0-i)| = 3$

Centre $(0\,,-1)$: $Radius$: 3

$|(x-0)+i(y+1)| = 3 \Rightarrow x^2+(y+2)^2 = 3^2$



2) *A point P represents a complex number $z = x+iy$ on an Argand diagram. Describe the locus of P if;*

a) $2|z+1| = |z-2|$

b) $|z+4i| = 3|z-4|$

c) $\left|\frac{z+i}{z-5-2i}\right| = 2$

d) $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3}$

e) $\left|\frac{z}{z+6}\right| = 5$

Solution

a) $2|z+1| = |z-2| \Rightarrow \frac{|z-2|}{|z+1|} = 2$

Let $z = x + yi$

$\frac{|z-2|}{|z+1|} = \frac{|(x-2)+iy|}{|(x+1)+iy|} = 2$

$\Rightarrow \frac{\sqrt{(x-2)^2+(y)^2}}{\sqrt{(x+1)^2+(y)^2}} = 2$, squaring both sides

$\Rightarrow \frac{(x-2)^2+(y)^2}{(x+1)^2+(y)^2} = 4 \ \therefore \frac{x^2-4x+4+y^2}{x^2+2x+1+y^2} = 4$

$\Rightarrow x^2 - 4x + 4 + y^2 = 4[x^2 + 2x + 1 + y^2]$

$\Rightarrow 3x^2 + 12x + 3y^2 = 0 \quad \therefore x^2 + y^2 + 4x = 0$

Circle with centre $(-2\,,0)$, Radius= $\sqrt{4+0-0} = 2\,units$

b) $|z+4i| = 3|z-4| \Rightarrow \frac{|z+4i|}{|z-4|} = 3$

$\frac{|z+4i|}{|z-4|} = \frac{|x+i(y+4)|}{|(x-4)+iy|} = 3$

$\Rightarrow \frac{\sqrt{x^2+(y+4)^2}}{\sqrt{(x-4)^2+y^2}} = 3$, squaring both sides

$\Rightarrow \frac{x^2+(y+4)^2}{(x-4)^2+y^2} = 9 \therefore \frac{x^2+y^2+8y+16}{x^2-8x+16+y^2} = 9$

$\Rightarrow x^2 + y^2 + 8y + 16 = 9[x^2 - 8x + 16 + y^2]$

$\Rightarrow 8x^2 - 72x - 8y + 128 + 8y^2 = 0 \ \therefore x^2 + y^2 - 9x - y + 16 = 0$

Circle with centre $\left(\frac{9}{2}\,,\frac{1}{2}\right)$, Radius= $\sqrt{\frac{81}{4}+\frac{1}{4}-16} = \frac{\sqrt{21}}{2}\ units$

c) $\left|\frac{z+i}{z-5-2i}\right| = 2 \Rightarrow \frac{|z+i|}{|z-5-2i|} = 2$

$\frac{|z+i|}{|z-5-2i|} = \frac{|x+i(y+1)|}{|(x-5)+i(y-2)|} = 2$

$\Rightarrow \frac{\sqrt{x^2+(y+1)^2}}{\sqrt{(x-5)^2+(y-2)^2}} = 2$, squaring both sides

$\Rightarrow \frac{x^2+(y+1)^2}{(x-5)^2+(y-2)^2} = 4 \therefore \frac{x^2+y^2+2y+1}{x^2-10x+25+y^2-4y+4} = 4$

$\Rightarrow x^2 + y^2 + 2y + 1 = 4[x^2 + y^2 - 10x - 4y + 29]$

$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 = 0 \quad \therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} = 0$

Circle with centre $\left(\frac{20}{3}\,,3\right)$, Radius= $\sqrt{\frac{400}{9}+9-\frac{115}{3}} = \frac{\sqrt{136}}{3} = \frac{2}{3}\sqrt{34}\,units$

d) $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3} \Rightarrow \frac{|z-1|}{|z+1-i|} = \frac{2}{3}$

$\frac{|z-1|}{|z+1-i|} = \frac{|(x-1)+iy|}{|(x+1)+i(y-1)|} = \frac{2}{3}$

$\Rightarrow \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+(y-1)^2}} = \frac{2}{3}$, squaring both sides

$\Rightarrow \frac{(x-1)^2+y^2}{(x+1)^2+(y-1)^2} = \frac{4}{9} \ \therefore \frac{x^2+y^2-2x+1}{x^2+2x+1+y^2-2y+1} = \frac{4}{9}$

$\Rightarrow 9[x^2 + y^2 - 2x + 1] = 4[x^2 + 2x + 1 + y^2 - 2y + 1]$

$\Rightarrow 9x^2 + 9y^2 - 18x + 9 = 4x^2 + 8x + 8 + 4y^2 - 8y$

$\therefore 5x^2 + 5y^2 - 26x + 8y + 1 = 0$

Circle with centre $\left(\frac{13}{5}, -\frac{4}{5}\right)$, Radius= $\sqrt{\frac{169}{25}+\frac{16}{25}-1} = \frac{\sqrt{184}}{5} = \frac{2}{5}\sqrt{46}\ units$

Task

If $z = x + iy$, determine the Cartesian equation of loci of the point z which moves in the Argand diagram so that;

i) $|z+2i|^2 + |z-2i|^2 = 40$
ii) $|z+2i|^2 - |z-2i|^2 = 24$
iii) $|z+ki|^2 + |z-ki|^2 = 30k^2$

ANSWER: $(\boldsymbol{i})\ \boldsymbol{x^2+y^2=16}\ \ (\boldsymbol{ii})\ \boldsymbol{y=3}\ \ (\boldsymbol{iii})\ \boldsymbol{x^2+y^2=4k^2}$

NOTE:

For $\left|\frac{z-a}{z-b}\right| = k\ and\ k = 1$, this is a case where locus is a *straight line*

Example

Sketch the locus of the point $P(x, y)$ representing the complex number $z = x + iy$ given that

i) $|z-1| = |z+i|$
ii) $\left|\frac{z+i}{z-5-2i}\right| = 1$

Write also the Cartesian equation of the locus.

Solution

i) $|z-1| = |z+i|\ ; z = x+iy$

$|(x-1)+iy| = |x+i(y+1)|$

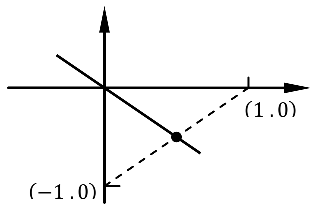
$\Rightarrow \sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y+1)^2}$;

squaring both sides

$\Rightarrow (x-1)^2+y^2 = x^2+(y+1)^2$

$\Rightarrow x^2+y^2-2x+1 = x^2+y^2+2y+1$

$\Rightarrow -2x = 2y\ \therefore y = -x\ (Line\ with\ slope = -1)$



ii) $\left|\frac{z+i}{z-5-2i}\right| = 1 \Rightarrow |z+i| = |z-5-2i|$

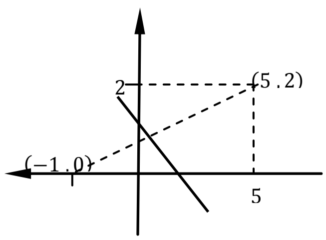
$|x+i(y+1)| = |(x-5)+i(y-2)|$

$\Rightarrow \sqrt{x^2+(y+1)^2} = \sqrt{(x-5)^2+(y-2)^2}$; squaring both sides

$\Rightarrow x^2+(y+1)^2 = (x-5)^2+(y-2)^2$

$\Rightarrow x^2+y^2+2y+1 = x^2+y^2-10x-4y+25+4$

$\Rightarrow 6y = -10x+28\ \therefore y = -\frac{5}{3}x+\frac{14}{3}\ \left(Line\ with\ slope = -\frac{5}{3}\right)$



LOCUS PROBLEMS FROM EQUALITY OF COMPLEX NUMBERS

1. *Given that* $Re\left(\frac{z-2i}{z+4}\right)=0$, *prove that in the Argand diagram, the locus of* z *is a circle. Find the centre and write as a complex number. Show that the radius is* $\sqrt{5}$

Solution

$\frac{z-2i}{z+4}=\frac{x+iy-2i}{x+iy+4}=\frac{x+i(y-2)}{(x+4)+iy}$; rationalizing the denominator,

$\Rightarrow \frac{[x+i(y-2)][(x+4)-iy]}{[(x+4)+iy][(x+4)-iy]}=\frac{x^2+4x-ixy+i(x+4)(y-2)-i^2(y^2-2y)}{(x+4)^2-i^2y^2}$

$\Rightarrow \frac{z-2i}{z+4}=\frac{x^2+4x-ixy+i(x+4)(y-2)+(y^2-2y)}{(x+4)^2+y^2}$

$\therefore \frac{z-2i}{z+4}=\frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2}+i\ \frac{(x+4)(y-2)-xy}{(x+4)^2+y^2}$

For $Real\ part=0$

$\Rightarrow \frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2}=0$

$\therefore x^2+4x+y^2-2y=0$, which is the required locus of a circle.

$Centre$: $\left(\frac{4}{-2}\ ,\ \frac{-2}{-2}\right)=(-2\ ,1)$

$\therefore Centre$: $-2+i$

$Radius$: $\sqrt{2^2+1^2-0}=\sqrt{5}$

2. *If* $z=x+iy$ *where* x *and* y *are real, show that*
   i) when $\frac{z+i}{z+2}$ is purely imaginary the locus of the point $P(x\ ,y)$ is a circle of radius $\frac{1}{2}\sqrt{5}$
   ii) When $\frac{z-2i}{2z-i}$ is purely imaginary the locus of a point $P(x\ ,y)$ in the Argand diagram is a circle. Write its centre as a complex number.
   iii) When $\frac{z-i}{z+2}$ is purely real the locus of the point $P(x\ ,y)$ is a straight line.

Solution

i) For purely imaginary complex number, $Real\ part=0$

$\frac{z+i}{z+2}=\frac{x+iy+i}{x+iy+2}=\frac{x+i(y+1)}{(x+2)+iy}$; rationalizing the denominator,

$\Rightarrow \frac{[x+i(y+1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}=\frac{x^2+2x-ixy+i(x+2)(y+1)-i^2(y^2+y)}{(x+2)^2-i^2y^2}$

$\Rightarrow \frac{z+i}{z+2}=\frac{x^2+2x-ixy+i(x+2)(y+1)+(y^2+y)}{(x+2)^2+y^2}$

$\therefore \frac{z+i}{z+2}=\frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2}+i\ \frac{(x+2)(y+1)-xy}{(x+2)^2+y^2}$

For $Real\ part=0$

$\Rightarrow \frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2}=0$

$\therefore x^2+y^2+2x+y=0$, which is the required locus of a circle.

$\therefore Centre$: $\left(-1\ ,\ -\frac{1}{2}\right)$

$\therefore Centre$: $1-\frac{1}{2}i$

$Radius$: $\sqrt{1^2+\left(\frac{1}{2}\right)^2-0}=\frac{1}{2}\sqrt{5}$

ii) $\frac{z-2i}{2z-i}=\frac{x+iy-2i}{2x+i2y-i}=\frac{x+i(y-2)}{2x+i(2y-1)}$; rationalizing the denominator,

$\Rightarrow \frac{z-2i}{2z-i}=\frac{[x+i(y-2)][2x-i(2y-1)]}{[2x+i(2y-1)][2x-i(2y-1)]}=\frac{2x^2-i(2xy-x)+i(2xy-4x)-i^2(y-2)(2y-1)}{(2x)^2-i^2(2y-1)^2}$

$\Rightarrow \frac{z-2i}{2z-i} = \frac{2x^2 - i(2xy-x) + i(2xy-4x) + 2y^2 - 5y + 2}{4x^2 + (2y-1)^2}$

$\therefore \frac{z-2i}{2z-i} = \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} + i \ \frac{-3x}{4x^2+(2y-1)^2}$

For $Real\ part = 0$

$\Rightarrow \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} = 0$

$\therefore 2x^2 + 2y^2 - 5y + 2 = 0\ or\ x^2 + y^2 - \frac{5}{2}y + 1 = 0$, which is the required locus of a circle.

$\therefore Centre: \left(0\ ,\ \frac{5}{4}\right)$

$\therefore Centre: 0 + \frac{5}{4}i$

iii) $\frac{z-i}{z+2}$ is purely Real if $Imaginary\ part = 0$

$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy}$; rationalizing the denominator,

$\Rightarrow \frac{[x+i(y-1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]} = \frac{x^2+2x-ixy+i(x+2)(y-1)-i^2(y^2-y)}{(x+2)^2-i^2y^2}$

$\Rightarrow \frac{z+i}{z+2} = \frac{x^2+2x-ixy+i(x+2)(y-1)+(y^2-y)}{(x+2)^2+y^2}$

$\therefore \frac{z+i}{z+2} = \frac{(x^2+2x+y^2-y)}{(x+2)^2+y^2} + i \ \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2}$

For $Imaginary\ part = 0$

$\Rightarrow \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2} = 0$

$\therefore (x+2)(y-1) - xy = 0$

$\Rightarrow xy - x + 2y - 2 - xy = 0$

$\therefore 2y = x + 2$, which is a straight line.

Task

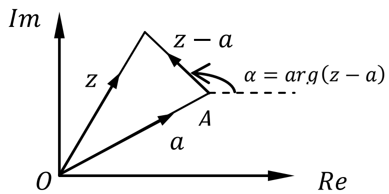
1) *If* $z = \frac{3}{3+cos\theta + i\ sin\theta}$, *Prove that the locus of z is a circle* $x^2 + y^2 = 4x - 3$
2) *If n is a variable and* $z = 4n + i3(1-n)$, *show that the locus of z is a straight line.*

LOCUS PROBLEMS FROM ARGUMENT OF COMPLEX NUMBER

Two major cases shall be considered.

<u>**Case I**</u>: $\boldsymbol{arg(z-a) = \alpha}$

P is a point representing complex number $z = x + iy$. A is a complex number $a = c + id$. $\overline{AP}$ is a straight line



Deduction

$arg(z-a) = \alpha$, is the equation of a *half line* with end point A inclined at an angle α to the $+ve$ Real axis.

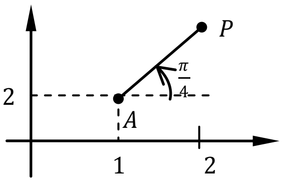
Example

1. *Sketch the loci defined by the equations*
   - i) $arg(z-1-2i)=\frac{\pi}{4}$
   - ii) $arg(z+2)=-\frac{2\pi}{3}$
   - iii) $arg(z+1-3i)=\frac{\pi}{6}$
   - iv) $arg(z+1-i)=-\frac{\pi}{4}$

Solution

i) $arg(z-1-2i)=\frac{\pi}{4}$, can be written as $arg[z-(1+2i)]=\frac{\pi}{4}\equiv\ arg(z-a)=\alpha$

This is a locus of half line with end point $(1,2)$ inclined at an angle $\frac{\pi}{4}$



Note:
Direction arrow of angle measurement is in anticlockwise since
$arg(z-1-2i)=\frac{\pi}{4}=+ve$

Finding Cartesian equation of line AP

Let $z=x+iy$

$\Rightarrow arg(z-1-2i)=arg\big((x-1)+(y-2)i\big)=\frac{\pi}{4}$

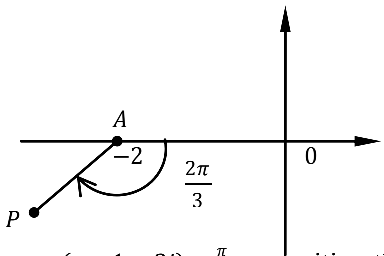
$\Rightarrow tan^{-1}\left(\frac{y-2}{x-1}\right)=\frac{\pi}{4}$

$\Rightarrow \frac{y-2}{x-1}=tan\frac{\pi}{4}=1$

$\therefore y-2=x-1\ \ or\ y=x+1$

ii) $arg(z+2)=-\frac{2\pi}{3}$, re-writing the equation

$\Rightarrow arg(z+2)=arg\left(z\text{—}(-2+0i)\right)=-\frac{2\pi}{3}\equiv arg(z-a)=\alpha$



Finding Cartesian equation of locus

Let $z=x+iy$

$\Rightarrow arg(z+2)=arg\big((x+2)+yi\big)=-\frac{2\pi}{3}$

$\Rightarrow tan^{-1}\left(\frac{y}{x+2}\right)=-\frac{2\pi}{3}$

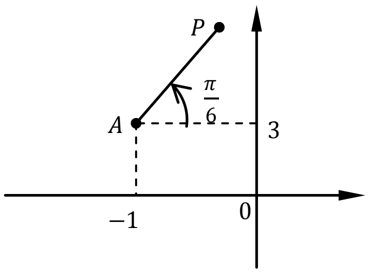
$\Rightarrow \frac{y}{x+2}=tan\left(-\frac{2\pi}{3}\right)=\sqrt{3}$

$\therefore y=x\sqrt{3}+2\sqrt{3}$

iii) $arg(z+1-3i)=\frac{\pi}{6}$, re-writing the equation

$\Rightarrow arg(z+1-3i)=arg\left(z\text{—}(-1+3i)\right)=\frac{\pi}{6}\equiv arg(z-a)=\alpha$

Half line with end point $A(-1,\ 3)$



Finding Cartesian equation of locus

Let $z=x+iy$

$\Rightarrow arg(z+1-3i)=arg\big((x+1)+(y-3)i\big)=\frac{\pi}{6}$

$\Rightarrow tan^{-1}\left(\frac{y-3}{x+1}\right)=\frac{\pi}{6}$

$\Rightarrow \frac{y-3}{x+1}=tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$

$\Rightarrow y-3=x\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}$

$\therefore y=x\frac{1}{\sqrt{3}}+\left(\frac{1}{\sqrt{3}}+3\right)$

iv) Left as an exercise for the student

<u>Case II</u>: $\boldsymbol{arg\left(\frac{z-a}{z-b}\right)=\gamma}$

Let $arg(z-a)=\alpha\,, arg(z-b)=\beta$ and $arg\left(\frac{z-a}{z-b}\right)=\gamma$

where $\gamma=\alpha-\beta\ (\pm 2\pi\ if\ necessary)$

NOTE:

γ can either be positive or negative depending on the turn of the half lines from the end points at a and b. *Anti clockwise turn* is $\overrightarrow{BP}$ to $\overrightarrow{AP}$ and γ is $+ve$ and for *clockwise turn*.$i.e$ $\overrightarrow{AP}$ to $\overrightarrow{BP}$. Two cases arise here

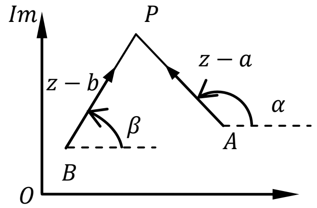


Fig $\boldsymbol{arg}\left(\frac{z-a}{z-b}\right)>0\,.i.e.\alpha>b$

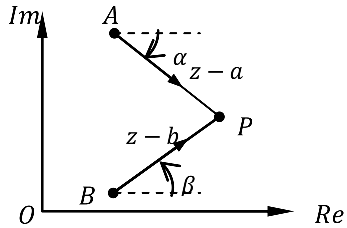


Fig $\boldsymbol{arg}\left(\frac{z-a}{z-b}\right)<0\,.i.e.\alpha<b$

In general, the equation $\boldsymbol{arg}\left(\frac{z-a}{z-b}\right)=\boldsymbol{\lambda}$, where $\lambda=constant$ represents a *circular arc* with end points A and B.

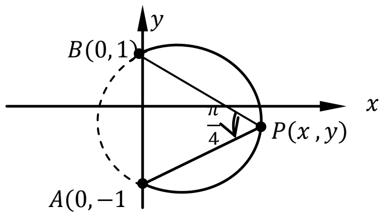
Example

1. *The point $P(x\,,y)$ represents $z=x+i\,y$ as the complex number on the Argand diagram. Describe and sketch the locus of P if;*
   a) $arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}$
   b) $arg\left(\frac{z-3}{z-1}\right)=\frac{\pi}{4}$
   c) $arg\left(\frac{z+2}{z-i}\right)=\frac{\pi}{3}$
   d) $arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$
   e) $arg\left(\frac{z-3-i}{z+5-3i}\right)=\frac{\pi}{3}$

Solution

a) $arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}$; re-writing the equation in the form $arg\left(\frac{z-a}{z-b}\right)=\lambda$

$arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}=arg\left[\frac{z-(0-i)}{z-(0-i)}\right]=\frac{\pi}{4}$, circular arc with end points $A(0\,,-1)$ and $B(0\,,1)$



Cartesian equation of locus:

$\frac{z+i}{z-i}=\frac{x+i(y+1)}{x+i(y-1)}=\frac{[x+i(y+1)][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]}$

$=\frac{x^2-i(xy-x)+i(xy+x)-i^2(y+1)(y-1)}{x^2-i^2(y-1)^2}$

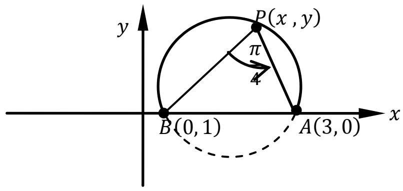
$=\frac{(x^2+y^2-1)+i(2x)}{x^2+(y-1)^2}$

$\therefore\ arg\left(\frac{z+i}{z-i}\right)=tan^{-1}\left[\frac{\frac{2x}{(x^2+(y-1)^2)}}{\frac{(x^2+y^2-1)}{(x^2+(y-1)^2)}}\right]=\frac{\pi}{4}$

$\Rightarrow \frac{2x}{x^2+y^2-1} = tan\frac{\pi}{4} = 1$

$\therefore x^2 + y^2 - 2x - 1 = 0$, centre $(1\,,0)$

b) $arg\left(\frac{z-3}{z-1}\right) = arg\left[\frac{z-(3+0i)}{z-(1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(3\,,0)$ and $B(0\,,1)$



Cartesian equation of locus:

$\frac{z-3}{z-1} = \frac{(x-3)+iy}{(x-1)+iy} = \frac{[(x-3)+iy][(x-1)-iy]}{[(x-1)+iy][(x-1)-iy]}$

$= \frac{(x-3)(x-1)-i(xy-3y)+i(xy-y)-i^2y^2}{(x-1)^2+(iy)^2}$

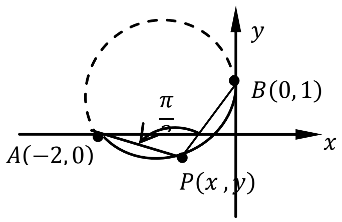
$= \frac{x^2-4x+3+i(3y-y)+y^2}{x^2-i^2(y-1)^2}$

$= \frac{(x^2+y^2-4x+3)+i(2y)}{x^2+(y-1)^2}$

$\therefore\ arg\left(\frac{z-3}{z-1}\right) = tan^{-1}\left[\frac{2y/(x^2+(y-1)^2)}{(x^2+y^2-4x+3)/(x^2+(y-1)^2)}\right] = \frac{\pi}{4}$

$\therefore x^2 + y^2 - 4x - 2y + 3 = 0$, centre $(2\,,1)$

c) $\arg\left(\frac{z+2}{z-i}\right) = \arg\left[\frac{z-(-2+0i)}{z-(0+i)}\right] = \frac{\pi}{3}$, circular arc with end points $A(-2\,,0)$ and $B(0\,,1)$



Cartesian equation of locus:

$\frac{z+2}{z-i} = \frac{(x+2)+iy}{x+i(y-1)} = \frac{[(x+2)+iy][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]}$

$\Rightarrow \frac{z+2}{z-i} = \frac{x^2+2x-i(y-1)(x+2)+ixy-i^2(y^2-y)}{x^2-i^2(y-1)^2}$

$= \frac{x^2+y^2+2x-y+i(xy+2y-x-2+xy)}{x^2+(y-1)^2}$

$= \frac{x^2+y^2+2x-y+i(-x+2y-2)}{x^2+(y-1)^2}$

$\therefore \arg\left(\frac{z+2}{z-i}\right) = \tan^{-1}\left[\frac{(-x+2y-2)/(x^2+(y-1)^2)}{(x^2+y^2+2x-y)/(x^2+(y-1)^2)}\right] = \frac{\pi}{3}$

$\Rightarrow \frac{-x+2y-2}{x^2+y^2+2x-y} = tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow\ -x + 2y - 2 = \sqrt{3}(x^2 + y^2 + 2x - y)$

$\Rightarrow x^2 + y^2 + 2x - y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y - \frac{2}{\sqrt{3}}$

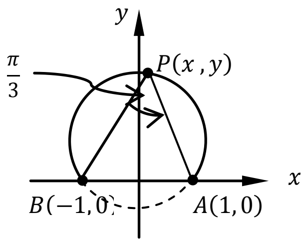
$\therefore x^2 + y^2 + \left(2 + \frac{1}{\sqrt{3}}\right)x + \left(-\frac{2}{\sqrt{3}} - 1\right)y + \frac{2}{\sqrt{3}} = 0$, centre $(-0.7\,,-0.1)$

d) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \arg\left[\frac{z-(1+0i)}{z-(-1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(1\,,0)$ and $B(-1\,,0)$

Note:

Since $\overrightarrow{BP}$ is rotating through $\frac{\pi}{4}$ ($Anticlock\,wise$) to $\overrightarrow{AP}$, arrow direction is as shown.

Cartesian equation of locus:



$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$
$$\Rightarrow \frac{z-1}{z+1} = \frac{x^2-1-i(xy-x)+i(xy+y)-i^2y^2}{(x-1)^2-i^2y^2}$$
$$\Rightarrow \frac{z-1}{z+1} = \frac{x^2+y^2-1+i(xy+y-xy+y)}{(x-1)^2+y^2}$$
$$= \frac{(x^2+y^2-1)+i(2y)}{(x-1)^2+y^2}$$

$\therefore arg\left(\frac{z-1}{z+1}\right) = tan^{-1}\left[\frac{2y}{(x^2+y^2-1)}\right] = tan\frac{\pi}{4} = 1$

$\therefore x^2+y^2-2y-1=0$, centre $(0\,,1)$

e) $\arg\left(\frac{z-3-i}{z+5-3i}\right) = \frac{\pi}{3} \Rightarrow \arg\left[\frac{z-(3+i)}{z-(-5+3i)}\right] = \frac{\pi}{3}$, circular arc with end points $A(3\,,1)$ and $B(-5\,,3)$

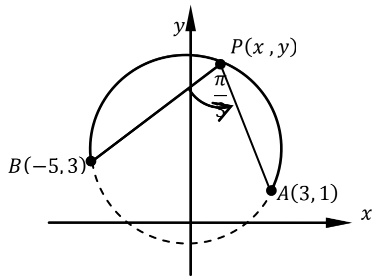
Cartesian equation of locus:

$$\frac{z-3-i}{z+5-3i} = \frac{(x-3)+i(y-1)}{(x+5)+i(y-3)} = \frac{[(x-3)+i(y-1)][(x+5)-i(y-3)]}{[(x+5)+i(y-3)][(x+5)-i(y-3)]}$$

$$= \frac{(x-3)(x+5)-i(x-3)(y-3)+i(y-1)(x+5)-i^2(y-3)(y-1)}{(x+5)^2+(y-3)^2}$$

$$= \frac{x^2+2x-15+i(xy+5y-x-5-xy+3y+3x-9)+y^2-4y+3}{(x+5)^2+(y-3)^2}$$

$$= \frac{(x^2+y^2+2x-4y-12)+i(2x+8y-14)}{(x+5)^2+(y-3)^2}$$



$\therefore arg\left(\frac{z-3-i}{z+5-3i}\right) = tan^{-1}\left[\frac{2x+8y-14}{x^2+y^2+2x-4y-12}\right] = \frac{\pi}{3}$

$\therefore \frac{2x+8y-14}{x^2+y^2+2x-4y-12} = tan\frac{\pi}{3} = \sqrt{3}$

$\therefore x^2+y^2+2x-4y-12 = \frac{2}{\sqrt{3}}x + \frac{8}{\sqrt{3}}y - \frac{14}{\sqrt{3}}$

$\therefore x^2+y^2+\left(2-\frac{2}{\sqrt{3}}\right)x + \left(-4-\frac{8}{\sqrt{3}}\right) + \left(\frac{14}{\sqrt{3}}-12\right) = 0$, centre $(-0.4\,,4.1)$

Task

Sketch the locus of the following

a) $arg\left(\frac{z+1}{z-1}\right) = -\frac{2\pi}{3}$

b) $arg\left(\frac{z-4}{z-2i}\right) = -\frac{\pi}{2}$

c) $arg\left(\frac{z-2i}{z-1}\right) = \frac{\pi}{3}$

2. *Represent each of the following loci on the Argand diagram*

a) $arg(z-1) = arg(z+1)$

b) $arg(z) = arg(z-1-i)$

c) $arg(z-2) = \pi + arg(z)$

d) $arg(z-1) = \pi + arg(z-i)$

e) $arg(z) = arg(z-4+2i) + \frac{\pi}{2}$

f) $arg(w-5+3i) = arg(w+2+3i) + \frac{\pi}{2}$

Solution

This example can be transformed in to forms as in example (1) above

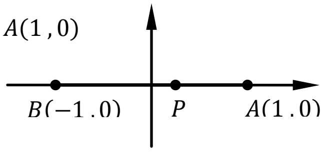
a) $arg(z-1) = arg(z+1) \Rightarrow arg(z-1) - arg(z+1) = 0$

$\therefore arg\left(\frac{z-1}{z+1}\right) = 0 \quad \left[\because arg(z_1) - arg(z_2) = arg\left(\frac{z_1}{z_2}\right)\right]$

$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$

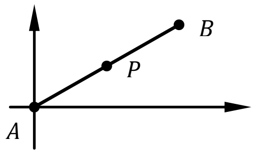
$= \frac{x^2-1+i(-xy+y+xy+y)+y^2}{(x+1)^2+y^2} = \frac{x^2+y^2-1-i2y}{(x+1)^2+y^2}$

$\Rightarrow tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = 0 \therefore y = 0$, is the locus of half straight line with end points $B(-1,0)$ and $A(1,0)$



b) $arg(z) = arg(z-1-i) \Rightarrow arg(z) - arg(z-1-i) = 0$

$\therefore arg\left(\frac{z}{z-1-i}\right) = arg\left[\frac{z-(0+0i)}{z-(1+i)}\right] = 0$



Since the angle $\overrightarrow{BP}$ is to turn to $\overrightarrow{AP}$ is zero, this is a straight line joining A and B (circular arc in form of a straight line).

Cartesian equation of locus:

$\frac{z}{z-1-i} = \frac{x+iy}{(x-1)+i(y-1)} = \frac{[x+iy][(x-1)-i(y-1)]}{[(x-1)+i(y-1)][(x-1)-i(y-1)]}$

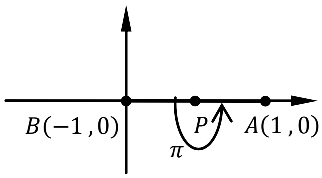
$\therefore \frac{z}{z-1-i} = \frac{x^2-x+i(-xy+x+xy-y)+y^2-y}{(x-1)^2+(y-1)^2} = \frac{x^2+y^2-x-y+i(x-y)}{(x-1)^2+(y-1)^2}$

$\therefore arg\left(\frac{z}{z-1-i}\right) = tan^{-1}\left(\frac{x-y}{x^2+y^2-x-y}\right) = 0$

$\therefore x = y$, is the locus of half straight line with end points $B(1,1)$ and $A(0,0)$

c) $arg(z-2) = \pi + arg(z) \Rightarrow arg(z-2) - arg(z) = \pi$

$\therefore arg\left(\frac{z-2}{z}\right) = arg\left[\frac{z-(2+0i)}{z-(0+0i)}\right] = \pi$, circular arc in form of a straight line with end points $B(0,0)$ and $A(2,0)$



Cartesian equation of locus:

$\frac{z-2}{z} = \frac{(x-2)+iy}{(x+iy)} = \frac{[(x-2)+iy][x-iy]}{[x+iy][x-iy]}$

$= \frac{x^2-2x+i(-xy-2y+xy)+y^2}{x^2+y^2}$

$= \frac{x^2+y^2-2x+i2y}{x^2+y^2}$

$\therefore arg\left(\frac{z-2}{z}\right) = tan^{-1}\left(\frac{2y}{x^2+y^2-2x}\right) = \pi$

$\Rightarrow \frac{2y}{x^2+y^2-2x} = tan\pi = 0 \therefore y = 0$

d) Left as an exercise for the students.

e) $arg(z) = arg(z-4+2i) + \frac{\pi}{2} \Rightarrow arg(z) - arg(z-4+2i) = \frac{\pi}{2}$

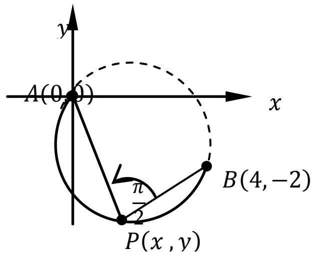
$\therefore arg\left(\frac{z}{z-4+2i}\right) = arg\left[\frac{z-(0+0i)}{z-(4-2i)}\right] = \frac{\pi}{2}$, circular arc with end points $B(4,-2)$ and $A(0,0)$

Cartesian equation of locus:

$\frac{z}{z-4+2i} = \frac{(x^2+y^2-4x+2y)+i(-2x-4y)}{(x-4)^2+(y+2)^2}$ $\left[\begin{matrix} \textit{show this as a} \\ \textit{student} \end{matrix}\right]$

$\therefore arg\left(\frac{z}{z-4+2i}\right) = tan^{-1}\left(\frac{-2x-4y}{x^2+y^2-4x+2y}\right) = \frac{\pi}{2}$

$\therefore x^2+y^2-4x+2y = 0$, since no value for $tan\frac{\pi}{2}$



Further examples on locus involving arguments

1) *Given that* $z = x+iy$, *find and sketch the locus* $arg(z) + arg(\lambda z) = \frac{\pi}{2}$ *if;*

a) $\lambda > 0$

b) $\lambda < 0$, where λ is a real number.

Solution

It can be shown that $\lambda = \lambda + 0i$

a) $arg(\lambda z) = arg(\lambda) + arg(z)$

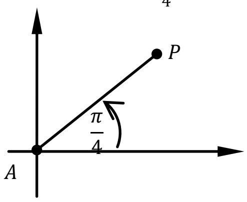
$\therefore arg(\lambda z) = 0 + arg(z) = arg(z)$

$\Rightarrow arg(\lambda z) = arg(z)$

$\therefore arg(z) + arg(z) = \frac{\pi}{2} \Rightarrow 2\,arg(z) = \frac{\pi}{2}$

$\therefore arg(z) = \frac{\pi}{4}$

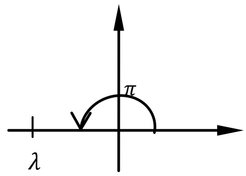
$\therefore arg(z) = arg[z-(0+0i)] = \frac{\pi}{4}$, which is half line with one end point as $(0,0)$



b) $arg(\lambda z) = arg(\lambda) + arg(z)$, for $\lambda < 0. i.e. \lambda\, is -ve$

$arg(\lambda) = \pi$ $(The\ principal\ argument)$

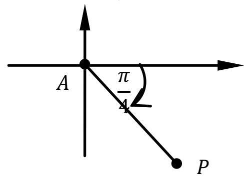
$\Rightarrow arg(\lambda z) = \pi + arg(z)$



$\therefore arg(z) + arg(\lambda z) = arg(z) + \pi + arg(z) = \frac{\pi}{2}$

$\Rightarrow 2\,arg(z) = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$\therefore arg(z) = -\frac{\pi}{4}$



2) *If* $arg(2z) + \frac{\pi}{4} = arg(z+i)$, *show that the locus of z is a circle.*

Solution

$arg(2z) + \frac{\pi}{4} = arg(z+i) \Rightarrow arg(z+i) - arg(2z) = \frac{\pi}{4}$

$\Rightarrow arg(z+i) - arg(z) = \frac{\pi}{4}$

$\Rightarrow arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$

$\frac{z+i}{z} = arg\left[\frac{z-(0-i)}{z-(0+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $B(0,0)$ and $A(0,-1)$

Cartesian equation of locus:

$\frac{z+i}{z} = \frac{x+i(y+1)}{(x+iy)} = \frac{[x+i(y+1)][x-iy]}{[x+iy][x-iy]}$

$\therefore \frac{z+i}{z} = \frac{x^2-ixy+i(xy+x)+y^2+y}{x^2+y^2} = \frac{x^2+y^2+y+ix}{x^2+y^2}$

$\therefore arg\left(\frac{z+i}{z}\right) = tan^{-1}\left(\frac{x}{x^2+y^2+y}\right) = \frac{\pi}{4}$

$\therefore x^2 + y^2 - x + y = 0$, which is a circle of centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius$\frac{\sqrt{2}}{2}$ $units$

Locus problems involving inequalities

This involves finding the equation of the locus, representing it on the Argand diagram and shading the unwanted region.

The knowledge portrayed in the previous examples shall be applied here.

Locus problems may involve arguments, modulus of inequality complex number.

Example

1. *Shade on separate Argand diagrams the regions in which;*
   i) $|z - 4 + 4i| < 4$
   ii) $|z - 3| > |z - 5|$
   iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$
   iv) $\left|\frac{z+2}{z-i}\right| > 3$
   v) $3 < |z| < 5$
   vi) $4 < |z + a| \leq 5$, where $a = 1 + i$

Solution

i) *Recall:* $|z - 4 + 4i| = 4$ represents a circle of centre $(4, -4)$ and radius 4 units.

Let $z = x + iy$

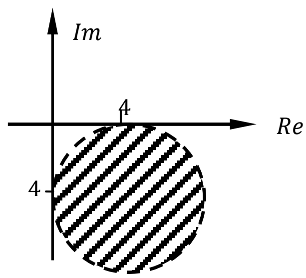
$|z - 4 + 4i| \Rightarrow |(x - 4) + i(y + 4)| < 4$

$\therefore (x - 4)^2 + (y + 4)^2 < 4^2$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore (4 - 4)^2 + (-4 + 4)^2 < 4^2$, which is true.

Hence shading the outside region of a circle,



ii) $|z - 3| > |z - 5|$

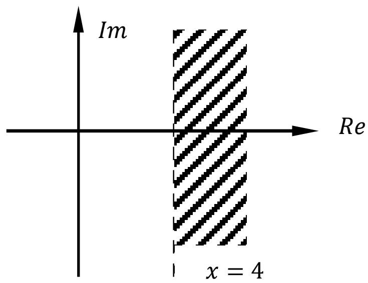
Finding the Cartesian inequality of locus

$|(x - 3) + iy| > |(x - 5) + iy|$, squaring both sides

$\Rightarrow (x - 3)^2 + y^2 > (x - 5)^2 + y^2$

$\Rightarrow x^2 - 6x + 9 > x^2 - 10x + 25$

$\Rightarrow 4x > 16 \;\therefore x > 4$



iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$

Finding the Cartesian inequality of locus

$\Rightarrow |z + i| \geq 2|z - 5 - 2i|$

$\therefore |x + i(y + 1)| \geq 2|(x - 5) + i(y - 2)|$, squaring both sides

$\Rightarrow x^2 + (y + 1)^2 \geq 4[(x - 5)^2 + (y - 2)^2]$

$\Rightarrow x^2 + y^2 + 2y + 1 \geq 4x^2 - 40x + 100 + 4y^2 - 16y + 16$

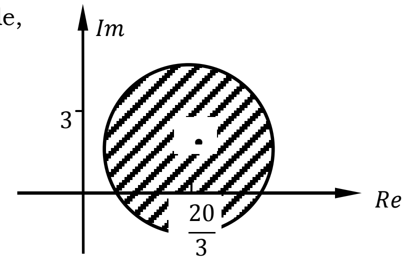
$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 \leq 0$

$\therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} \leq 0$, which is a circle of centre $\left(\frac{20}{3}, 3\right)$ and radius $\sqrt{\left(\frac{20}{3}\right)^2 + 3^2 - \frac{115}{3}} = \frac{2}{3}\sqrt{34}$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore \left(\frac{20}{3}\right)^2 + (3)^2 - \frac{40}{3}\left(\frac{20}{3}\right) - 6(3) + \frac{115}{3} = -\frac{274}{9} \leq 0$, which satisfies the inequality.

Shading the inside of the circle,



iv) $\left|\frac{z+2}{z-i}\right| > 3$

Finding the Cartesian inequality of locus

$\Rightarrow |z+2| > 3|z-i|$

$\therefore |(x+2)+iy| > 3|x+i(y-1)|$, squaring both sides

$\Rightarrow (x+2)^2 + y^2 > 9[x^2 + (y-1)^2]$

$\Rightarrow x^2 + 4x + 4 + y^2 > 9x^2 + 9y^2 - 18y + 9$

$\Rightarrow 8x^2 + 8y^2 - 4x - 18y + 5 > 0$

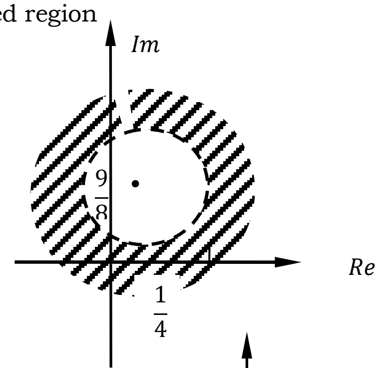
$\therefore x^2 + y^2 - \frac{1}{2}x - \frac{9}{4}y + \frac{5}{8} > 0$, which is a circle of centre $\left(\frac{1}{4}, \frac{9}{8}\right)$ and radius $\sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{9}{8}\right)^2 - \frac{5}{8}} = \frac{3}{8}\sqrt{5}$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore \left(\frac{1}{4}\right)^2 + \left(\frac{9}{8}\right)^2 - \frac{1}{8} - \frac{81}{32} + \frac{5}{8} = -\frac{45}{64} \leq 0$,

, which does not satisfy the inequality.

Shading the outside of the circle,



v) $3 < |z| < 5$, for $3 < |z|$, $3^2 < x^2 + y^2$

For $|z| < 5$, $x^2 + y^2 \leq 5^2$

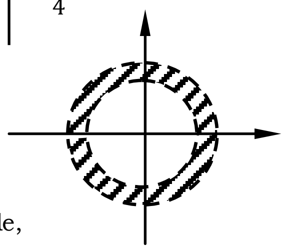
Using the centre coordinate to find the wanted region to be shaded,

For $9 < x^2 + y^2 \Rightarrow 3^2 < 0^2 + 0^2$, this is not true.

Shading the outside of circle,

Now for $|z| < 5$, $x^2 + y^2 < 5^2$

$\Rightarrow 0^2 + 0^2 < 5^2$, this is true. Shading the inside of circle,



vi) $4 < |z+a| \leq 5$, $a = 1 + i$

$\Rightarrow 4^2 < (x+1)^2 + (y+1)^2 \leq 5^2$

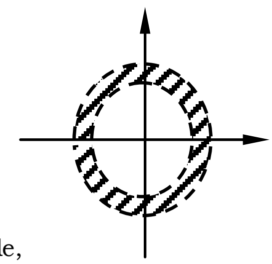
Using the centre $(-1, -1)$ to find the wanted region to be shaded,

For $4^2 < (x+1)^2 + (y+1)^2 \Rightarrow 4^2 < 0^2 + 0^2$

, this is not true shading the outside of circle,

Now for $(x+1)^2 + (y+1)^2 \leq 5^2$

$\Rightarrow 0^2 + 0^2 \geq 5^2$, this is true. Shading the inside of circle,

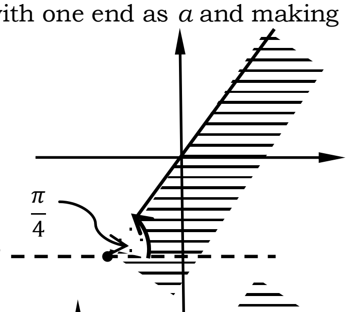


NOTE:
When testing region that is to be shaded, always use the centre to represent the inner region of the locus.

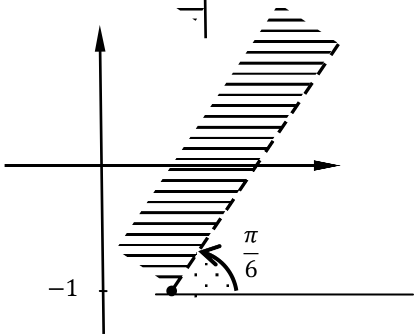
2. *Shade the region represented by;*
   a) $arg(z + a) \leq \frac{\pi}{4}$, $a = 1 + i$
   b) $arg(z - 1 + i) > \frac{\pi}{6}$
   c) $arg(z + 2 - i) < -\frac{2\pi}{3}$
   d) $arg(z - i) \geq -\frac{\pi}{3}$
   e) $-\frac{\pi}{2} \leq arg(z + i) \leq \frac{2\pi}{3}$
   f) $0 < arg(z - a) \leq \frac{\pi}{2}$, where $a = 1 + i$
   g) $-\frac{\pi}{6} < arg(z + a) < \frac{\pi}{4}$; $a = 2 + i$

Solution

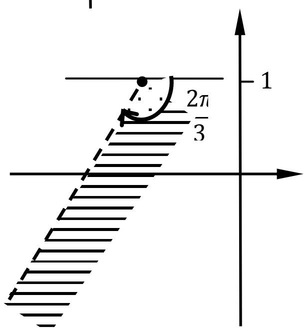
a) $arg(z + a) \leq \frac{\pi}{4}$, $a = 1 + i$
Recall: $arg(z - a) = \alpha$, represents a half a line with one end as a and making an angle of α with $+ve$ real axis.
Cartesian inequality:
$arg(z + 1 + i) \leq \frac{\pi}{4}$
$arg[(x + 1) + i(y + 1)] \leq \frac{\pi}{4}$
$\Rightarrow tan^{-1}\left(\frac{y+1}{x+1}\right) \leq \frac{\pi}{4} \Rightarrow y + 1 \leq (x + 1)tan\frac{\pi}{4}$
$\therefore y + 1 \leq x + 1$ *or* $y \leq x$, shading below the line,



b) $arg(z - 1 + i) > \frac{\pi}{6} \Rightarrow arg[z - (1 - i)] > \frac{\pi}{6}$
Cartesian inequality:
$arg[(x - 1) + i(y + 1)] > \frac{\pi}{6}$
$\Rightarrow tan^{-1}\left(\frac{y+1}{x-1}\right) > \frac{\pi}{6} \Rightarrow y + 1 > (x - 1)tan\frac{\pi}{6}$
$\therefore y + 1 > (x - 1)\frac{1}{\sqrt{3}}$ *or* $y > \frac{1}{\sqrt{3}}x + \left(-1 - \frac{1}{\sqrt{3}}\right)$
, shading the above region of the line,



c) $arg(z + 2 - i) < -\frac{2\pi}{3} \Rightarrow arg\big[z - (-2 + i)\big] < -\frac{2\pi}{3}$
Cartesian inequality:
$arg[(x + 2) + i(y - 1)] < -\frac{2\pi}{3}$
$\Rightarrow tan^{-1}\left(\frac{y-1}{x+2}\right) < -\frac{2\pi}{3}$
$\Rightarrow y - 1 < (x + 2)tan\left(-\frac{2\pi}{3}\right)$
$\therefore y - 1 < (x + 2)\sqrt{3}$ *or* $y < \sqrt{3}x + \left(1 + 2\sqrt{3}\right)$
, shading the below region of the line,



d) *left as an exercise*

e) $-\frac{\pi}{2} \leq arg(z+i) \leq \frac{2\pi}{3}$

$\Rightarrow -\frac{\pi}{2} \leq arg(z+i)\ ;\ arg(z+i) \leq \frac{2\pi}{3}$

Representing these on the Argand diagram,

For $-\frac{\pi}{2} \leq arg(z+i) \Rightarrow -\frac{\pi}{2} \leq arg[z-(0-i)]$

Cartesian inequality:

$-\frac{\pi}{2} \leq arg[x+i(y+1)]$

$\Rightarrow -\frac{\pi}{2} \leq tan^{-1}\left(\frac{y+1}{x}\right)\ \ or\ \ tan\left(-\frac{\pi}{2}\right) \leq \frac{y+1}{x}$

$\therefore 0 \leq x\ \ or\ x \geq 0$, shading the **R.H.S** of the line,

For $arg(z+i) \leq \frac{2\pi}{3} \Rightarrow arg[z-(0-i)] \leq \frac{2\pi}{3}$

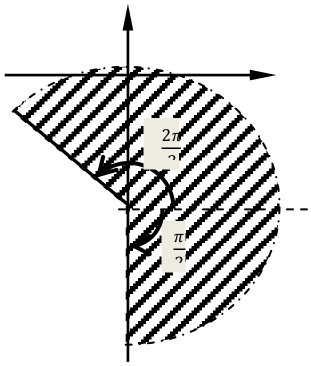
Cartesian inequality:

$arg[x+i(y+1)] \leq \frac{2\pi}{3}$

$\Rightarrow tan^{-1}\left(\frac{y+1}{x}\right) \leq \frac{2\pi}{3}\ \ or\ \ \frac{y+1}{x} \leq tan\left(\frac{2\pi}{3}\right)$

$\therefore \frac{y+1}{x} \leq -\sqrt{3}\ \ or\ y+1 \leq -\sqrt{3}x$

$\therefore y \leq -\sqrt{3}x - 1$, shading the below region of the line,



f) Left as an exercise

g) Left as an exercise